Firm Dynamics and Finance: Distinguishing Information Regimes[*]

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Abstract

We formulate and numerically simulate various dynamic models of endogenously incomplete credit markets that allow for moral hazard and unobservable capital/investment. We compare them to exogenously incomplete autarky, saving only, and borrowing and lending environments. We characterize the optimal allocations implied by the regimes from both cross-sectional and dynamic perspective. The paper develops computational methods based on mechanism design theory and linear programming methods that are used to structurally estimate, compare and distinguish between the structural models. Our results match several stylized facts from the empirical firm dynamic literature as listed by Cooley and Quadrini, 1999. The compared financing regimes are demonstrated to differ significantly in qualitative and/or quantitative sense with respect to their implications for investment, consumption, financial flows, and insurance in cross-section, transitions, and long-run outcomes.

Keywords: financial constraints, dynamic mechanism design, structural estimation and testing

JEL Classifications: C61, D82

1   Introduction

Small and medium enterprise in both emerging markets and developed economies are typically single proprietorships run by households as family businesses. Yet, with some important exceptions, the literature maintains a dichotomy embedded in the conceptualization of the national income accounts: households are consumers and suppliers of market inputs, whereas firms do the production and hire labor and other factors[1]. This gives rise, on the one hand, to a development literature which studies household consumption smoothing, recognizing a wide range of possible financial regimes (full risk sharing, limited commitment, moral hazard, permanent income, buffer stock[2]. On the other hand, there is a literature in finance which studies investment of firms and sensitivity to cash flow; risk neutral firms maximize the discounted expected present value of profits and access credit according a pecking order or hierarchy of funds hypothesis (first internal funds, then credit, then equity (Myers and Majuf, 1984). In Bond and Meghir (1994), firms follow Euler equations derived from a cost-of-adjustment model, and this fits the data well only if the firm is essentially not constrained. Many empirical studies find rejections for small firms (Fazzari, Hubbard and Petersen, 1988) though this field is not without controversy (Kaplan and Zingales, 2000).

The purpose of this paper is contribute to a literature which is attempting to bring these two consumption and investment strands of empirical research together[3]. We contrast the consumption and investment behavior of risk averse households running business under various possible financial regimes, both exogenously incomplete (autarky, savings only, borrowing and lending) and endogenously constrained by information considerations (moral hazard and observed investment, moral hazard and unobserved capital, both relative to full insurance). We compare the predictions of each of the financial regimes to the stylized facts reported in the literature (e.g., Cooley and Quadrini, 1999) and discuss in what circumstances they might be distinguished in data. Indeed, we develop methods for empirical implementation of mechanism design models and test the various models against each other, naturally using data generated from the models themselves. This is an important step toward implementation in actual data.

A brief listing of the kind of data we have in mind may help to clarify both what we are trying to accomplish in this paper and the longer run research agenda. A Townsend survey of households in villages in Thailand in 1997 turns up 22% running small businesses (see Paulson and Townsend, 2004). Some of these have incomes below the poverty line, while others are large enough to show up in a comprehensive Industrial Census (Townsend, 2007). Eighty percent use family labor, only. In a corresponding monthly panel, consumption among households is quite smooth (Chiappori, Shulhulfer-Wohl, Samphantharak, and Townsend, 2006) as if in a risk-sharing network, though business and agricultural investment seems sensitive to cash flow (Samphantharak and Townsend, 2007). Credit is available in principle from friends and family, money lenders, traders, store owners, village-level savings and loan funds, a government agricultural development bank (BAAC), and commercial banks. Quite a few households use multiple sources, but a few do not borrow at all. On the asset side, savings is in cash, rice storage, and formal financial accounts (Seiler, 1998; Kaboski and Townsend, 1998).

Ideally, one might like to distinguish the role of each of these financial providers and/or financial

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[1]See Dept of Commerce (l985) for the construction and a critique of its application in India, also Srinivasan (2003).

[2]There is a vast empirical literature on the buffer stock, permanent income model risk sharing model, in many countries, replete with rejections. For incorporation of private information and limited commitment see Ligon (1998), Ligon Thomas and Worall (2002), Albarran and Attanasio (2003).

[3]From theory point of view, the separation between consumption and production is not justified when assuming market imperfections. Thus, we look at consumption and production at the same time.

instruments[4]. Here though we consider credit and savings as if from a single competitive financial sector as this allow us to focus on the overall net effect, the overall financial regime in place and the salient obstacles[5]. Specifically, we consider dynamic, investment versions of Paulson, Townsend and Karaivanov (2006), hereafter PTK, who distinguish moral hazard from limited commitment in the (retrospective) l997 Thai data on firm formation and initial wealth, and Karaivanov (2006) who cannot distinguish in the same data moral hazard from a simple model of debt with default. On the other hand, in Karaivanov, Socio-Economic Survey (SES) data from the 1970's in Thailand indicate that a savings-only regime fits best, consistent historically with a less developed financial system.

Related to our goals in this paper are the ENAMIN surveys of small enterprise in Mexico. Much as in Thailand, these come from a household-based enumeration, in this case an urban labor survey, ENEU, which picks up family business, in turn based on the Population Census. Among the 10,000 firms in the follow-up ENAMIN surveys, the maximum number of paid employees is 15, but a majority use only unpaid family labor. Still, these firms account for a nontrivial fraction of the Mexican workforce, and in Woodruff and McKenzie (2006), high rates of return suggests these firms are credit constrained. Starting on the other end, from a firm based enumeration, a survey of registered firms in Ghana, conducted by the Rural Program for Enterprise Development (in collaboration with the World Bank/Oxford), many Ghanaian manufacturing firms have (only) 1-5 employees. Rates of return are high yet investment is rare, especially for the micro/small firms, again a symptom of credit constraints. This is modeled by Schundeln (2006) as a costly adjustment model with a reduced-form equation capturing an exogenous and increasing cost of finance. The point is that small firms are a large part of poor and emerging market economies, yet we are left with many questions about the nature of the financial markets, institutions and constraints.

