

## Econ 101A – Problem Set 5 Solutions

Due in class on Tu 26 November. No late Problem Sets accepted, sorry!

This Problem set tests the knowledge that you accumulated mainly in lectures 20 to 24. Some of the material will only be covered on Lecture 24, but you should be able to do most of the problem set already [as of Tu 17 November]. The problem set is focused on monopoly, oligopoly, and static game theory. General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

**Problem 1. Monopoly, Oligopoly, and Perfect Competition** (46 points) In this problem you are asked to compare the outcomes of monopoly, oligopoly, and perfect competition in one market. We are going to assume very simple functional forms in order to simplify the algebra. We assume that the firm has a very simple cost function:  $c(y) = cy$ , with  $c > 0$ . The marginal cost of production therefore is constant. As for the market demand, we assume that it takes the simple linear form  $p(Y) = a - bY$ , with  $a > c > 0$  and  $b > 0$ , where  $Y$  is the total production on the industry.

1. Consider first the case of **perfect competition**. Derive the marginal and average cost curves. How does the supply curve look like for each firm? What about in the industry? (aggregate the individual supply curve over  $J$  firms). (4 points)
2. Equate supply and demand to obtain the industry-level production under perfect competition  $Y_{PC}^*$ , as well as the price level under perfect competition  $p_{PC}^*$ . (3 points)
3. How do perfect competition price and quantity vary as the cost of production  $c$  increases? How do they vary if there is a positive demand shock ( $a$  increases)? (3 points)
4. We consider now the **monopoly** case. Write down the profit maximization problem and the first order conditions with respect to  $y$ . [In the case of monopoly,  $y = Y$ ] (2 points)
5. Solve for  $y_M^*$  and  $p_M^*$ . How does  $p_M^*$  vary as  $a$  increases? Why is this comparative statics different from the one under perfect competition? (2 points)
6. Compare the total output and prices of perfect competition and monopoly. Compute the monopoly profits and compare them to the profits under perfect competition. (3 points)
7. Consider now the case of **duopoly**, that is, an oligopoly with two firms,  $i = 1, 2$ . Write down the profit maximization problem of firm  $i$  as a function of the quantity produced by firm  $-i$ ,  $y_{-i}$ . (3 points)
8. We are now looking for Nash equilibria (in pure strategies) in the quantity produced  $y_1, y_2$ . Each firm  $i$ , holding fixed the quantity produce by the other firm at  $y_{-i}^*$ , maximizes profits with respect to  $y_i$ . Write down the first order conditions for firms 1 and 2. (3 points)
9. In a Nash equilibrium each firm must choose the optimal quantity produced given the production choice of the other firm. Combine the two first order conditions in point 5 to obtain the Nash equilibrium quantity produced by firm 1,  $y_1^*$ , and by firm 2,  $y_2^*$ . Derive also the industry production  $Y_D^* = y_1^* + y_2^*$ , the price  $p_D^*$ , the profit level of each firm  $\pi_{i,D}^*$ , and the aggregate profit level  $\Pi_D^* = \pi_{1,D}^* + \pi_{2,D}^*$  (6 points)
10. Compare  $Y_D^*$ ,  $\Pi_D^*$ , and  $p_D^*$  with  $Y_{PC}^*$ ,  $\Pi_{PC}^*$ , and  $p_{PC}^*$  and  $Y_M^*$ ,  $\Pi_M^*$ , and  $p_M^*$ . (3 points)
11. Finally, the general case of **oligopoly**. Assume that there are  $I$  firms, all identical, with production costs as above. Write down the profit maximization problem and the first order condition of a firm. (2 points)

12. We now solve for the oligopoly production using a trick. We look for a symmetric solution, that is, a solution where each firm produces the same. In particular, impose the condition  $y_{-i}^* = \sum_{j \neq i} y_j^* = (I - 1)y_O^*$ . Find the solution for the Nash equilibrium quantity  $y_O^*$ . Derive also the industry production  $Y_O^* = Iy_O^*$ , the price  $p_O^*$ , the profit level of each firm  $\pi_O^*$ , and the aggregate profit level  $\Pi_O^* = I\pi_O^*$ . (4 points)
13. What is nice about this general oligopoly solution is that embeds the previous cases: for  $I = 1$  we go back to monopoly, for  $I = 2$  we get the duopoly solution. Most interestingly, show that for  $I \rightarrow \infty$ , the prices, the total quantity produced, and the total industry profits converge to the perfect competition ones. Compare this to Bertrand competition. How many companies did we need there to get the same outcomes as perfect competition? (8 points)

### Solution to Problem 1.

1. The marginal cost and average cost both equal  $c$ . It follows that the supply curve is as follows:

$$y_i^*(p) = \begin{cases} \infty & \text{if } p > c \\ \text{any } y \in [0, \infty] & \text{if } p = c \\ 0 & \text{if } p < c \end{cases}$$

The aggregate supply curve coincides with the individual supply curve (this is the case since supply is horizontal, perfectly elastic).

