

Econ 101A – Problem Set 4 Solutions

Due in class on Tu 4 November. No late Problem Sets accepted, sorry!

This Problem set tests the knowledge that you accumulated mainly in lectures 15 to 19. Some of the material will only be covered on Lecture 18, but you should be able to do most of the problem set already [as of Tu 29 October]. The problem set is focused on production. General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

Problem 1. Production: cost minimization and profit maximization (11 points) In class we introduced two different characterizations of firm decisions: cost minimization and profit maximization. In this exercise you are asked to show that the two optimization problems are equivalent ways of solving the same problem. Consider a mango farm in the Philippines. The farm has a production function $y = f(L, K, F)$, where L is labor, K is capital, and F is farm land. The price of labor is w , the price of capital is r , the price of land is s , with all of the prices positive. Assume also $f'_i > 0$ and $f''_{i,i} < 0$, with $i = L, K, F$.

1. Write down the first step of the cost minimization problem and the first order conditions with respect to L , K , and F . (2 points)
2. The first order conditions in step 1, together with the constraint $y = f(L, K, F)$, allow us to solve for $L^*(w, r, s, y)$, $K^*(w, r, s, y)$ and $F^*(w, r, s, y)$. We can therefore write the cost function $c(w, r, s, y) = wL^*(w, r, s, y) + rK^*(w, r, s, y) + sF^*(w, r, s, y)$. In the second step of the cost minimization, the firm maximizes $py - c(w, r, s, y)$. Write down the first order conditions with respect to y . (1 point)
3. Now, instead of going down the cost-minimization path, we go down the one-stage profit-maximization path. Write down the profit-maximization problem and the first order conditions with respect to L , K , and F . (2 points)
4. Your task now is to show that the four first order conditions in the cost minimization (from points 1 and 2) are equivalent to the three first order condition from the profit maximization (from point 3). In other words, profit maximization and cost minimization are two different ways to do the same thing. To do this, you will need to use the envelope theorem to find an alternative expression for $\partial c(w, r, s, y) / \partial y$. [Hint: $c(w, r, s, y)$ is the objective function of the minimization program $\min wL + rK + sF$ s.t. $f(L, K, F) - y = 0$.] (6 points)

Solution to Problem 1.

1. First, the firm minimizes the cost of the inputs:

$$\begin{aligned} \min_{L, K, F} \quad & wL + rK + sF \\ \text{s.t.} \quad & f(L, K, F) \geq y \end{aligned}$$

We can assume that the budget constraint is satisfied with equality given the assumption $f'_i > 0$ for all i . The Lagrangean is

$$\mathcal{L}(L, K, F, \lambda) = wL + rK + sF - \lambda[f(L, K, F) - y]$$

and the three first order conditions are

$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda f'_L(L^*, K^*, F^*) = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda f'_K(L^*, K^*, F^*) = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial F} = s - \lambda f'_F(L^*, K^*, F^*) = 0 \tag{3}$$

2. Given the cost function $c(w, r, s, y) = wL^*(w, r, s, y) + rK^*(w, r, s, y) + sF^*(w, r, s, y)$, we can write the second maximization problem as

$$\max_y py - c(w, r, s, y).$$

The first order condition is

$$p - c'_y(w, r, s, y^*) = 0. \quad (4)$$

3. The profit maximization problem is

$$\max_{L, K, F} pf(L, K, F) - wL - rK - sF$$

with first order conditions

$$pf'_L(L^*, K^*, F^*) - w = 0 \quad (5)$$

$$pf'_K(L^*, K^*, F^*) - r = 0 \quad (6)$$

$$pf'_F(L^*, K^*, F^*) - s = 0 \quad (7)$$

4. Equation (4) in the cost minimization states that the price p equals the marginal cost $c'_y(w, r, s, y^*)$. But we can rewrite the marginal cost. Consider the cost-minimization problem, of which the cost function $c(w, r, s, y^*)$ is the value function. By the envelope theorem, we know that

$$\begin{aligned} \partial c(w, r, s, y^*) / \partial y &= \partial \mathcal{L}(L, K, F, \lambda) / \partial y = \\ &= \partial [wL + rK + sF - \lambda [f(L, K, F) - y]] / \partial y = \\ &= \lambda. \end{aligned}$$

The marginal cost equals the multiplier λ ! A related result is that in a utility maximization problem the multiplier λ represents the marginal utility of money. Equation (4) then implies $p = \lambda$. If we substitute λ for p in equations (1) to (3), we obtain equations (5) to (7). Therefore, the first-order conditions for the two-stage cost minimization process coincide with the first order conditions for the one-stage profit maximization. We are done!