Of course, OECD counties have small enterprise too. In Spain, based on the Sabi-Informa database of the commercial registry, newly established (plus merged) firms have 493,000 Euro sales (and some are much smaller). For these firms, the number of banking relationships increase with age (and size), and about half of the new firms (and small firms) have no credit from formal financial institutions. Thus it appears that age (size) may alleviate constraints. On the other hand, firms without formal financial institution credit do get trade and other credit; the ratio of banking credit to total credit is only 16% for the smallest firms. Growth and the variance of growth decrease with age of the firm (and sometimes with size), while debt/asset ratios are somewhat flat if not declining, replicating some, but not all the stylized facts in the US literature (Cooley and Quadrini, 1999). Some firms seem to be sensitive to cash flow, as Euler equation implications are rejected for the lowest decile of firms ordered by (lagged) cash/investment ratios (Ruano, 2006). We want to see if this is ameliorated with credit (Ruano, Saurina, and Townsend, in progress). Finally, and related, many financial institutions in Spain have nontrivial amounts of their portfolio in household proprietorships (as distinct from legally registered firms). The point is that household proprietorships and small firms remain a big part of the financial system of an industrialized country, yet we do not understand well the financial arrangements that they have.

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[4]For example, using the annual panel , Alem and Townsend (2006) find that the BAAC seems to help households smooth consumption while commercial banks help to smooth investment from cash flow. Kaboski and Townsend (2007a, b) using the same data focus on the impact of movement in consumption, investment, savings, and credit from of a government-induced innovation in village funds. Gine (2005) tests a theory of selection, why some borrow from both formal and informal sources and others not.

[5]In a neoclassical world credit from one source can substitute for another and, more generally, there would be indifference across various credit providers in a Miller-Modigliani world. A mechanism design, optimal contracting model of credit can reduce to pure borrowing and lending when clients have unlimited access to outside funds or internal savings at the same rate as the principal, e.g. Cole and Kocherlakota (1999), Allen (1985).

Similarly, small firms are a very important part of the US economy: the Small Business Database of manufacturing firms gathered by Dunn and Bradstreet (l980), used by Brock and Evans (1986) and others, finds 1.1 and 2.5 million firms with sales of 0-25,000 and 25,000-999,999, respectively (the IRS numbers are 2.7 and 7.8 million, respectively). The Fed's l998 Survey of Small Business Finance selects from a population of 2.6 million firms with fewer than 500 employees and finds a preponderance of owner equity, low debt/asset ratios (median .35), yet many loans with personal guarantees. An attempt is made to distinguish business from non-business wealth, but on the other end of the distribution and starting with a household based survey, the Federal Reserve Board's Survey of Consumer Finances shows the personal side of the wealthier of the business owners, some with substantial positions in the stock market. Indeed wealth transmission, enterprise formation, and inequality in the US income distribution are the focus of Cagetti and De Nardi (2007). Similarly, Heaton and Lucas (2000) and Moskowitz and Vissing-Jorgensen (2002), focus on proprietorships as a portfolio choice while finding rate of return anomalies. But typically, this literature imposes a financial regime, typically some combination of debt and equity, not a full set of contingent claims markets.

Using numerical methods based on mechanism design theory, this paper characterizes investment, financial flows and insurance in a wide range of incomplete and complete markets dynamic models of asymmetric information as opposed to a single, exogenously assumed financial regime. Financial market imperfections affect agents' and firm's investment and consumption choices in static and dynamic sense. The goal of this paper is to distinguish across various dynamic models of financial market imperfections affecting access to finance and the ability to smooth consumption. To that end we put together a unified theoretical framework in which we analyze, compare across, estimate and test both exogenously and endogenously incomplete market environments aiming to provide insights as to the source and nature of financial constraints[6].

Specifically, we study the following alternative financial regimes under which firms may operate: (1) autarky (no access to credit markets), (2) saving only, (3) borrowing and lending in a single risk-free asset, (4) contingent claims market under moral hazard and adverse selection (due to unobserved effort and unobserved capital), (5) contingent claims market under moral hazard (unobserved effort but observed capital), and (6) contingent claims market under full information. Regimes (2) and (3) are relatively standard "permanent income" type models of self-insurance by saving and/or borrowing, while the endogenously incomplete regimes (4) and (5) belong to the class of dynamic moral hazard models pioneered by Townsend (1982), Rogerson (1985) or Spear and Srivastava (1987).

We model an economy in which infinitely-lived agents (interpreted as running small businesses) produce output using two inputs: labor effort and capital. Output is stochastic, i.e. a given input combination induces a probability distribution over a finite set of output levels. Capital can be accumulated over time. An important difference with most of the existing empirical literature on dynamic moral hazard is our emphasis on investment and capital accumulation (firm dynamics) in a production economy[7]. By assuming that the capital stock (and investment) which enters production can be unobserved, but the agent's access to outside finance is fully controlled by the principal, our analysis is also different to most of the "hidden savings" literature[8] (Allen, 1985;

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[6]In a similar vein Attanasio and Pavoni (2005) derive methods for testing private information models with asset accumulation concentrating on their predictions about the distribution of consumption over time.

[7]In contrast, the previous theory literature has mostly looked at either endowment or exogenous income economies (Green, 1987; Thomas and Worrall, 1990), taste shocks (Atkeson and Lucas, 1991), or models with stochastic output affected only by effort (Spear and Srivastava, 1987; Phelan and Townsend, 1991).