2. From point 1, it is clear that in perfect competition the price  $p_{PC}^*$  must equal  $c$  (remember, price must equal marginal cost). Given this, it is easy to find  $y_{PC}^*$ :

$$p_{PC}^* = a - by_{PC}^*$$

or

$$y_{PC}^* = \frac{a - c}{b}$$

3. An increase in the cost of production  $c$  induces an increase in the perfect competition price and a decrease in the quantity produced. An exogenous increase in demand (increase in  $a$ ) increases the quantity produced, but it does not vary the price.
4. A monopolistic firm maximizes profits:

$$\max_y (a - by)y - cy.$$

and the first order condition is

$$a - 2by_M^* - c = 0.$$

5. It follows that

$$y_M^* = \frac{a - c}{2b}$$

and

$$p_M^* = a - by_M^* = a - b \frac{a - c}{2b} = \frac{a + c}{2}.$$

In this case, as demand increases, ( $a$  increases), the monopoly price goes up. While under perfect competition an increase in demand does not increase prices, it does so under monopoly. The monopolistic firm takes advantage of the increased demand in a way that a perfectly competitive firm cannot do.

6. The total output under monopoly  $y_M^*$  equals  $y_{PC}^*/2$  and the price  $p_M^*$  is higher than  $p_{PC}^* = c$ . Under monopoly the firm produces less output than under perfect competition in order to raise prices. The profits equal

$$(p_M^* - c)y_M^* = \left(\frac{a + c}{2} - c\right) \frac{a - c}{2b} = \frac{(a - c)^2}{4b} > 0$$

These profits are large than the profits under perfect competition, which are zero.

7. A duopolistic firm maximizes profits:

$$\max_{y_i} (a - b(y_i + y_{-i})) y_i - cy_i. \quad (1)$$

8. The first order condition of problem (1) is

$$a - 2by_i^* - by_{-i}^* - c = 0.$$

It follows that the first order conditions for the two firms are

$$a - 2by_1^* - by_2^* - c = 0. \quad (2)$$

and

$$a - 2by_2^* - by_1^* - c = 0. \quad (3)$$

9. From the first order condition (2), we get  $y_2^* = (a - c) / b - 2y_1^*$ . We substitute this expression into (3) to get  $a - 2(a - c) + 4by_1^* - by_1^* - c = 0$  or  $3by_1^* = a - c$ , so

$$y_1^* = \frac{a - c}{3b}$$

and

$$y_2^* = \frac{a - c}{b} - 2y_1^* = \frac{a - c}{b} - 2\frac{a - c}{3b} = \frac{a - c}{3b}.$$

Not surprisingly, the quantities produced by firms 1 and 2 are equal. The total quantity produced is

$$Y_D^* = y_1^* + y_2^* = 2\frac{a - c}{3b}$$

The duopoly price is

$$p_D^* = a - bY_D^* = a - b\left(2\frac{a - c}{3b}\right) = \frac{1}{3}a + \frac{2}{3}c.$$

The firm profits equal

$$\pi_D^* = (p_D^* - c)y_D^* = \left(\frac{1}{3}a + \frac{2}{3}c - c\right) \frac{a - c}{3b} = \frac{(a - c)^2}{9b} > 0$$

and the aggregate profits equal

$$\Pi_D^* = 2\pi_D^* = 2\frac{(a - c)^2}{9b}$$

10. When we compare prices across monopoly, duopoly and perfect competition, we find

$$c = p_{PC}^* < p_D^* < p_M^*.$$

As for the total quantity produced, we obtain

$$Y_{PC}^* > Y_D^* > Y_M^*.$$

Finally, for profits, we have

$$0 = \Pi_{PC}^* < \Pi_D^* < \Pi_M^*.$$

A monopolistic firm extracts most profits by charging a higher price, and therefore selling less. Firms in a duopoly would like to collude and do the same, but collusion is not sustainable. They end up producing too much (relative to the profit maximizing level) and as a consequence the price and the industry profits decline. Still, those profits are higher than under perfect competition.

11. In the case of oligopoly, the maximization problem of firm  $i$  is

$$\max_{y_i} (a - b(y_i + y_{-i})) y_i - cy_i$$

with  $y_{-i} = \sum_{j \neq i} y_j$ . The first order condition of problem (1) is

$$a - 2by_i^* - by_{-i}^* - c = 0. \quad (4)$$

12. Impose symmetry, we can write (4) as

$$a - 2by_i^* - b(I-1)y_O^* - c = 0.$$

We can then sum over the  $I$  such conditions to get

$$a - 2b \sum_{i=1}^I y_i^* - b(I-1)y_O^* - c = 0$$

or

$$Ia - 2bIy_O^* - bI(I-1)y_O^* - Ic = 0.$$

This implies

$$y_O^* = \frac{a-c}{(I+1)b}$$

The total quantity produced is

$$Y_O^* = Iy_O^* = I \frac{a-c}{(I+1)b}$$

The oligopoly price is

$$p_O^* = a - bY_O^* = a - b \left( I \frac{a-c}{(I+1)b} \right) = \frac{1}{I+1}a + \frac{I}{I+1}c.$$

