Economics 101A
(Lecture 15)

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Outline

1. Production Function II
2. Returns to Scale
3. Two-step Cost Minimization
4. Cost Minimization: Example

1 Production Function II

- Isoquants $Q(y) = \{\mathbf{x} | f(\mathbf{x}) = y\}$

- Set of inputs $\mathbf{z}$ required to produce quantity y

- Special case. Two inputs:

  - $z_1 = L$ (labor)

  - $z_2 = K$ (capital)

- Isoquant: $f(L, K) - y = 0$

- Slope of isoquant $dK/dL = MRTS$

- Convex production function if convex isoquants

- Reasonable: combine two technologies and do better!

- Mathematically, convex isoquants if $d^2K/d^2L > 0$

- Solution:

$$\frac{d^2K}{d^2L} = -\frac{f''_{L,L}f'_K - 2f''_{L,K}f'_L + f''_{K,K}\left(f'_L\right)^2/f'_K}{\left(f'_K\right)^2}$$

- Hence, $d^2K/d^2L > 0$ if $f''_{L,K} > 0$ (inputs are complements in production)

2 Returns to Scale

- Nicholson, Ch. 9, pp. 302-305 (Ch. 7, pp. 190–193, 9th)

- Effect of increase in labor: f'_L

- Increase of all inputs: $f(t\mathbf{z})$ with t scalar, $t > 1$

- How much does input increase?

  – Decreasing returns to scale: for all $\mathbf{z}$ and $t > 1$,

$$f(t\mathbf{z}) < tf(\mathbf{z})$$

– Constant returns to scale: for all $\mathbf{z}$ and $t > 1$,

$$f(t\mathbf{z}) = tf(\mathbf{z})$$

– Increasing returns to scale: for all $\mathbf{z}$ and $t > 1$,

$$f(t\mathbf{z}) > tf(\mathbf{z})$$

- Example: $y = f(K, L) = AK^{\alpha}L^{\beta}$

- Marginal product of labor: $f'_L =$

- Decreasing marginal product of labor: $f''_{L,L} =$

- $MRTS =$

- Convex isoquant?

- Returns to scale: $f(tK, tL) = A(tK)^{\alpha}(tL)^{\beta} = t^{\alpha+\beta}AK^{\alpha}L^{\beta} = t^{\alpha+\beta}f(K, L)$

3 Two-step Cost minimization

- Nicholson, Ch. 10, pp. 323-330 (Ch. 12 , pp. 212–220, 9th)

- Objective of firm: Produce output that generates maximal profit.

- Decompose problem in two:

  – Given production level y, choose cost-minimizing combinations of inputs

  – Choose optimal level of y.

- *First step.* Cost-Minimizing choice of inputs

- Two-input case: Labor, Capital

- Input prices:
  - Wage w is price of L
  - Interest rate r is rental price of capital K

- Expenditure on inputs: $wL + rK$

- Firm objective function:

$$\min_{L,K} wL + rK$$
$$s.t. f(L, K) \geq y$$

- Equality in constraint holds if:

  1. $w > 0$, $r > 0$;

  2. f strictly increasing in at least L or K.

- Counterexample if ass. 1 is not satisfied

- Counterexample if ass. 2 is not satisfied

- Compare with expenditure minimization for consumers

- First order conditions:

$$w - \lambda f'_L = 0$$

and

$$r - \lambda f'_K = 0$$

- Rewrite as

$$\frac{f'_L(L^*, K^*)}{f'_K(L^*, K^*)} = \frac{w}{r}$$

- MRTS (slope of isoquant) equals ratio of input prices

- Graphical interpretation

- Derived demand for inputs:
  - $L = L^* (w, r, y)$
  - $K = K^* (w, r, y)$
- Value function at optimum is **cost function:**

$$c(w, r, y) = wL^* (r, w, y) + rK^* (r, w, y)$$

- *Second step.* Given cost function, choose optimal quantity of y as well

- Price of output is p.

- Firm's objective:

$$\max py - c(w, r, y)$$

- First order condition:

$$p - c'_y(w, r, y) = 0$$

- Price equals marginal cost – very important!

- Second order condition:

$$-c''_{y,y}(w, r, y^*) < 0$$

- For maximum, need increasing marginal cost curve.

4 Cost Minimization: Example

- Continue example above: $y = f(L, K) = AK^{\alpha}L^{\beta}$

- Cost minimization:

$$\begin{array}{l} \min wL + rK \\ s.t. AK^{\alpha}L^{\beta} = y \end{array}$$

- Solutions:

  – Optimal amount of labor:

$$L^{*}(r, w, y) = \left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}}$$

  – Optimal amount of capital:

$$\begin{aligned} K^{*}(r, w, y) &= \frac{w}{r}\frac{\alpha}{\beta}\left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}} = \\ &= \left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \end{aligned}$$

- Check various comparative statics:
  - $\partial L^*/\partial A < 0$ (technological progress and unemployment)
  - $\partial L^*/\partial y > 0$ (more workers needed to produce more output)
  - $\partial L^*/\partial w < 0$, $\partial L^*/\partial r > 0$ (substitute away from more expensive inputs)

- Parallel comparative statics for K^*

- Cost function

$$\begin{aligned} c(w,r,y) &= wL^*(r,w,y) + rK^*(r,w,y) = \\ &= \left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}} \left[\begin{array}{c} w\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}} + \\ +r\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \end{array} \right] \end{aligned}$$

- Define $B := w\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}} + r\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}$

- Cost-minimizing output choice:

$$\max py - B\left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}}$$

- First order condition:

$$p - \frac{1}{\alpha+\beta}\frac{B}{A}\left(\frac{y}{A}\right)^{\frac{1-(\alpha+\beta)}{\alpha+\beta}} = 0$$

- Second order condition:

$$-\frac{1}{\alpha+\beta}\frac{1-(\alpha+\beta)}{\alpha+\beta}\frac{B}{A^2}\left(\frac{y}{A}\right)^{\frac{1-2(\alpha+\beta)}{\alpha+\beta}}$$

- When is the second order condition satisfied?

- Solution:
  - $\alpha + \beta = 1$ (CRS):
    - S.o.c. equal to 0
    - Solution depends on p
    - For $p > \frac{1}{\alpha+\beta}\frac{B}{A}$, produce $y^* \to \infty$
    - For $p = \frac{1}{\alpha+\beta}\frac{B}{A}$, produce any $y^* \in [0, \infty)$
    - For $p < \frac{1}{\alpha+\beta}\frac{B}{A}$, produce $y^* = 0$

- $\alpha + \beta > 1$ (IRS):
  - S.o.c. positive
  - Solution of f.o.c. is a minimum!
  - Solution is $y^* \to \infty$.
  - Keep increasing production since higher production is associated iwth higher returns

- $\alpha + \beta < 1$ (DRS):
  - s.o.c. negative. OK!
  - Solution of f.o.c. is an interior optimum
  - This is the only "well-behaved" case under perfect competition
  - Here can define a supply function

5 Next Lectures

- Geometry of Cost Curves
- Profit Maximization
- Aggregation
- Market Equilibrium