[8]Hidden access to outside credit affects consumption in these models but not production and they typically abstract

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Cole and Kocherlakota, 2001; Abraham and Pavoni, 2005).

The numerical approach that we employ allows us to look at the model implications from virtually all possible angles, both static and dynamic. We concentrate on the behavior of investment, firm size and insurance. We first compare the implications of the various regimes with five "stylized facts" about firm growth and investment sensitivity to cashflow as listed in Cooley and Quadrini (1999) using data on the joint distribution of firm size today and next period and cashflow. In contrast with most of the finance literature, we do not need to resort to the assumption of risk neutrality as our methodology allows us to consider general preferences and technologies.

Overall, our model is shown to be successful in matching the empirical regularities from the finance literature. However, this success should be taken with caution. We find that most of the stylized facts are matched qualitatively by financial regimes from the whole spectrum between autarky and complete markets, suggesting that these regularities perhaps need to be qualified if we are to use them to elicit information about firm's credit market circumstances. On the other hand, we find significant quantitative differences between the regimes and test whether this could serve as a basis of which to distinguish between the alternative financial environments.

We also search for other testable predictions of the model that can be used to differentiate between regimes and hence give us insights into the major sources of financial market incompleteness in a given dataset. Specifically, we look at cross-sectional and intertemporal consumption profiles, which theory predicts should differ across the regimes, using data on the cross-sectional distribution of consumption and output/cashflow. We also demonstrate that the model regimes differ significantly in the dynamics they generate. The exogenously incomplete regimes are shown to converge relatively quickly (within 20-50 periods) to non-degenerate consumption, investment and asset distributions. This implies time-invariant cross-sectional consumption variance in the long run.[9] In contrast, the endogenously incomplete regimes are shown to converge more slowly. Indeed, panel data or cross sectional histograms separated by a large time interval can serve to distinguish moral hazard with and without observed capital. Specifically, we compute and use as a basis to distinguish across the regimes the dynamics of consumption smoothing over time (using data on the distributions of consumption and cashflow one or fifty periods apart).

We take one of our models as a baseline and generate data (potentially subject to measurement error) and we use that data to estimate and statistically test and distinguish across all the financial regimes we study. We use structural maximum likelihood estimation methods (see PTK, 2006 in a static setting) and Vuong's (1989) model selection procedure to statistically test if the regimes can be distinguished based on each of the listed dimensions of the simulated data.

One remarkable finding is that the various regimes are rank ordered in the likelihoods exactly as the nature of incompleteness or endogenous constraints might suggest (they are not rank ordered by the variance of the measurement error). That is, if we generate data from the moral hazard observed capital regime , then the likelihoods are ordered from best to worst by: moral hazard with observed capital (MH), moral hazard with unobserved capital (UC), full information (FI), borrowing and lending (BL), savings only (S), and autarky (A). Indeed, in pairwise tests of the regimes, the regime which dominates is the one which is closest, even if we start with a counterfactual. For example, if we generate data from MH, but test borrowing/lending we find that BL is dominated by FI and

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from investment.

[9]This is an important observation in view of the literature on testing the full vs. partial insurance hypotheses (e.g. Blundell, Pistaferri and Preston, 2002). Observing time varying variance in the consumption distribution is certainly a sufficient condition to reject the full insurance hypothesis but, as our results show, the opposite is not true - observing zero time variation in the cross-sectional variance is perfectly consistent with partial insurance as well as is observing non-zero variation.

UC, but BL dominates S and A . Not all these dominance pairings are statistically significant, however.

Naturally, the ability to distinguish between the regimes depends on both the type of data used and the amount of measurement error allowed for, as well as underlying parameters. As expected, higher levels of measurement error in the data reduce the power of the model comparison test to such an extent that many of the models cannot be distinguished from the data generating baseline as well as from each other. This is more pronounced for the cases of firm size data (especially under full depreciation). For the consumption/cashflow data, we typically cannot distinguish between the moral hazard and full information regimes, and we cannot distinguish between the autarky and saving only regimes. In contrast, in virtually all cases we are able to distinguish between regime groups, that is autarky/ saving only versus the less constrained moral hazard with observed or unobserved capital and full information regimes.

When using repeated cross-sections data, we are generally able to distinguish between the exogenously incomplete vs. endogenously constrained regimes, even under high measurement error, although when the data are fifty periods apart, the distinction between the exogenously incomplete regimes is blurred. That is, if the hypothesized null regime is true, then we can distinguish the truth from other regimes. But if the researcher guesses incorrectly, and the null is a counterfactual regime, then there is less ability to distinguish the null of other regimes nearby using time-separated data. This happens in other instances, i.e., under incomplete depreciation, the (consumption, output) data allow one to correctly pick the MH regime more reliably than using (capital / investment data), but less able statistically to reject and distinguish incorrect guesses of S and BL. Thus researchers should be cautious when testing a given regime against an alternative when they fail to reject it in the data. Both the null and the alternatives may not be the true regime.

2 The Model

We consider an environment consisting of agents (firms) which are heterogeneous in their initial endowments (firm size), k_0 of the single consumption and investment good in the economy. The agents live T periods, where T can be infinity. They can potentially interact with a financial intermediary, entering into saving, debt, or insurance contracts.

We characterize the optimal dynamic financial contracts that will arise between the agents and the financial intermediary under the different information regimes described in the introduction (all details are provided in the next section). We model these financial contracts as probability distributions over assigned or implemented allocations of consumption, effort and investment. There are two possible ways to interpret this. First, we can either think of a principal (the intermediary) contracting with a single agent (a firm) at a time, in which case the optimal contract specifies a mixed strategy over various allocations. Alternatively, we can think of a continuum of agents where the optimal contract specifies the fraction of agents of given type that receive a specific deterministic allocation. It is assumed that there are no aggregate shocks, there are no technological links between the agents, and they cannot collude.

