Econ 101A – Solution to Problem Set 2
No late Problem Sets accepted, sorry!

This Problem set tests the knowledge that you accumulated in the lectures 5 to 8. It is focused on preferences, utility functions, and utility maximization. General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

Problem 1. Addictive goods. (23 points) In this exercise, we propose a generalization of Cobb-Douglas preferences that incorporates the concept of reference point. We use it to model the consumption of addictive goods. Consider the following utility function:

$$u(x_1, x_2; r_1) = (x_1 - r_1)^{\alpha} {x_2}^{\beta}$$

with $\alpha + \beta = 1$, $0 < \alpha < 1$, $0 < \beta < 1$, and $r_1 > 0$. Notice that the above utility is only defined for $x_1 \geq r_1$ and $x_2 \geq 0$. Assume that for $x_1 < r_1$ or $x_2 < 0$ the utility is zero. Good x_1 is an addictive good with addiction level r_1. Examples of addictive goods are alcohol, drugs or... chocolate. The more you have consumed of these goods in the past, the higher the addiction level r_1.

1. Draw an approximate map of indifference curves for the case $\alpha = \beta = .5$. (2 points)
2. How does the utility function change as r_1 changes? In other words, compute $\partial u(x_1, x_2; r_1)/\partial r_1$. Why is this term negative? [Hint: If I have gotten used to drinking a lot of alcohol, my utility of drinking three bottles of beer...] (3 points)
3. Compute now the marginal utility with respect to x_1. In other words, compute $\partial u(x_1, x_2; r_1)/\partial x_1$ for $x_1 > r_1$. How does this marginal utility change as r_1 changes? In other words, compute $\partial^2 u(x_1, x_2; r_1)/\partial x_1 \partial r_1$ for $x_1 > r_1$. Why is this term positive? [Hint: If I have gotten used to drinking a lot of alcohol, my desire to drink one *more* bottle of beer...] (3 points)
4. Consider now the maximization subject to a budget constraint. The agent maximizes
$$\begin{aligned} \max_{x_1, x_2} u(x_1, x_2) &= (x_1 - r_1)^{\alpha} {x_2}^{\beta} \\ \text{s.t. } p_1 x_1 + p_2 x_2 &= M. \end{aligned}$$
Write down the Lagrangean function. (1 point)
5. Write down the first order conditions for this problem with respect to x_1, x_2, and λ. (1 point)
6. Solve explicitely for x_1^* and x_2^* as a function of p_1, p_2, M, r_1, α, and β. [You do not have to check the second order conditions. I guarantee that they are satisfied :-), provided that the condition in point 7 is satified] (3 points)
7. What is the minimum level of income in order for the solution to make sense, i.e., so that $x_1^* \geq r_1$ and $x_2^* \geq 0$? (for a lower level of income the agent would have zero utility) (2 points)
8. Is good x_1 a normal good, i.e., is $\partial x_1^*/\partial M > 0$ for all values of M above the minimum level of income in point 7? (2 points)
9. Compute the change in x_1^* as r_1 varies. In order to do so, use directly the expressions that you obtained in point 6, and differentiate x_1^* with respect to r_1. Does your result make sense? Why do I consumer more of good 1 if I am more addicted to it (higher r_1)? (2 points)
10. Compute the change in x_2^* as r_1 varies: differentiate x_2^* with respect to r_1. Does your result make sense? Think of the case of drug addicts that spend virutally all of their income into buying drugs. (2 points)

11. Use the envelope theorem to calculate $\partial u(x_1^* (p_1, p_2, M; r_1), x_2^* (p_1, p_2, M; r_1); r_1)/\partial r_1$. What happens to utility at the optimum as the level of addition increases? (2 points)

Solution to Problem 1.

1. See Figure in the other file.

2. $\partial u(x_1, x_2; r_1)/\partial r_1 = -\alpha(x_1 - r_1)^{\alpha-1} x_2^\beta < 0$. The intuition here is that having a higher reference point is bad. Higher r_1 is associated to a lower level of utility. An alcoholist is less happy after drinking three bottles of beer than someone with normal driking habits.

3. $\partial u(x_1, x_2; r_1)/\partial x_1 = \alpha(x_1 - r_1)^{\alpha-1} x_2^\beta > 0$. As for the cross second derivative, $\partial^2 u(x_1, x_2; r_1)/\partial x_1 \partial r_1 = -\alpha (\alpha - 1) (x_1 - r_1)^{\alpha-2} x_2^\beta > 0$. The intuition here is that a higher reference point increases the marginal utility of consumption. Higher r_1 is associated to a higher desire for one additional unit. An alcoholist craves the fourth bottle of beer more than someone with normal driking habits.

