Economics 101A
(Lecture 4)

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Outline

1. Constrained Maximization
2. Envelope Theorem II
3. Preferences
4. Properties of Preferences
5. From Preferences to Utility

1 Constrained Maximization

- Ch. 2, pp. 38–44 [OLD, 39-46]

- So far unconstrained maximization on R (or open subsets)

- What if there are constraints to be satisfied?

- Example 1: $\max_{x,y} x * y$ subject to $3x + y = 5$

- Substitute it in: $\max_{x,y} x * (5 - 3x)$

- Solution: $x^* =$

- Example 2: $\max_{x,y} xy$ subject to $x \exp(y) + y \exp(x) = 5$

- Solution: ?

- Graphical intuition on general solution.

- Example 3: $\max_{x,y} f(x,y) = x * y$ s.t. $h(x,y) = x^2 + y^2 - 1 = 0$

- Draw $0 = h(x,y) = x^2 + y^2 - 1$.

- Draw $x * y = K$ with $K > 0$. Vary K

- Where is optimum?

- Where dy/dx along curve $xy = K$ equals dy/dx along curve $x^2 + y^2 - 1 = 0$

- Write down these slopes.

- Idea: Use implicit function theorem.

- Heuristic solution of system

$$\max_{x,y} f(x,y)$$
$$\text{s.t. } h(x,y) = 0$$

- Assume:
  - continuity and differentiability of h
  - $h'_y \neq 0$ (or $h'_x \neq 0$)

- Implicit function Theorem: Express y as a function of x (or x as function of y)!

- Write system as $\max_x f(x, g(x))$

- f.o.c.: $f'_x(x, g(x)) + f'_y(x, g(x)) * \frac{\partial g(x)}{\partial x} = 0$

- What is $\frac{\partial g(x)}{\partial x}$?

- Substitute in and get: $f'_x(x, g(x)) + f'_y(x, g(x)) * (-h'_x/h'_y) = 0$ or

$$\frac{f'_x(x, g(x))}{f'_y(x, g(x))} = \frac{h'_x(x, g(x))}{h'_y(x, g(x))}$$

- **Lagrange Multiplier Theorem, necessary condition.** Consider a problem of the type

$$\max_{x_1,\ldots,x_n} f\left(x_1, x_2, \ldots, x_n; \mathbf{p}\right)$$

$$\text{s.t.} \quad \left\{ \begin{array}{c} h_1\left(x_1, x_2, \ldots, x_n; \mathbf{p}\right) = 0 \\ h_2\left(x_1, x_2, \ldots, x_n; \mathbf{p}\right) = 0 \\ \ldots \\ h_m\left(x_1, x_2, \ldots, x_n; \mathbf{p}\right) = 0 \end{array} \right.$$

with $n > m$. Let $\mathbf{x}^* = \mathbf{x}^*(\mathbf{p})$ be a local solution to this problem.

- Assume:

  - f and h differentiable at x^*

  - the following Jacobian matrix at $\mathbf{x}^*$ has maximal rank

$$J = \left(\begin{array}{ccc} \frac{\partial h_1}{\partial x_1}(\mathbf{x}^*) & \ldots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}^*) \\ \ldots & \ldots & \ldots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}^*) & \ldots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}^*) \end{array} \right)$$

- Then, there exists a vector $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_m)$ such that $(\mathbf{x}^*, \boldsymbol{\lambda})$ maximize the Lagrangean function

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}; \mathbf{p}) - \sum_{j=0}^{m} \lambda_j h_j(\mathbf{x}; \mathbf{p})$$

- Case $n = 2, m = 1$.

- First order conditions are

$$\frac{\partial f(\mathbf{x}; \mathbf{p})}{\partial x_i} - \lambda \frac{\partial h(\mathbf{x}; \mathbf{p})}{\partial x_i} = 0$$

for $i = 1, 2$

- Rewrite as

$$\frac{f'_{x_1}}{f'_{x_2}} = \frac{h'_{x_1}}{h'_{x_2}}$$

- **Constrained Maximization, Sufficient condition for the case $n = 2, m = 1$.**

- If $\mathbf{x}^*$ satisfies the Lagrangean condition, and the determinant of the bordered Hessian

$$H = \begin{pmatrix} 0 & -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial^2 x_1}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2 \partial x_1}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2 \partial x_2}(\mathbf{x}^*) \end{pmatrix}$$

is positive, then $\mathbf{x}^*$ is a constrained maximum.

- If it is negative, then $\mathbf{x}^*$ is a constrained minimum.

- Why? This is just the Hessian of the Lagrangean L with respect to λ, x_1, and x_2

- Example 4: $\max_{x,y} x^2 - xy + y^2$ s.t. $x^2 + y^2 - p = 0$

- $\max_{x,y,\lambda} x^2 - xy + y^2 - \lambda(x^2 + y^2 - p)$

- F.o.c. with respect to x:

- F.o.c. with respect to y:

- F.o.c. with respect to λ:

- Candidates to solution?