The agents are risk-averse and have time-separable preferences defined over consumption, c, and labor effort, z represented by $U(c, z)$ where $U_1 > 0$, $U_2 < 0$. They discount future utility using a discount factor, β where $\beta \in (0, 1)$. While we use concave utility in our applications below, we should mention that our methods are valid for *any* (possibly non-convex) preferences and technologies. For computational reasons we assume that c and z belong to the finite discrete sets (grids) C, Z accordingly.

The agents have access to a stochastic output technology, $P(q|z,k) : Q \times Z \times K \rightarrow [0,1]$ interpreted as the probability of obtaining output q from effort level z and investment (capital) level k[10]. The sets Q and K are assumed to be finite and discrete[11]. In all information regimes we study output is assumed to be fully observable and verifiable. However, one or both of the inputs, k and z may be unobservable to the principal introducing moral hazard and/or adverse selection problems. Capital, k depreciates at a rate δ every period. Depending on the application we have in mind, the lowest capital level ($k = 0$) could be interpreted as a "worker" occupation (similar to PTK, 2006) or as a "firm exit" state when computing the optimal contract.

The financial intermediary is risk neutral and has an access to an outside credit market at an opportunity cost of funds of R. In the endogenously constrained information regimes, the contract between the principal and the agent allows for any optimal *transfer*, τ (possibly contingent on output history) between the two parties. As with all other variables, we assume $\tau \in T$ where T is a discrete finite set. In contrast, in the saving only and borrowing and lending regimes, the transfer (the amount saved or borrowed, b) is exogenously constrained to be non-contingent on output (no default is allowed). Finally, the principal and the agent can fully commit to the ex-ante optimal contract in each regime (although our methods allow us to relax this assumption - see last section).

3   Information and Credit Access Regimes

This section describes in detail the various information and credit access regimes we study. For each regime we write down the recursive dynamic optimization problem determining the optimal contract and characterize the respective state space. In solving for the optimal contracts under incomplete information we use the revelation principle looking only at direct mechanisms in which the agents announce truthfully their type, k and obey the principal's recommendations for effort, z and next period capital, k'. The proof that the revelation principle applies in our setting is a relatively simple extension of the proofs in Doepke and Townsend (2005) and hence is not reproduced here. Demonstrating that the optimal contracting problem can be written in a recursive form also follows easily from Doepke and Townsend's results.

As mentioned above, the information / credit access regimes we analyze can be classified into two broad classes. The first class consists of exogenously incomplete market regimes - the *autarky (A)*, the *savings only (S)*, and *borrowing and lending (BL)* regimes. In these regimes the reason for the financial market incompleteness is exogenously given, i.e. it is assumed that the feasible contracts that agents have access to take a specific form similar to some typical financial arrangements observed in reality (autarky, a deposit contract, or a debt contract).

We also look at a second class of information regimes where the optimal financial contract is endogenously determined, subject only to the constraint of asymmetric information. We look at two such regimes - *moral hazard (MH)* where agents' effort is unobserved by the bank and hence non-contractible but their investment decisions, k and k' are observed, and a *moral hazard with unobserved capital (UC)* regime where both agents' efforts, z, and capital stock, k and investment, $k' - (1-\delta)k$ are unobservable. As a benchmark those regimes are compared to the *full information (FI)* regime (the first best) where all variables are observable and contractible upon.

In terms of numerical methods[12], we employ the linear programming (LP) solution approach

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[10]We can easily incorporate heterogeneity in entrepreneurial ability across agents as in Paulson et al. (2006), for instance by adding a talent parameter θ multiplying output q in the production function $P(q|z,k)$.

[11]This can be interpreted as either a technological or purely computational assumption depending on the particular application.

[12]Given our goal is to develop an empirically applicable methodology to distinguish between financial regimes,

pioneered by Prescott and Townsend (1984) and Phelan and Townsend (1991). Building on the theoretical results of Doepke and Townsend (2005) below we show how one can write the dynamic problems corresponding to each regime as recursive linear programs (LP). An alternative to our LP methodology is the "first order approach" (Rogerson, 1985) used for instance by Abraham and Pavoni (2005) and Kapicka (2005). A potential problem is that the resulting solution may not be a maximum in the original problem due to non-convexities introduced by the incentive constraints (e.g. see Kocherlakota, 2004). In contrast, the LP approach is extremely general and valid by construction for any possible preference and technology specifications, as it convexifies the original problem by allowing for any possible lotteries over allocations. The price of this generality is that the LP method suffers from the "curse of dimensionality" due to the need to use discrete grids for all variables. We develop computational methods to minimize this deficiency.

3.1 Exogenously Incomplete Markets Regimes

3.1.1 Autarky

In this regime there is no interaction with a financial intermediary i.e. the agent is assumed to have no access to credit or savings / storage. The timeline is as follows. The agent starts the current period with initial capital k carried over from last period which he invests into production. At this time the agent also decides on his effort level z. At the end of the period output q is realized, the agent decides on the next period capital level, k', and consumes $c = (1 - \delta)k + q - k'$. Capital, k is the state variable in the recursive formulation of the agent's optimization problem. This is a trivial problem and can be solved by standard non-linear dynamic programming techniques. To be consistent with our solution methods used for the endogenously incomplete regimes where such techniques may be inapplicable due to non-convexities introduced by the incentive and truth-telling constraints, we reformulate the problem as a linear program with respect to the joint probabilities of obtaining allocations (q, z, k') given the state k.