4. Lagrangean is $L(x_1, x_2, \lambda) = (x_1 - r_1)^\alpha x_2{}^\beta - \lambda (p_1 x_1 + p_2 x_2 - M)$.

5. First order conditions:
$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= \alpha(x_1 - r_1)^{\alpha-1} x_2^\beta - \lambda p_1 = 0 \\
\frac{\partial L}{\partial x_2} &= \beta(x_1 - r_1)^\alpha x_2^{\beta-1} - \lambda p_2 = 0 \\
\frac{\partial L}{\partial \lambda} &= p_1 x_1 + p_2 x_2 - M = 0
\end{aligned}$$

6. Using the first two equations, we find
$$\frac{\alpha x_2}{\beta (x_1 - r_1)} = \frac{p_1}{p_2}$$
or $x_2 p_2 = \frac{\beta}{\alpha} (x_1 - r_1) p_1$. We substitute this into the budget constraint to get $p_1 x_1 + \frac{\beta}{\alpha} (x_1 - r_1) p_1 = M$ or $x_1^* = \alpha \left(M + \frac{\beta}{\alpha} r_1 p_1\right) / p_1$ (we used $\alpha + \beta = 1$) or
$$x_1^* = \alpha \frac{M}{p_1} + (1 - \alpha) r_1. \tag{1}$$
The demand for good 1 thus is a convex combination of the addiction level r_1 and the maximal number of units of good 1 that the consumer could afford, M/p_1. Using the budget constraint, we then get
$$x_2^* = (M - \alpha M - (1 - \alpha) r_1 p_1) / p_2 = (1 - \alpha) (M - r_1 p_1) / p_2. \tag{2}$$

7. This solution makes sense if $x_1^* \geq r_1$ and $x_2^* \geq 0$, or $M/p_1 \geq r_1$. In words, the agent must have at least enough income to afford r_1 units of good 1.

8. From expression (1) it is clear that $\partial x_1^*/\partial M = \alpha/p_1 > 0$ for all M. This good is a normal good. The presence of addictive effects does not alter the fact that goods with Cobb-Douglas preference are normal.

9. From expression (1) we obtain $\partial x_1^*/\partial r_1 = (1 - \alpha) > 0$. If I am more addicted to a good (higher r_1), I have a higher marginal utility for the good (see point 3) and therefore I consumer more of that good in equilibrium, compared to someone who is not addicted.

10. From expression (2) we obtain $\partial x_2^*/\partial r_1 = -(1 - \alpha) p_1/p_2 < 0$. If I am more addicted to a good (higher r_1), I have a higher taste for the addictive good than for the 'normal' good 2. Therefore the addicted agent in equilibrium consumes less of good 2 compared to a non-addicted agent.

11. Use the envelope theorem to compute $\partial v(p_1, p_2, M; r_1)/\partial r_1 = \partial \left[(x_1^* - r_1)^\alpha (x_2^*)^\beta - \lambda^* (p_1 x_1^* + p_2 x_2^* - M)\right] / \partial r_1 = -\alpha(x_1^* - r_1)^{\alpha-1} (x_2^*)^\beta < 0$ since $x_1^* > r_1$ and $x_2^* > 0$. Therefore, as addiction goes up (higher r_1) utility in equilibrium goes down. Beware, addiction is bad!

Problem 2. Quasi-linear preferences (25 points) In economics, it is often convenient to write the utility function in a quasi-linear form. These utility functions have the following form:

$$u(x_1, x_2) = \phi(x_1) + x_2$$

with $\phi'(x) > 0$, and $\phi''(x) < 0$. These preferences are called quasi-linear because the utility function is linear in good 2. In this exercise we explore several convenient properties of this utility function. We will do so at first without assuming a particular functional form for $\phi(x)$.

Consider the maximization subject to a budget constraint. The agent maximizes

$$\max_{x_1, x_2} \ \phi(x_1) + x_2$$
$$\text{s.t. } p_1 x_1 + p_2 x_2 = M$$

with $p_1 > 0$, $p_2 > 0, M > 0$.

