Economics 101A
(Lecture 6)

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Outline

1. From Preferences to Utility (and viceversa)

2. Common Utility Functions

3. Utility maximization

1 From preferences to utility

- Nicholson, Ch. 3

- Economists like to use utility functions $u : X \to R$

- $u(x)$ is 'liking' of good x

- $u(a) > u(b)$ means: I prefer a to b.

- **Def.** Utility function u represents preferences $\succeq$ if, for all x and y in X, $x \succeq y$ if and only if $u(x) \geq u(y)$.

- **Theorem.** If preference relation $\succeq$ is rational and continuous, there exists a continuous utility function $u : X \to R$ that represents it.

- [Skip proof]

- Example:

$$(x_1, x_2) \succeq (y_1, y_2) \text{ iff } x_1 + x_2 \geq y_1 + y_2$$

- Draw:

- Utility function that represents it: $u(x) = x_1 + x_2$

- But... Utility function representing $\succeq$ is not unique

- Take $3u(x)$ or $\exp(u(x))$

- $u(a) > u(b) \iff \exp(u(a)) > \exp(u(b))$

- If $u(x)$ represents preferences $\succeq$ and f is a strictly increasing function, then $f(u(x))$ represents $\succeq$ as well.

- If preferences are represented from a utility function, are they rational?
  - completeness
  - transitivity

- Indifference curves: $u(x_1, x_2) = \bar{u}$

- They are just implicit functions! $u(x_1, x_2) - \bar{u} = 0$

$$\frac{dx_2}{dx_1} = -\frac{U'_{x_1}}{U'_{x_2}} = MRS$$

- Indifference curves for:
  - monotonic preferences;
  - strictly monotonic preferences;
  - convex preferences

2 Common utility functions

- Nicholson, Ch. 3, pp. 100-104 (82-86, 9th)

1. Cobb-Douglas preferences: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$
   - $MRS = -\alpha x_1^{a-1} x_2^{1-\alpha} / (1-a) x_1^{\alpha} x_2^{-\alpha} = \frac{\alpha}{1-\alpha} \frac{x_2}{x_1}$

2. Perfect substitutes: $u\left(x_1, x_2\right) = \alpha x_1 + \beta x_2$
   - $MRS = -\alpha / \beta$

3. Perfect complements: $u(x_1, x_2) = \min(\alpha x_1, \beta x_2)$

   - MRS discontinuous at $x_2 = \frac{\alpha}{\beta} x_1$

4. Constant Elasticity of Substitution: $u(x_1, x_2) = \left(\alpha x_1^\rho + \beta x_2^\rho\right)^{1/\rho}$

   - $MRS = -\frac{\alpha}{\beta}\left(\frac{x_1}{x_2}\right)^{\rho-1}$

   - if $\rho = 1$, then...

   - if $\rho = 0$, then...

   - if $\rho \to -\infty$, then...

3 Utility Maximization

- Nicholson, Ch. 4, pp. 114–124 (94–105, 9th)

- $X = R^2_+$ (2 goods)

- Consumers: choose bundle $x = (x_1, x_2)$ in X which yields highest utility.

- Constraint: income $= M$

- Price of good 1 $= p_1$, price of good 2 $= p_2$

- Bundle x is feasible if $p_1 x_1 + p_2 x_2 \leq M$

- Consumer maximizes

$$\max_{x_1, x_2} u(x_1, x_2)$$
$$s.t.\ p_1 x_1 + p_2 x_2 \leq M$$
$$x_1 \geq 0,\ x_2 \geq 0$$

- Maximization subject to inequality. How do we solve that?

- Trick: u strictly increasing in at least one dimension. ($\succeq$ strictly monotonic)

- Budget constraint always satisfied with equality

- Ignore temporarily $x_1 \geq 0$, $x_2 \geq 0$ and check afterwards that they are satisfied for x_1^* and x_2^*.

- Problem becomes

$$\max_{x_1, x_2} u(x_1, x_2)$$
$$s.t.\ p_1 x_1 + p_2 x_2 - M = 0$$

- $L(x_1, x_2) = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 = M)$

- F.o.c.s:

$$\begin{aligned} u'_{x_i} - \lambda p_i &= 0 \text{ for } i = 1, 2 \\ p_1 x_1 + p_2 x_2 - M &= 0 \end{aligned}$$

- Moving the two terms across and dividing, we get:

$$MRS = -\frac{u'_{x_1}}{u'_{x_2}} = -\frac{p_1}{p_2}$$

- Graphical interpretation.

- Second order conditions:

$$H = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{x_1,x_1} & u''_{x_1,x_2} \\ -p_2 & u''_{x_2,x_1} & u''_{x_2,x_2} \end{pmatrix}$$

$$\begin{aligned} |H| &= p_1 \left(-p_1 u''_{x_2,x_2} + p_2 u''_{x_2,x_1}\right) \\ &\quad -p_2 \left(-p_1 u''_{x_1,x_2} + p_2 u''_{x_1,x_1}\right) \\ &= -p_1^2 u''_{x_2,x_2} + 2p_1 p_2 u''_{x_1,x_2} - p_2^2 u''_{x_1,x_1} \end{aligned}$$

- Notice: $u''_{x_2,x_2} < 0$ and $u''_{x_1,x_1} < 0$ usually satisfied (but check it!).

- Condition $u''_{x_1,x_2} > 0$ is then sufficient

- Example with CES utility function.

$$\max_{x_1,x_2} \left(\alpha x_1^\rho + \beta x_2^\rho\right)^{1/\rho}$$
$$s.t.\ p_1 x_1 + p_2 x_2 - M = 0$$

- Lagrangean =

- F.o.c.:

- Solution:

$$\begin{aligned} x_1^* &= \frac{M}{p_1\left(1 + \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\rho-1}}\left(\frac{p_2}{p_1}\right)^{\frac{\rho}{\rho-1}}\right)} \\ x_2^* &= \frac{M}{p_2\left(1 + \left(\frac{\beta}{\alpha}\right)^{\frac{1}{\rho-1}}\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{\rho-1}}\right)} \end{aligned}$$

- Special case 1: $\rho = 0$ (Cobb-Douglas)

$$\begin{aligned} x_1^* &= \frac{\alpha}{\alpha+\beta}\frac{M}{p_1} \\ x_2^* &= \frac{\beta}{\alpha+\beta}\frac{M}{p_2} \end{aligned}$$

- Special case 1: $\rho \to 1$ (Perfect Substitutes)

$$\begin{aligned} x_1^* &= \begin{cases} 0 & \text{if } \; p_1/p_2 \geq \alpha/\beta \\ M/p_1 & \text{if } \; p_1/p_2 < \alpha/\beta \end{cases} \\ x_2^* &= \begin{cases} M/p_2 & \text{if } \; p_1/p_2 \geq \alpha/\beta \\ 0 & \text{if } \; p_1/p_2 < \alpha/\beta \end{cases} \end{aligned}$$

- Special case 1: $\rho \to -\infty$ (Perfect Complements)

$$x_1^* = \frac{M}{p_1 + p_2} = x_2^*$$

- Parameter ρ indicates substition pattern between goods:
  - $\rho > 0$ –> Goods are (net) substitutes
  - $\rho < 0$ –> Goods are (net) complements

4 Next Class

- Utility Maximization – Tricky Cases
- Indirect Utility Function
- Comparative Statics:
  - with respect to price
  - with respect to income