The firm profits equal

$$\pi_O^* = (p_O^* - c)y_O^* = \left( \frac{1}{I+1}a + \frac{I}{I+1}c - c \right) \frac{a-c}{(I+1)b} = \frac{(a-c)^2}{(I+1)^2 b} > 0$$

and the aggregate profits equal

$$\Pi_O^* = I\pi_O^* = I \frac{(a-c)^2}{(I+1)^2 b}$$

13. For  $I \rightarrow \infty$ , the total quantity produced  $Y_O^*$  converges to  $(a-c)/b$ , which is the production under perfect competition. Price  $p_O^*$  converges to  $c$ , that is, to marginal cost pricing. Finally, industry-level profits  $\Pi_O^*$  converge to zero, as in perfect competition. Therefore, when there are many firms, we converge to perfect competition. But it takes an awful lot of firms, not just 2 as in the Bertrand case!

**Problem 2. Nash Equilibria in a simple game** (19 points) Two firms are deciding simultaneously whether to enter a market. If neither enters, they make zero profits. If both enters, they make profits -1, since the market is too small for two firms. If only one enters, that firm makes high profits. This game is summarized in the following matrix:

1 \ 2	Enter	Do not Enter
Enter	-1, -1	10, 0
Do not Enter	0, 5	0, 0

1. What are the pure-strategy Nash Equilibria of this game? (3 points)
2. Now assume that firm 1 can enter the market with probability  $p_1$  and firm 2 can enter the market with probability  $p_2$ . Write down the expected utility of each firm as a function of the strategy of the other player, and find the best response correspondence for firms 1 and 2. (8 points)
3. Graph these best response correspondences and find the Nash equilibria in mixed strategies. (5 points)
4. Is there one equilibrium out of these that seems more plausible to you? (3 points)

**Solution to Problem 2.**

1. The pure-strategy Nash Equilibria in this game are (Enter, Do Not Enter) and (Do Not Enter, Enter). That is, in the two equilibria only one firm will enter the market, but we cannot predict which one.
2. The expected utility for firm 1 is as follows:

$$u_1(\text{Enter}, \sigma_2) = p_2 u_1(\text{Enter}, \text{Enter}) + (1 - p_2) u_1(\text{Enter}, \text{Do Not Enter}) = -p_2 + 10(1 - p_2) = 10 - 11p_2$$

and

$$u_1(\text{Do Not Enter}, \sigma_2) = p_2 u_1(\text{Do Not Enter}, \text{Enter}) + (1 - p_2) u_1(\text{Do Not Enter}, \text{Do Not Enter}) = 0$$

Similarly, the expected utility for firm 2 is as follows:

$$u_2(\text{Enter}, \sigma_1) = p_1 u_2(\text{Enter}, \text{Enter}) + (1 - p_1) u_2(\text{Enter}, \text{Do Not Enter}) = -p_1 + 5(1 - p_1) = 5 - 6p_1$$

and

$$u_2(\text{Do Not Enter}, \sigma_1) = p_1 u_2(\text{Do Not Enter}, \text{Enter}) + (1 - p_1) u_2(\text{Do Not Enter}, \text{Do Not Enter}) = 0.$$

It follows that the best response correspondence of firm 1 is

$$BR_1(\sigma_2) = \begin{cases} p_1 = 1 & \text{if } p_2 < 10/11 \\ p_1 \in [0, 1] & \text{if } p_2 = 10/11 \\ p_1 = 0 & \text{if } p_2 > 10/11 \end{cases}$$

and the best response correspondence of firm 2 is

$$BR_2(\sigma_1) = \begin{cases} p_2 = 1 & \text{if } p_1 < 5/6 \\ p_2 \in [0, 1] & \text{if } p_1 = 5/6 \\ p_2 = 0 & \text{if } p_1 > 5/6. \end{cases}$$

3. See figure. The mixed-strategy equilibrium is  $(p_1^*, 1-p_1^*); (p_2^*, 1-p_2^*) = (5/6, 1/6); (10/11, 1/11)$ . Notice that we also find the two equilibria in pure strategies.
4. It is not obvious which one of the three equilibria is the best predictor of behavior. Between the pure strategy equilibria, the one where only firm 1 enters seems more likely since the payoff from entering is higher for firm 1 (10) than for firm 2 (5). This is what Thomas Schelling has called a *focal point*. On the other hand, this is a game in which the potential gains from entering are very large, while the potential losses are not large. It is hard to imagine that firm 2 will accept to stay out. If I was to predict the outcome, I would probably predict that both firms enter, although this is not a Nash Equilibrium! (you do not have to agree with me:)

# PROBLEM 2.3 - P. Set 5