1. Write down the Lagrangean function (1 point)
2. Write down the first order conditions for this problem with respect to x_1, x_2, and λ. (1 point)
3. What do the first order conditions tell you regarding the value of λ? **(Hint: Use the first order condition with respect to x_2)** Does the value of λ depend on p_1 or M? (usually it does) Why is this the case? Think of λ as the marginal utility of wealth. (3 points)
4. Plug the value of λ into the first order condition for x_1. You now have an equation that implicitely defines x_1^* as a function of the parameters p_1, p_2, M. Does the optimal quantity of x_1^* depend on income M? Is good 1 a normal good ($\partial x_1^*/\partial M > 0$), an inferior good ($\partial x_1^*/\partial M < 0$), or a neutral good ($\partial x_1^*/\partial M = 0$)? [If $\partial x_1^*/\partial M = 0$, we say that it is a neutral good, i.e., that there is no income effect] (5 points)
5. Use the implicit function theorem to compute $\partial x_1^*/\partial p_1$ from the first order condition with respect to x_1 (remember, you have already substituted for the value of λ). You should find that $\partial x_1^*/\partial p_1 < 0$. You should know this already from the answer to point 4. Why? (Hint: think about the Slutzky equation) (5 points)
6. Continue now under the assumption $u(x_1, x_2) = x_1^{1/2} + x_2$. Explicitely solve for x_1^* and then, using the budget constraint, solve for x_2^*. (2 points)
7. Under what conditions for p_1, p_2, and M is $x_2^* \geq 0$? (2 points)
8. The indifference curves satisfy equation $x_1^{1/2} + x_2 = \overline{u}$ or $x_2 = \overline{u} - x_1^{1/2}$. Draw a map of indifference curves in the space (x_1, x_2). What is special about this indifference curves? (compare them, for example, to the ones for Cobb-Douglas preferences) (3 points)
9. Write down two budget lines: for $(p_1 = 1, p_2 = 1, M = 1)$ and for $(p_1 = 1, p_2 = 1, M = 2)$. Find graphically the optimal consumption bundles by tangency of the budget set and the indifference curve. You should find that $x_1^*(1, 1, 1) = x_1^*(1, 1, 2)$. This means that there is no income effect in good 1. The increase in income goes all toward good 2. This should be a graphical confirmation of what you found at point 4 (3 points)

Solution to Problem 2.

1. Lagrangean is $L(x_1, x_2, \lambda) = \phi(x_1) + x_2 - \lambda (p_1 x_1 + p_2 x_2 - M)$.

2. First order conditions:

$$\frac{\partial L}{\partial x_1} = \phi'(x_1) - \lambda p_1 = 0$$
$$\frac{\partial L}{\partial x_2} = 1 - \lambda p_2 = 0$$
$$\frac{\partial L}{\partial \lambda} = p_1 x_1 + p_2 x_2 - M = 0$$

3. Using the second equation, we easily obtain $\lambda^* = 1/p_2$. This is a remarkable result: in general, λ depends from all the variable, and it is not possible to determine its value before one has solved for x_1^* and x_2^*. Remember, λ is the marginal effect of income on the indirect utility v. In this particular case λ does not depend on income. This means that the marginal effect of an increase in income on the v is constant. As we will see better in the following points, with quasi-linear utility function, there are no income effects on good 1. This means that, as income increases, the agent simply purchases more of good 2. Since good 2 is linear, the increment in utility due to higher income is constant and equal to $1/p_2$. [We did not expect a full explanation like this for this point, do not worry]

4. After plugging in for the value of λ, we get

$$\phi'(x_1^*) - p_1/p_2 = 0. \tag{3}$$

   From this implicit equation, it is clear that x_1^* does not depend on M. (If you do not see this immediately, apply the implicit function theorem, compute $\partial x_1^*/\partial M$, and discover that it is zero). Therefore good 1 ia a neutral good. In the case of quasi-linear utility, there is no income effect on the purchase of good 1. Whenever income increases, the agent just purchases more of good 2.

5. We apply the implicit function theorem on (3) and get

$$\partial x_1^*/\partial p_1 = -\frac{-1/p_2}{\phi''(x_1^*)}$$

   which is negative since $\phi'' < 0$ by assumption. Therefore, as the price goes up, the quantity consumer decreases. We should have guessed this alread using the Slutsky equation: $\partial x_1^*/\partial p_1 = \partial h_1^*/\partial p_1 - x_1^* \partial x_1^*/\partial M$. We know that $\partial x_1^*/\partial M = 0$ from point 4, and we know that $\partial h_1^*/\partial p_1 \leq 0$ always holds (compensated demand always has negative derivative with respect to own price).