The agent's problem, given his current capital level, k is represented recursively as:[13]

$$V(k) = \max_{\pi(q,z,k'|k)} \sum_{QxZxK'} \pi(q, z, k'|k)[u((1 - \delta)k + q - k', z) + \beta V(k')] \tag{1}$$

The policy variables $\pi(q, z, k'|k)$ represent the solution to the above maximization problem and determine the optimal effort and investment level z and k'. The maximization in (1) is subject to a set of constraints on the choice variables π. First, we need to ensure that for each $k \in K$, the $\pi's$ are Bayes consistent with the production function (the probability distribution over outputs):

$$\sum_{K'} \pi(q, z, k'|k) = P(\bar{q}|\bar{z}, k) \sum_{Q \times K'} \pi(q, \bar{z}, k'|k) \text{ for all } (\bar{q}, \bar{z}) \in Q \times Z \tag{2}$$

The choice variables π must also form a valid probability function, i.e. we must have that, given k, $\pi(q, z, k'|k) \geq 0$ for all $(q, z, k') \in Q \times Z \times K'$ and adding-up:

$$\sum_{QxZxK'} \pi(q, z, k'|k) = 1 \tag{3}$$

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we have chosen general functional forms sacrificing analytical tractability. We are well aware of the limitations of this approach and the fact that our computed examples do not constitute proofs. We have tried our best to check robustness by using numerous parameter specifications and initial conditions. The full set of numerical computations is readily available from the authors.

[13]Here and everywhere later in the paper we use notation K', B', etc. to denote the set of next period values. In terms of values, these sets coincide with K, B, etc.

3.1.2 Saving and/or Borrowing

In this financial environment the agent is assumed to be able to either only save (which we call the *saving only, (S)* regime) or both borrow and save/lend (called the *borrowing and lending, (BL)* regime) through a competitive financial intermediary. Clearly then the agent has more opportunities for smoothing consumption than under autarky. Still, his consumption smoothing ability is restricted as the standard "deposit" or "debt"-like contracts that this regime features do not allow state-contingent repayments. More specifically, if the agent borrows (saves) an amount b, next period he has to repay (collect) an amount Rb (where R is the gross interest rate) independently of the state of the world. Default is ruled out by assuming that the bank refuses to lend if a borrower who is at risk of not repaying[14]. By shutting down all contingencies in the debt contracts we aim for better differentiation with the endogenously constrained contingent transfer regimes described next. Finally, to be consistent with the other regimes, b is assumed to take values on the finite discrete set (grid), B.

The basic setting of the S and BL regimes is similar to the autarky regime. The main difference is that, in addition to all he could do before, the agent can now hold debt b and carry debt b' in the next period (a negative value of b represents savings). The timeline is as follows: the agent starts with capital k carried from last period and uses it in production together with effort z. In the end of the period output q is realized, the agent repays Rb and borrows (rolls-over) or saves b'. He also puts aside next period's capital, k' and consumes $c = (1 - \delta)k + q + b' - Rb - k'$. Thus, the agent can use debt/savings to smooth consumption over time on top of what he could do under autarky. In the saving only regime b is constrained to be non-positive, i.e. the upper bound of the grid B is zero - the agent can only accumulate and run down a buffer stock. The two assets k and b can be freely converted into one another each period when a decision of how much capital to carry over and invest in production is made. Under the BL regime the upper bound of the set B is positive.

Given the above, the problem of an agent with current capital stock k and debt/savings level b can be written recursively as:

$$V(k,b) = \max_{\pi(q,z,k',b'|k,b)} \sum_{QxZxK'xB'} \pi(q,z,k',b'|k,b)[U((1 - \delta)k + q + b' - Rb - k',z) + \beta V(k',b')] \quad (4)$$

subject to the Bayes consistency and adding up constraints analogous to (2) and (3) and subject to $\pi(q,z,k',b'|k,b) \geq 0$ for all $(q,z,k',b') \in Q \times Z \times K' \times B'$.

3.2 Full Information and Endogenously Incomplete Markets Regimes

3.2.1 Full Information

In this regime the assumption is that the principal observes and can contract upon the agent's effort, z as well as investment, k and k'. We write the corresponding dynamic principal-agent problem as an extension of Phelan and Townsend (1991) adding capital accumulation. As is standard in such settings (see Spear and Srivastava, 1987; Doepke and Townsend, 2005) to obtain a recursive formulation of the mechanism design problem we need an additional state variable - *promised utility* (i.e. discounted future utility), w.

The optimal contract for an agent with current promised utility w and firm size k consists of effort and investment levels, z, k', next period's promised utility w', as well as a transfer, τ between the principal and the agent (a positive value of τ is taken to mean that the direction of the transfer

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[14]Technically this is achieved by assigning a very low utility value for the borrower so default is never optimal for the agent.

is from the principal to the agent). The timing of events is the same as above with the addition that the transfer occurs after output is observed.

The principal's objective function, $V(w, k)$ when contracting with an agent (w, k) represents the expected value of output net of transfers to the agent, plus the discounted value of future outputs and transfers. As above, we write the problem as an linear program using the joint probabilities, $\pi(\tau, q, z, k', w'|w, k)$ of an allocation (τ, q, z, k', w') occurring:

$$V(w, k) = \max_{\{\pi(\tau, q, z, k', w'|w, k)\}} \sum_{T \times Q \times Z \times K \times W} \pi(\tau, q, z, k', w'|w, k)[q - \tau + (1/R)V(w', k')] \qquad (5)$$

The maximization in (5) is subject to the familiar Bayes consistency and adding up constraints on the probabilities π:

$$\sum_{TxK'xW'} \pi(\tau, \bar{q}, \bar{z}, k', w'|w, k) = P(\bar{q}|\bar{z}, k) \sum_{TxQ \times K'xW'} \pi(\tau, q, \bar{z}, k', w'|w, k) \text{ for all } (\bar{q}, \bar{z}) \in Q \times Z \quad (6)$$

$$\sum_{T \times Q \times Z \times K \times W} \pi(\tau, q, z, k', w'|w, k) = 1 \qquad (7)$$

and non-negativity: $\pi(\tau, q, z, k', w'|w, k) \geq 0$ for all $(\tau, q, z, k', w') \in T \times Q \times Z \times K \times W$.