Problem 2. Production: perfect competition (42 points) In this exercise, we characterize the short-run solution to the Philippino farmers problem for the case of perfect competition. We assume that the firm has the production function $y = AL^\alpha K^\beta F^\gamma$. In the short-run, however, the quantity of land farmed is fixed to \bar{F} , so there effectively are only two factors of production with respect to which the firm maximizes.

1. Write down the cost minimization problem with respect to L and K and the first order conditions with respect to L and K (2 points)
2. Solve for $L^*(w, r, s, y, \bar{F})$ and $K^*(w, r, s, y, \bar{F})$. (3 points)
3. How does L^* vary as w increases? Compute $\partial L^*/\partial w$ using the solution in point 2. Does it make sense? How about if there is technological progress and A increases? What happens to L^* ? (3 points)
4. Write down the cost function $c(w, r, s, y, \bar{F})$ and derive the expressions for the marginal cost $c'_y(w, r, s, y, \bar{F})$ and the average cost $c(w, r, s, y, \bar{F})/y$. (4 points)
5. Assume $\alpha + \beta < 1$ and plot the marginal cost $c'_y(w, r, s, y, \bar{F})$ and the average cost $c(w, r, s, y, \bar{F})/y$ as a function of y . [I am interested in the shape of the curves, not in exact drawings]. Draw the supply curve for the firm, that is, $y = y^*(w, r, s, \bar{F})$. Is the supply curve (weakly) increasing in p ? (5 points)
6. Assume $\alpha + \beta = 1$ and plot the marginal cost $c'_y(w, r, s, y, \bar{F})$ and the average cost $c(w, r, s, y, \bar{F})/y$ as a function of y . How does the supply curve for the firm look like? Write it down analytically. Why does the firm never want to produce a positive and finite quantity of output despite constant returns to scale in labor and capital? (5 points)
7. Assume $\alpha + \beta > 1$ and plot the marginal cost $c'_y(w, r, s, y, \bar{F})$ and the average cost $c(w, r, s, y, \bar{F})/y$ as a function of y . How does the supply curve for the firm look like? (3 points)
8. We now consider market supply and demand for the only well-behaved case, that is, for $\alpha + \beta < 1$. First, let's look at aggregation of the supply side. There are J producers with production function as above. Each one of them, therefore, has a supply function as at point 5. Assuming that we are on the increasing portion of the supply curve, we can write the equation for the supply function as

$$\frac{(y^j)^{\frac{1-(\alpha+\beta)}{\alpha+\beta}}}{\alpha+\beta} D = p \quad (8)$$

where

$$D = \left(\frac{1}{A(\bar{F})^\gamma} \right)^{\frac{1}{\alpha+\beta}} \left[w \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} + r \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}} \right].$$

We can invert expression (8) to get

$$y^j = \left[\frac{(\alpha + \beta)p}{D} \right]^{\frac{\alpha+\beta}{1-(\alpha+\beta)}}$$

This is the firm supply function provided that the price p is high enough (we are not checking this, enough algebra!). Now, your turn. Aggregate over the J firms to generate the industry supply function Y^S . (2 points)

9. Now, the consumer side. Assume that consumers have Cobb-Douglas preferences for mangoes (good y with price p) and all the other goods (good m with price 1). Their utility function is

$$u(y, m) = y^\alpha m^{1-\alpha}.$$

There are J consumers, each with income M_j , $j = 1, \dots, J$. From the many exercises we did on Cobb-Douglas utility it follows that in equilibrium

$$y_j^* = \alpha \frac{M_j}{p}.$$

This is the individual demand function for good y . Compute now the aggregate demand function Y^D for the whole economy. Notice that in this special case aggregate demand does not depend on the distribution of income. (3 points)