6. If $\phi(x_1) = x_1^{1/2}$, we get from (3) $1/2\,(x_1^*)^{-1/2} = p_1/p_2$ or $x_1^* = (2p_1/p_2)^{-2} = \frac{1}{4}\left(\frac{p_2}{p_1}\right)^2$. For x_2^* we get $x_2^* = \left(M - p_1 \frac{1}{4}\left(\frac{p_2}{p_1}\right)^2\right)/p_2 = M/p_2 - \frac{1}{4}\frac{p_2}{p_1}$. Notice that for both goods increases in own price are associated with decreases in quantity purchased.

7. In order to guarantee $x_2^* \geq 0$, we require $M \geq \frac{1}{4}\frac{(p_2)^2}{p_1}$.

8. The indifference curves in this example have the special feature that they are parallel, i.e., they are all vertical shifts one from the other. Mathematically, the slope of the indifference curve does not depend on the level of x_2 at which it is evaluated, but only on the level of x_1. (Convince yourself that this property does not hold for Cobb-Douglas). This depends on the fact that the utility function is linear in x_2 (hence the name quasi-linear), so the quantity of x_2 does not affect the MRS.

9. See the figure in the other file. There is no income effect on good 1. As income M increases, the income effect is all on good 2. With quasi-linear preferences, prices determine the quantity of x_1^* consumed, and then the income left over determines the quantity consumed of x_2^*. This is a very peculiar feature that we do not normally find (again, compare with Cobb-Douglas, where increases in income translate into higher consumption of both goods).

Problem 3. Expenditure minimization–Tricky cases. (11 points) Here are two expenditure minimization problems in which you are not supposed to take derivatives. Use your intuition and graphical methods. The solution is similar to the ones that we explored in class for the case of utility maximization.

$$\min_{x_1,x_2} p_1x_1 + p_2x_2$$
$$\text{s.t. } u(x_1,x_2) = \overline{u},$$

1. Assume $u(x_1,x_2) = \min(x_1,x_2)$.

   (a) What is the solution for h_1^* and h_2^*, that is, for the Hicksian compensated demand functions? (do not write the Lagrangean, try graphically). (4 points)

   (b) Show that the Hicksian compensated demand function does not depend on p_1 or p_2, that is, $\partial h_i^*(p,\overline{u})/\partial p_i = \partial h_i^*(p,\overline{u})/\partial p_j = 0$. In this case, the substitution effect of a change in price is null. (3 points)

2. Assume $u(x_1,x_2) = x_1^2 + x_2^2$. What is the solution for h_1^* and h_2^*, that is, for the Hicksian compensated demand functions? (do not write the Lagrangean, try graphically). (4 points)

Solution to Problem 3

1. We wish to solve for x_1^* and x_2^* such that $min\{x_1^*,\, x_2^*\} = \overline{u}$ and $p_1x_1 + p_2x_2$ is as small as possible. Graphing the indifference curve corresponding to utility level $\overline{u}$ we claim that the expenditure minimizing budget occurs when $x_1^* = x_2^* = \overline{u}$. Otherwise, if say $x_1^* > x_2^*$, then we would not be minimizing expenditure since we could decrease x_1^* a small amount so that the utility $min\{x_1^*,\, x_2^*\} = x_2^*$ remains unchanged. Then, while maintaining the same level of utility we have decreased our expenditure. Hence, at an optimum we cannot have $x_1^* > x_2^*$. Similarly, at an optimum we cannot have $x_2^* > x_1^*$. It follows that $x_1^* = x_2^*$. Since $min\{x_1^*,\, x_2^*\} = \overline{u} \Rightarrow x_1^* = x_2^* = \overline{u}$.