Finally, because of the extra state there is an additional constraint, namely the *promise keeping* constraint, which ensures that the agent's expected utility equals his current utility promise, w. Notice that by varying the initial value of w one can trace the whole Pareto frontier of expected utilities for the principal and the agent (see section 6 for computed examples). The promise keeping constraint is:

$$\sum_{T \times Q \times Z \times K \times W} \pi(\tau, q, z, k', w'|w, k)[U(\tau + (1 - \delta)k - k', z) + \beta w'] = w \qquad (8)$$

The optimal FI contract consists of maximizing (5) subject to the constraints (6), (7) and (8).

As mentioned above, we also need to characterize the set of utility promises W[15]. Similarly to Phelan and Townsend (1991) the grid of promised utilities we use has a lower bound, $w_{\min}$ corresponding to the lowest possible consumption for the agent, $c_{\min}$ (obtained by choosing the lowest possible τ and highest k') and the highest possible assigned effort, $z_{\max}$ promised forever and an upper bound, $w_{\max}$ corresponding to the highest possible consumption, $c_{\max}$ and lowest possible effort promised with certainty forever:

$$w_{\min}^{FI} = \frac{U(c_{\min}, z_{\max})}{1 - \beta} \text{ and } w_{\max}^{FI} = \frac{U(c_{\max}, z_{\min})}{1 - \beta}$$

We solve the above dynamic linear program numerically. The solution is a vector of probabilities $\pi^*(\tau, q, z, k', w'|w, k)$ representing the optimal contract between the bank and the agent. Typically most of the $\pi's$ are zeros. We know by theory that the optimal solution under full information features full consumption insurance and no intertemporal tie-ins. We chose not to solve the simpler problem that arises after imposing these equilibrium properties on $\pi(.)$ in order to be able to measure any numerical distortions introduced by the grids and thus obtain a valid benchmark for the asymmetric information regimes below where such simplification is not possible.

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[15]In principle, some values in this set may be infeasible. The set of feasible values is determined along with iterating on the value function using the methods proposed by Abreu, Pearce and Stacchetti (1990).

3.2.2 Moral Hazard With Observed Investment

Under this regime we deviate from the first best world of the full information and assume that while the principal can still observe and control capital and investment (k and k') he can no longer observe (or verify) agent's effort, z. Note that with capital observed and controlled, it can serve in a sense as collateral on a loan. The transfer back to the bank can tax what is left of capital after production, and this may happen when output is low, i.e. the project is not successful. All other assumptions regarding the timing, preferences, technologies, etc. used in the previous section are kept unchanged.

The unobservability of effort implies that the principal must now induce effort from the agent as the contract between the two parties is subject to moral hazard. Technically this is achieved by requiring that the optimal contract, $\pi(\tau, q, z, k', w'|w, k)$ satisfy an incentive compatibility constraint (ICC) in addition to (8)-(10). The ICC states that, given agent's state (w, k), a recommended effort level, $\bar{z}$, and known and enforced capital level k' and transfer τ, the agent must not be able to achieve higher expected utility by deviating to any effort level $\hat{z}$ different from $\bar{z}$. In our notation this requires that for all $(\bar{z}, \hat{z}) \in Z \times Z$:

$$\sum_{T \times Q \times W' \times K'} \pi(\tau, q, z, k', w'|w, k)[U(\tau + (1-\delta)k - k', \bar{z}) + \beta w'] \geq$$

$$\geq \sum_{T \times Q \times W' \times K'} \pi(\tau, q, z, k', w'|w, k)\frac{P(q|\hat{z}, k)}{P(q|\bar{z}, k)}[U(\tau + (1-\delta)k - k', \hat{z}) + \beta w'] \qquad (9)$$

For details on the derivation of the ICC in our linear programming framework see Prescott and Townsend (1984a,b) and Phelan and Townsend (1991). The critical term in the above inequality is the "likelihood ratio", $\frac{P(q|\hat{z}, k)}{P(q|\bar{z}, k)}$ present on the r.h.s. since if the agent deviates he changes the probability distribution of output and the contract joint probabilities π must be adjusted to preserve Bayes rule consistency.

Apart from the added constraint (9), the moral hazard setting also differs from the full information case in the set of feasible promised utilities. Specifically, the lowest possible promise under moral hazard can no longer be the value $w_{\min}^{FI}$ from the FI regime. Indeed, if the agent is assigned minimum consumption forever he would not supply any effort level above the minimum possible. Thus, the range of promised utilities for the MH setting is:

$$w_{\min}^{MH} = \frac{U(c_{\min}, z_{\min})}{1 - \beta} \text{ and } w_{\max}^{MH} = \frac{U(c_{\max}, z_{\min})}{1 - \beta}$$

The formal derivation of the expression for $w_{\min}^{MH}$ follows Phelan and Townsend (1991). The basic intuition is that the principal cannot promise a little bit higher consumption in exchange for much higher effort such that agent's utility falls below $w_{\min}^{MH}$ since this is not incentive compatible. The reason is that if the agent does not follow the principal's recommendations but deviates to $z_{\min}$ the worst punishment he can receive is $c_{\min}$ forever.

The (constrained) optimal contract in the moral hazard regime with observed k is the solution to the linear program in (7)-(11). We know by theory (e.g. see Townsend, 1982) that the optimal contract features incomplete consumption insurance and intertemporal tie-ins, i.e. the optimal dynamic contract is not simply a repetition of the optimal one-period contracts.