10. Finally, the market equilibrium! Equate the demand Y^D and supply Y^s to solve for the equilibrium price p_M . What is p_M ? Plug back the value of p_M into your expression for Y^D to get the market level of production Y (4 points). You should get the following solutions

$$p_M = \left[\alpha \frac{\sum_{j=1}^J M_j}{J} \right]^{1-(\alpha+\beta)} \frac{(\alpha + \beta)^{-(\alpha+\beta)}}{A(\bar{F})^\gamma} \left[w \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} + r \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}} \right]^{\alpha+\beta}$$

and

$$Y = \left[\alpha \sum_{j=1}^J M_j \right]^{\alpha+\beta} J^{1-(\alpha+\beta)} A(\bar{F})^\gamma (\alpha + \beta)^{\alpha+\beta} \left[w \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} + r \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}} \right]^{-(\alpha+\beta)}$$

11. Determine now the sign of the following changes on p_M and Y . Provide intuition. Also, make sure that the results square with your graphical intuition about moving supply and demand function. (8 points)
- an increase in total income $\sum_{j=1}^J M_j$;
 - an increase in productivity A .

Solution to Problem 2.

1. The cost minimization problem is

$$\begin{aligned} \min_{L, K} wL + rK + s\bar{F} \\ \text{s.t. } AL^\alpha K^\beta (\bar{F})^\gamma \geq y \end{aligned}$$

where we can assume that the constraint is satisfied with equality since $\partial [AL^\alpha K^\beta (\bar{F})^\gamma] / \partial L > 0$ and $\partial [AL^\alpha K^\beta (\bar{F})^\gamma] / \partial K > 0$. The Lagrangean is

$$\mathcal{L}(L, K, F, \lambda) = wL + rK + s\bar{F} - \lambda [AL^\alpha K^\beta (\bar{F})^\gamma - y]$$

and the first order conditions with respect to the inputs L and K are (remember that land is fixed)

$$\begin{aligned} w - \lambda \alpha AL^{\alpha-1} K^\beta (\bar{F})^\gamma &= 0 \\ r - \lambda \beta AL^\alpha K^{\beta-1} (\bar{F})^\gamma &= 0 \end{aligned}$$

2. We can easily transform the first order conditions in the usual way to get

$$\frac{\alpha K^*}{\beta L^*} = \frac{w}{r}$$

or

$$K^* = \frac{\beta w}{\alpha r} L^*. \tag{9}$$

We can substitute this expression for K^* into the constraint to get

$$\begin{aligned} y &= A(L^*)^\alpha \left(\frac{\beta w}{\alpha r} L^* \right)^\beta (\bar{F})^\gamma \text{ or} \\ L^* &= \left(\frac{y}{A(\bar{F})^\gamma} \right)^{\frac{1}{\alpha+\beta}} \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} \end{aligned}$$

Using expression (9), one similarly finds

$$\begin{aligned} K^* &= \frac{\beta w}{\alpha r} \left(\frac{y}{A(\bar{F})^\gamma} \right)^{\frac{1}{\alpha+\beta}} \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} = \\ &= \left(\frac{y}{A(\bar{F})^\gamma} \right)^{\frac{1}{\alpha+\beta}} \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}} \end{aligned}$$

3. It is straightforward to compute the comparative statics:

$$\frac{\partial L^*}{\partial w} = \left(\frac{y}{A(\bar{F})^\gamma} \right)^{\frac{1}{\alpha+\beta}} \left(-\frac{\beta}{\alpha+\beta} \right) \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}-1} \frac{\beta}{\alpha r} < 0.$$

Unsurprisingly, as the wage increases, the demand for labor L^* decreases as the firm substitutes away from labor into capital. Similarly,

$$\frac{\partial L^*}{\partial A} = \left(\frac{1}{\alpha+\beta} \right) \left(\frac{y}{A(\bar{F})^\gamma} \right)^{\frac{1}{\alpha+\beta}-1} - \frac{y}{A^2(\bar{F})^\gamma} \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} < 0.$$

As the firm becomes more productive, the firm needs fewer inputs to produce the same quantity of outputs.