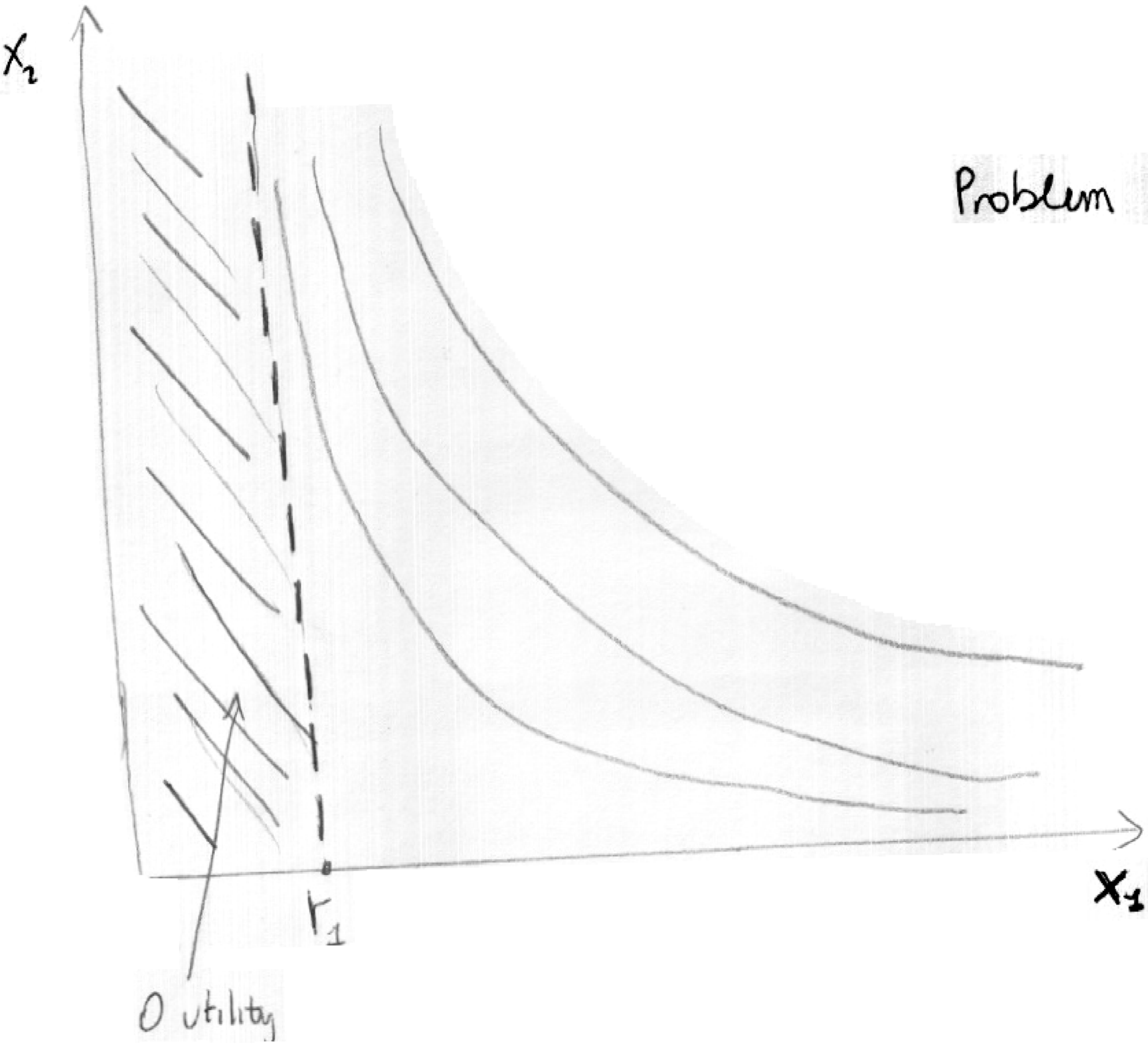
2. In the previous part we just computed that the Hicksian was precisely equal to the utility level $\overline{u}$. In other words, for ***any*** level of prices p_1, p_2 we found that the solution to the expenditure minimization problem:
$$h_1(p_1,p_2,\overline{u}) = \overline{u},\; h_2(p_1,p_2,\overline{u}) = \overline{u}$$
Thus, the Hicksian is independent of prices. It follows that the derivative of h_i with respect to p_i is zero, and hence that the substitution effect for both goods is zero. Of course, we could have anticipated this result since we previously established that these goods were perfect complements. Since consumers with these preferences need to have $x_1 = x_2$, changes in price do not induce them to change their allocation (except for income effects). Nevertheless, it is nice to have a mathematical argument to vindicate our heuristics.