3.2.3 Moral Hazard with Unobserved Investment

Now suppose there are two sources of asymmetric information in the financial environment. First, let the effort exerted by the agent be still unobservable by the principal. In addition, assume that

the principal also cannot observe the capital level, k that the agent has and also the level of capital investment planned for next period. Thus, within each period there is both the adverse selection problem of *unobserved information* about the agent (his "type", k) as well as the moral hazard problem of two *unobserved actions*, namely z and k'.

Assume that the agent sends a message about his capital level k to the principal who offers a contract conditional on the agent's message which consists of a transfer τ, recommended effort, z and investment, k' as well as future promised utility. Doepke and Townsend (2005) have shown that the revelation principle still applies in this asymmetric information setting, thus we continue to restrict attention to direct mechanisms.

Due to the physical linkage between time periods and the dynamic adverse selection problem caused by the unobservable state k, we follow the approach of Fernandes and Phelan (2000) and use as a state variable a *function (vector) of utility promises*, $\mathbf{w}$ rather than the scalar w from the MH regime. The reason why utility promises now cannot be independent of k in general is that the incentives of agents coming with different k' in the next period are different (see Kocherlakota, 2004). Thus, to induce incentive compatibility, the principal needs to offer an optimal schedule of promises one for each possible firm size, $\mathbf{w} \equiv \{w(k_1), w(k_2), ...w(k_{\#K})\} \in \mathbf{W}$ where k_1, k_2, etc. are the elements of the grid K[16]. Note that this introduces a much higher number of state variables into the problem which is exponentially increasing in the number of points in the capital grid. The set $\mathbf{W}$ is endogenously determined and iterated upon together with the value and policy functions (see Abreu, Pierce and Stachetti, 1990).

We study two different scenarios regarding investment. First, we study the case where capital depreciates fully during the production process within the period, i.e. $\delta = 1$ or we can interpret k as "materials". This case is easier to analyze because time periods are linked only through the choice of k' last period similar to the setups in Fernandes and Phelan (2000) and Doepke and Townsend (2005). Matters become more complicated when capital is allowed to depreciate incompletely, i.e. $\delta < 1$ (e.g. "machines"). In this case the level of k' chosen would be dependent on the level of capital k with which the period started, creating an additional interdependence between time periods. Writing the optimal contracting problem in recursive form now is considerably harder than in the full-depreciation case as one has to control for joint deviations in reporting the state k, as well as the action k', within each period. We manage to resolve this problem by judicious usage of extra state variables and *utility bounds* (see Prescott, 2003).

The computational methods we propose in this section require separability in consumption and leisure, $U(c, z) = u(c) - d(z)$[17]. Notice that this is not needed in the MH, FI or the exogenously incomplete regimes. The separability allows to split each time period into two separate stages and use dynamic programming not only across but also within time periods. We emphasize that this is just a computational construct (no economic content is implied) that helps us keep the curse of dimensionality in check as the resulting two stage problems are of lower dimensionality.

The two sub-periods used in the computation are as follows. The first stage is up to the moment of output realization and includes the announcement of k by the agent, the principal's recommendation on effort, z, the agent's effort supply and production and the realization of the output q. The second stage includes the transfer τ, the announcement of the promised utility vector, $\mathbf{w}'$, as well as the investment recommendation, k', agent's consumption and investment. As a purely mathematical artifact, to tie the two sub-periods together, we use an extra variable that we call *interim utility*, representing the expected utility for the agent from the end of sub-period 1 onwards.

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[16]We use bold font to denote vector variables. The notation $\#X$ means "the number of elements of vector X".

[17]Our methods allow the functions u and d to take any form although in our simulations we use standard concave u and convex d.

Full Depreciation   We start with the case of full depreciation. The first sub-period problem, Program UC1 for computing the optimal contract between the bank and an agent who has announced k and is promised $\mathbf{w}$ is:

Program UC1:

$$V(\mathbf{w}, k) = \max_{\{\pi(q,z,w_m|\mathbf{w},k)\}} \sum_{Q \times Z \times W_m} \pi(q, z, w_m|\mathbf{w},k)[q + V_m(w_m)] \tag{10}$$

In this first stage, the choice variables are the probabilities over allocations $(q, z, w_m) \in Q \times Z \times W_m$. The set of interim utilities, W_m is a discrete finite set with lower and upper bounds consistent with those on W. The function $V_m(w_m)$ will be defined in the second stage problem (see below).

The optimization proceeds subject to a number of constraints. First, we must ensure that the optimal contract delivers the promised utility, $w(k)$:

$$\sum_{Q \times Z \times W_m} \pi(q, z, w_m|\mathbf{w},k)[-d(z) + w_m] = w(k) \tag{11}$$

Notice that the utility from consumption as well as discounted future utility are implicitly kept track of through w_m[18]. Second, effort is unobservable so the optimal contract must satisfy incentive compatibility. That is, for all $(z, \hat{z}) \in Z \times Z$:

$$\sum_{Q \times W_m} \pi(q, z, w_m|\mathbf{w},k)[-d(z) + w_m] \geq \sum_{Q \times W_m} \pi(q, z, w_m|\mathbf{w},k)[-d(\hat{z}) + w_m]\frac{P(q|\hat{z}, k)}{P(q|z, k)} \tag{12}$$

Third, since the state k is also private information, agents would also need incentives to reveal it. On top of that, they can presumably consider joint deviations in their announcements about k and effort, z. Such behavior is ruled out by the following *truth-telling constraints* that must hold for all $\hat{k} \neq k$ and $\delta(z) \in Z$:

$$w(\hat{k}) \geq \sum_{Q \times Z \times W_m} \pi(q, z, w_m|\mathbf{w},k)[-d(\delta(z)) + w_m]\frac{P(q|\delta(z), \hat{k})}{P(q|z, k)} \tag{13}$$