4. The cost function equals

$$\begin{aligned} c(w, r, s, y) &= wL^*(w, r, s, y) + rK^*(w, r, s, y) + s\bar{F} = \\ &= \left(\frac{y}{A(\bar{F})^\gamma} \right)^{\frac{1}{\alpha+\beta}} \left[w \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} + r \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}} \right] + s\bar{F} \end{aligned}$$

and the marginal cost equals

$$c'_y(w, r, s, y) = \frac{y^{\frac{1-(\alpha+\beta)}{\alpha+\beta}}}{\alpha+\beta} \left(\frac{1}{A(\bar{F})^\gamma} \right)^{\frac{1}{\alpha+\beta}} \left[w \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} + r \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}} \right]$$

and the average cost equals

$$c(w, r, s, y)/y = y^{\frac{1-(\alpha+\beta)}{\alpha+\beta}} \left(\frac{1}{A(\bar{F})^\gamma} \right)^{\frac{1}{\alpha+\beta}} \left[w \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} + r \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}} \right] + s \frac{\bar{F}}{y}$$

5. For $\alpha + \beta < 1$ the marginal cost is increasing in y , and the average cost is first decreasing and then increasing. The supply function is the part of the marginal cost function above the average cost function, and is otherwise zero. See picture.
6. For $\alpha + \beta = 1$ the marginal cost is constant, while the average cost is always decreasing and higher than the marginal cost. The supply function is degenerate: it is zero for prices below \bar{p} , with

$$\bar{p} = \left(\frac{1}{A(\bar{F})^\gamma} \right)^{\frac{1}{\alpha+\beta}} \left[w \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} + r \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}} \right]$$

and ∞ for $p > \bar{p}$. See picture. This happens because in the short-run the choice of land is fixed, and therefore suboptimal. This generates a fixed cost $s\bar{F}$ that the firm has to bear, and that induces the firm not to want to produce for $p \leq \bar{p}$.

7. For $\alpha + \beta > 1$ the marginal cost is decreasing in y , and the average cost is decreasing as well and always lies above the marginal cost. The supply function is equal to ∞ for any price $p > 0$. See picture. Given the increasing returns to scale, the firm wants to produce more and more to achieve a more efficient scale. Definetely, this is not compatible with perfect competition! (under perfect competition each firm ought to produce a small quantity).

8. The market supply function is simply the sum of the J supply functions of the firms:

$$Y^S = \sum_{j=1}^J \left[\frac{(\alpha + \beta)p}{D} \right]^{\frac{\alpha + \beta}{1 - (\alpha + \beta)}} = J \left[\frac{(\alpha + \beta)p}{D} \right]^{\frac{\alpha + \beta}{1 - (\alpha + \beta)}}$$

9. The market demand function Y^D is just obtained by summing over the individual demand functions

$$Y^D = \sum_{j=1}^J y_j^* = \alpha \frac{\sum_{j=1}^J M_j}{p}. \quad (10)$$

This aggregate demand function does not depend on the distribution of income. This depends on the fact that with Cobb-Douglas preferences each agent spends a share α of income on mangoes.

10. In order to find the market-clearing price, we equate demand Y^D and supply Y^S to obtain

$$Y^S = J \left[\frac{(\alpha + \beta)p_M}{D} \right]^{\frac{\alpha + \beta}{1 - (\alpha + \beta)}} = \alpha \frac{\sum_{j=1}^J M_j}{p_M} = Y^D$$

or

$$p_M^{\frac{1}{1 - (\alpha + \beta)}} = \alpha \frac{\sum_{j=1}^J M_j}{J} \left[\frac{(\alpha + \beta)}{D} \right]^{-\frac{\alpha + \beta}{1 - (\alpha + \beta)}}$$

or

$$\begin{aligned} p_M &= \left[\alpha \frac{\sum_{j=1}^J M_j}{J} \right]^{1 - (\alpha + \beta)} \left[\frac{(\alpha + \beta)}{D} \right]^{-(\alpha + \beta)} = \\ &= \left[\alpha \frac{\sum_{j=1}^J M_j}{J} \right]^{1 - (\alpha + \beta)} \frac{(\alpha + \beta)^{-(\alpha + \beta)}}{A(\bar{F})^\gamma} \left[w \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha + \beta}} + r \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha + \beta}} \right]^{\alpha + \beta} \end{aligned}$$