3. Remember that this was a case where the Langrangean provided a constrained minimum, not a maximum. To find the Hicksian in this case, we again need to refer to the graph of the given utility function, for fixed level of utility $\overline{u}$. First note that at the optimum, the bundle x_1^*, x_2^* lies at the corners of the budget set. To prove this claim, we consider three cases:

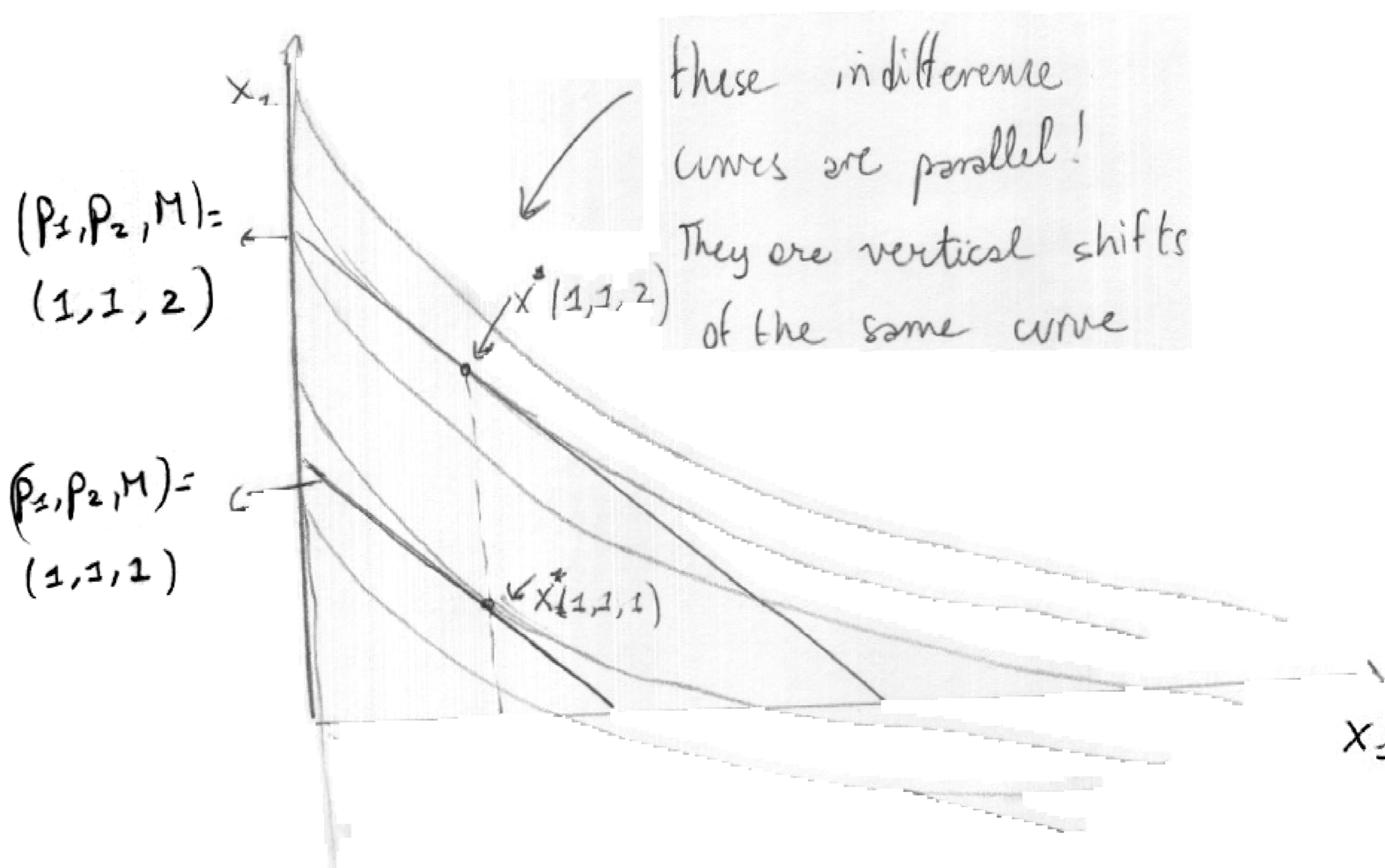
   (1) $p_1 > p_2$
   (2) $p_1 < p_2$
   (3) $p_1 = p_2$