- Maxima and minima?

2 Envelope Theorem II

- Envelope Theorem II: Ch. 2, pp. 44 [OLD, 46-47]

- **Envelope Theorem for Constrained Maximization.** In problem above consider $F(p) \equiv f(\mathbf{x}^*(\mathbf{p}); \mathbf{p})$. We are interested in $dF(p)/dp$. We can neglect indirect effects:

$$\frac{dF}{dp_i} = \frac{\partial f(\mathbf{x}^*(\mathbf{p}); \mathbf{p})}{\partial p_i} - \sum_{j=0}^{m} \lambda_j \frac{\partial h_j(\mathbf{x}^*(\mathbf{p}); \mathbf{p})}{\partial p_i}$$

- Example 4 (continued). $\max_{x,y} x^2 - xy + y^2$ s.t. $x^2 + y^2 - p = 0$

- $df(x^*(p), y^*(p))/dp$?

- Envelope Theorem.

3 Preferences

- Part 1 of our journey in microeconomics: *Consumer Theory*

- Choice of consumption bundle:

  1. Consumption today or tomorrow
  2. work, study, and leisure
  3. choice of government policy

- Starting point: preferences.

  1. 1 egg today $\succ$ 1 chicken tomorrow
  2. 1 hour doing problem set $\succ$ 1 hour in class $\succ$ ... $\succ$ 1 hour out with friends
  3. War on Iraq $\succ$ Sanctions on Iraq

4 Properties of Preferences

- Nicholson, Ch. 3, pp. 69-70 [OLD: 66]

- Commodity set X (apples vs. strawberries, work vs. leisure, consume today vs. tomorrow)

- Preference relation $\succeq$ over X

- A preference relation $\succeq$ is *rational* if

  1. It is *complete*: For all x and y in X, either $x \succeq y$, or $y \succeq x$ or both

  2. It is *transitive*: For all x, y, and z, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$

- Preference relation $\succeq$ is *continuous* if for all y in X, the sets $\{x : x \succeq y\}$ and $\{x : y \succeq x\}$ are closed sets.

- Example: $X = R^2$ with map of indifference curves

- Counterexamples:

  1. Incomplete preferences. Dominance rule.

  2. Intransitive preferences. Quasi-discernible differences.

  3. Discontinuous preferences. Lexicographic order

- Indifference relation $\sim$: $x \sim y$ if $x \succeq y$ and $y \succeq x$
- Strict preference: $x \succ y$ if $x \succeq y$ and not $y \succeq x$
- Exercise. If $\succeq$ is rational,
  - $\succ$ is transitive
  - $\sim$ is transitive
  - Reflexive property of $\succeq$. For all x, $x \succeq x$.

- Other features of preferences

- Preference relation $\succeq$ is:

  - *monotonic* if $x \geq y$ implies $x \succeq y$.

  - *strictly monotonic* if $x \geq y$ and $x_j > y_j$ for some j implies $x \succ y$.

  - *convex* if for all x, y, and z in X such that $x \succeq z$ and $y \succeq z$, then $tx + (1 - t)y \succeq z$ for all t in $[0, 1]$

5 From preferences to utility

- Nicholson, Ch. 3

- Economists like to use utility functions $u : X \to R$

- $u(x)$ is 'liking' of good x

- $u(a) > u(b)$ means: I prefer a to b.

- **Def.** Utility function u represents preferences $\succeq$ if, for all x and y in X, $x \succeq y$ if and only if $u(x) \geq u(y)$.

- **Theorem.** If preference relation $\succeq$ is rational and continuous, there exists a continuous utility function $u : X \to R$ that represents it.

- [Skip proof]

- Example:

$$(x_1, x_2) \succeq (y_1, y_2) \text{ iff } x_1 + x_2 \geq y_1 + y_2$$

- Draw:

- Utility function that represents it: $u(x) = x_1 + x_2$

- But... Utility function representing $\succeq$ is not unique

- Take $\exp(u(x))$

- $u(a) > u(b) \Longleftrightarrow \exp(u(a)) > \exp(u(b))$

- If $u(x)$ represents preferences $\succeq$ and f is a strictly increasing function, then $f(u(x))$ represents $\succeq$ as well.

- If preferences are represented from a utility function, are they rational?
  - completeness
  - transitivity

- Indifference curves: $u(x_1, x_2) = \bar{u}$
- They are just implicit functions! $u(x_1, x_2) - \bar{u} = 0$

$$\frac{dx_2}{dx_1} = -\frac{U'_{x_1}}{U'_{x_2}} = MRS$$

- Indifference curves for:
  - monotonic preferences;
  - strictly monotonic preferences;
  - convex preferences

6 Next Class

- Common Utility Functions

- Utility Maximization