and, using (10),

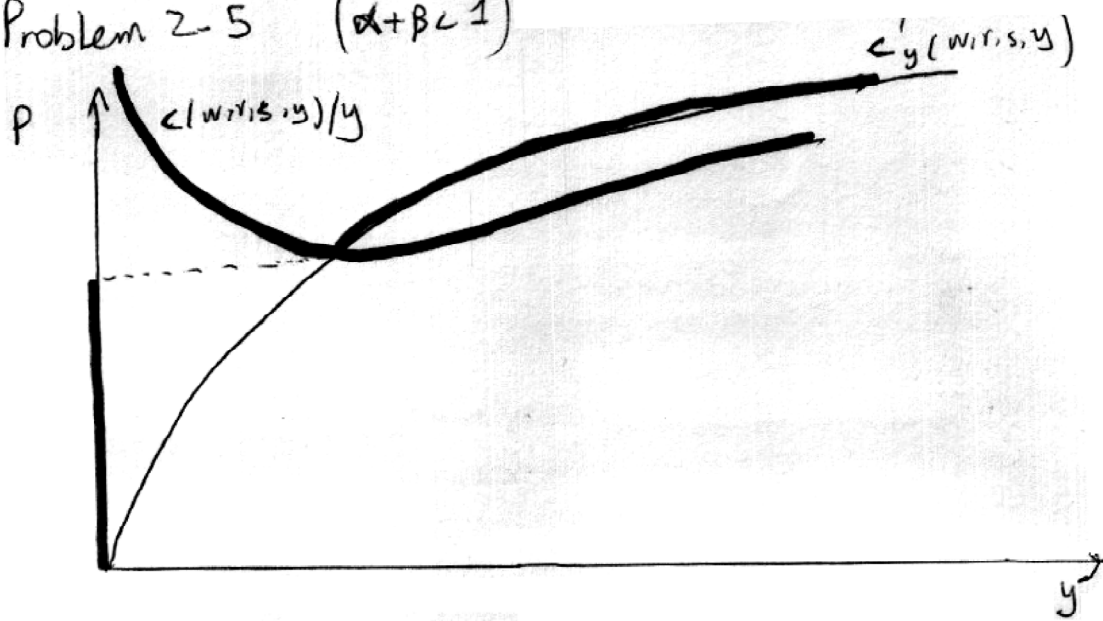
$$\begin{aligned} Y &= Y^S = Y^D = \alpha \sum_{j=1}^J M_j \left[\alpha \frac{\sum_{j=1}^J M_j}{J} \right]^{-(1 - (\alpha + \beta))} \left[w \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha + \beta}} + r \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha + \beta}} \right]^{-(\alpha + \beta)} = \\ &= \left[\alpha \sum_{j=1}^J M_j \right]^{\alpha + \beta} J^{1 - (\alpha + \beta)} A(\bar{F})^\gamma (\alpha + \beta)^{\alpha + \beta} \left[w \left(\frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha + \beta}} + r \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha + \beta}} \right]^{-(\alpha + \beta)} \end{aligned}$$


11. Here are the comparative statics:

- (a) An increase in total income $\sum_{j=1}^J M_j$ increases both the equilibrium price and the equilibrium quantity produced. Higher income corresponds to a positive demand shock. This increases demand and, since the supply curve is upward sloping, increases price as well.
- (b) An increase in productivity A decreases the equilibrium price and increases the equilibrium quantity. Higher productivity induces a positive supply shock. This increases demand and, since the demand curve is downward sloping, decreases price as well. Think at the effect of increased productivity of computers: quality of computers goes up and price goes down.

PROBLEM SET 4 - DRAWINGS

Problem 2-5 ($\alpha + \beta < 1$)



 = SUPPLY FUNCTION

Problem 2-6 ($\alpha + \beta = 1$)