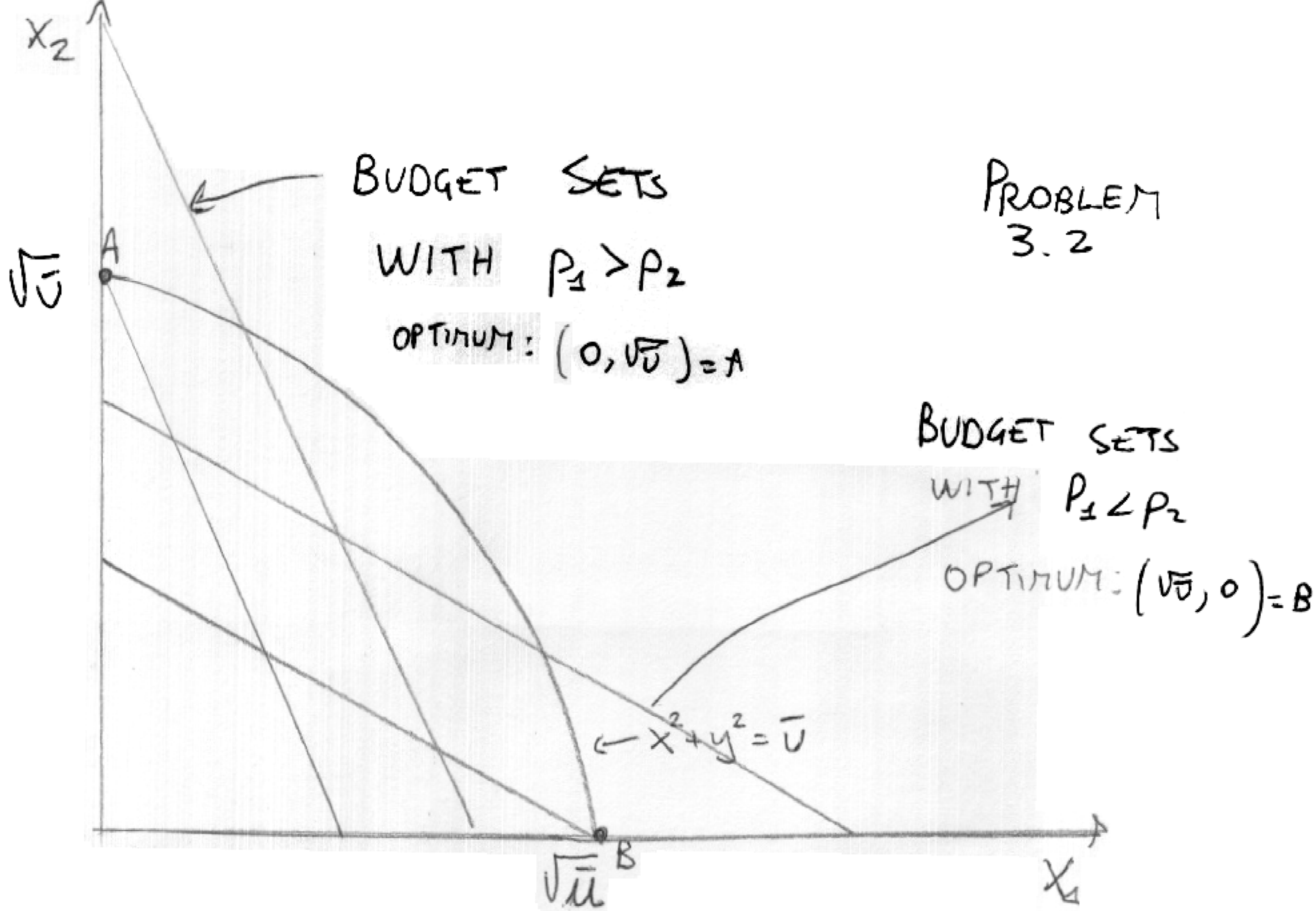
   In case (1), the cheapest budget that allows us to attain utility level $\overline{u}$ intersects the indifference curve on the x_2-axis, at the point $(x_1^* = 0, x_2^* = \sqrt{\overline{u}})$. In case (2), the cheapest budget intersects the indifference curve at the x_1 axis, at the point $(x_1^* = \sqrt{\overline{u}}, x_2^* = 0)$. Finally, in case (3), the cheapest budget touches the indifference curve in two points, the x_1 and x_2 axes, respectively. This yields two bundles that solve the expenditure minimization problem. Bundle one is $(x_1^* = 0, x_2^* = \sqrt{\overline{u}})$ and bundle two is $(x_1^* = \sqrt{\overline{u}}, x_2 = 0)$.

Problem 1.1

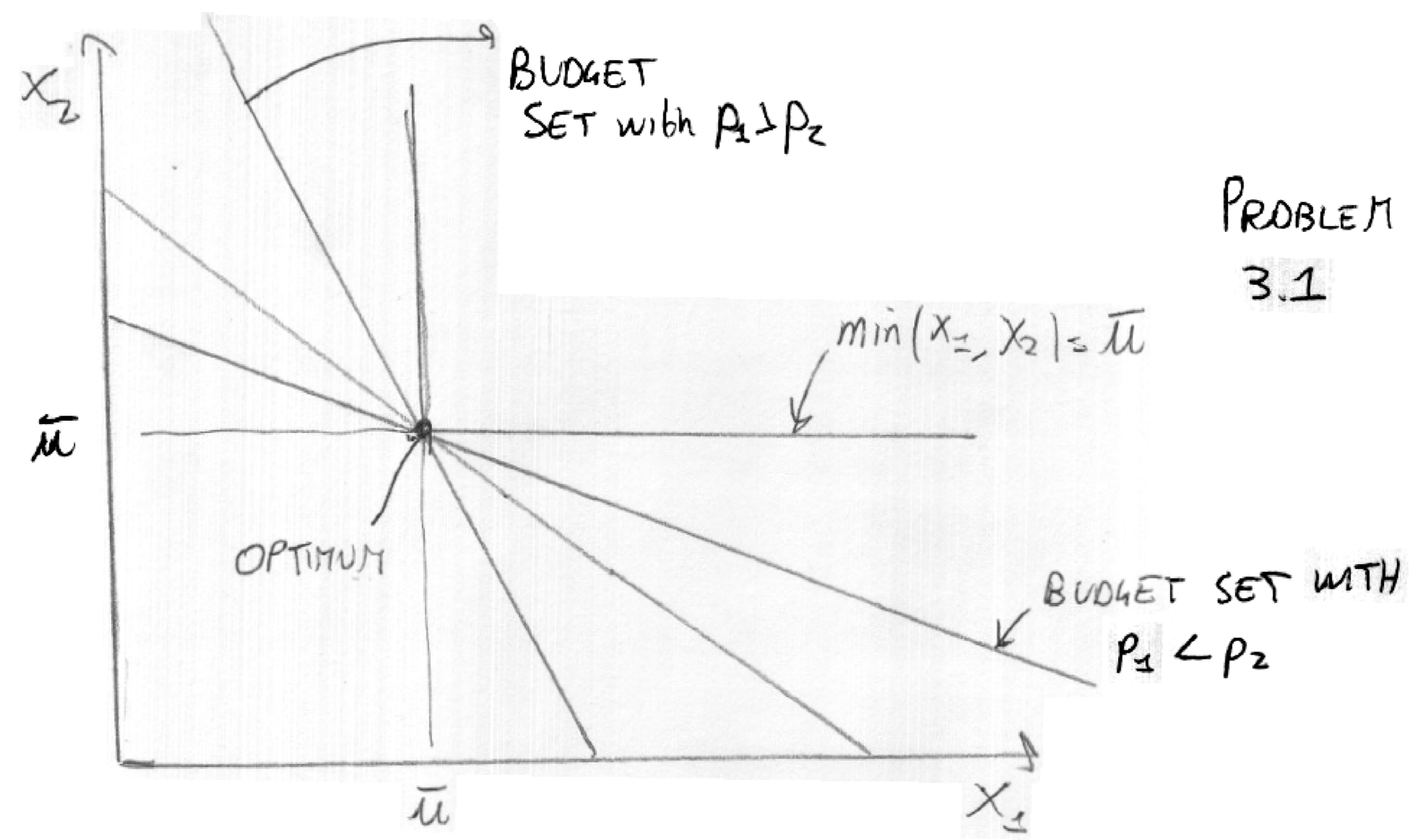


Problem 2.8





X_2
BUDGET SETS
WITH $p_1 > p_2$
OPTIMUM: $(0, \sqrt{\bar{U}}) = A$
PROBLEM 3.2
A
$\sqrt{\bar{U}}$
BUDGET SETS
WITH $p_1 < p_2$
OPTIMUM: $(\sqrt{\bar{U}}, 0) = B$
$x^2 + y^2 = \bar{U}$
B
$\sqrt{\bar{u}}$
X_1



X_2
BUDGET SET with $p_1 > p_2$
PROBLEM 3.1
$\min(X_1, X_2) = \bar{u}$
$\bar{u}$
OPTIMUM
BUDGET SET WITH $p_1 < p_2$
$\bar{u}$
X_1