In words, any agent who actually has capital $\hat{k}$ but considers announcing k should find such deviation unattractive. Notice that to rule out joint deviations in k and z, (13) is required to hold regardless of whether the agent decides to follow the effort recommendation, z or considers a deviation to another effort level, $\delta(z)$ where $\delta(z)$ denotes all possible mappings from Z to Z. There are $(\#K-1)\#Z^{\#Z}$ such constraints in total. As before, the utility obtained when the agent considers a deviation in either his announcement of k or his action is weighed by the likelihood ratio, $\frac{P(q|\delta(z),\hat{k})}{P(q|z,k)}$ to preserve Bayes consistency. Finally, the contract must satisfy the familiar Bayes consistency, adding up and non-negativity constraints for $\pi(q, z, w_m)$.

To solve Program UC1, we first need to compute the function $V_m(w_m)$ giving the principal's value at an interim utility w_m. Thus, we compute, *for each $w_m \in W_m$* the following:

Program UC2:

$$V_m(w_m) = \max_{\{\pi(\tau,k',\mathbf{w}'|w_m)\}} \sum_{T \times K' \times \mathbf{W}'} \pi(\tau, k', \mathbf{w}'|w_m)[-\tau + (1/R)V(k', \mathbf{w}')] \tag{14}$$

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[18]Remember that the interim utility is an artifact of the computation algorithm and hence not a part of the optimal contract.

The maximization in (14) is subject to the following constraints. First, we impose the definition of interim utility:

$$w_m = \sum_{T \times K' \times \mathbf{W}'} \pi(\tau, k', \mathbf{w}'|w_m)[u(\tau - k') + \beta w'(k')] \tag{15}$$

Next, obedience in the investment decision must be ensured by providing incentives that the agent does not deviate from the recommended value, k' to some alternative value, $\hat{k}'$. Due to our timing assumptions this has to be true for any value of the transfer τ, i.e. we must have that for all $\tau \in T$, $k', \hat{k}' \in K', \hat{k}' \neq k'$:

$$\sum_{\mathbf{W}'} \pi(\tau, k', \mathbf{w}'|w_m)[u(\tau - k') + \beta w'(k')] \geq \sum_{\mathbf{W}'} \pi(\tau, k', \mathbf{w}'|w_m)[u(\tau - \hat{k}') + \beta w'(\hat{k}')] \tag{16}$$

Finally, adding up and non-negativity must hold for $\pi(\tau, k', \mathbf{w}'|w_m)$. There are no Bayes consistency constraints since production occurs in the first sub-period.

Incomplete Depreciation   We now generalize the model to include the possibility of incomplete depreciation, $\delta < 1$. This creates an extra link between the two sub-periods featured in our computational algorithm by making the interim principal's value, V_m also dependent on k since capital does not expire in production. Similarly, the interim utility variable should now take into account that the agent might have deviated in his announcement of k when entering the second stage, i.e. we need to define it as the vector $\mathbf{w}_m = \{w_m(k_1), w_m(k_2), ...\} \in \mathbf{W}_m$ by the same logic as for the vector of promises, $\mathbf{w}$. The set $\mathbf{W}_m$ is endogenously determined during the value function iteration, similarly to the set $\mathbf{W}$.

As in the previous section, start with the first sub-period problem where a principal faces an agent with promised utility vector $\mathbf{w}$ who has announced capital k.

Program UC3:

$$V(\mathbf{w}, k) = \max_{\{\pi(q,z,\mathbf{w}_m|\mathbf{w},k)\}} \sum_{Q \times Z \times \mathbf{W}_m} \pi(q, z, \mathbf{w}_m|\mathbf{w},k)[q + V_m(\mathbf{w}_m, k)] \tag{17}$$

As usual, the maximization is subject to several constraints. First, promises should be kept and incentives for following the effort recommendation must be given, corresponding to the constraints (11) and (12) only replacing w_m with $w_m(k)$ and $\pi(q, z, w_m|\mathbf{w},k)$ with $\pi(q, z, \mathbf{w}_m|\mathbf{w},k)$. Second, truth-telling needs to be induced, taking into account any possible joint deviations in effort which is again same as (13) above, only replacing w_m with $w_m(\hat{k})$ and $\pi(q, z, w_m|\mathbf{w},k)$ with $\pi(q, z, \mathbf{w}_m|\mathbf{w},k)$. The interim utility the agent expects must be consistent with his true type $(\hat{k})$ based on which he makes decisions in sub-period 2. The rest of the constraints are the familiar Bayes consistency, adding up and non-negativity.

Now move on to the second sub-period, after the output realization. The state variables are different from the $\delta = 1$ case, namely the *vector* of interim utilities, $\mathbf{w}_m$ as well as the announcement k. The fact that the state $\mathbf{w}_m$ is now a vector introduces extra truth-telling and obedience constraints in the second sub-period program The reason is that now we need to ensure that, in the second stage, when deciding on k' the agent cannot get more than his interim utility, $w_m(k)$ for *any* announcement k. Due to the higher dimension of the state space, we need to compute a higher number of linear programs as the second-stage Program UC4 below has to be computed for all possible $(k, \mathbf{w}_m) \in K \times \mathbf{W}_m$.

Program UC4:

$$V_m(\mathbf{w}_m, k) = \max_{\{\pi(\tau,k',\mathbf{w}'|\mathbf{w}_m,k)\},\{v(k,\hat{k},k',\tau)\}} \sum_{T \times K' \times \mathbf{W}'} \pi(\tau, k', \mathbf{w}'|\mathbf{w}_m,k)[-\tau + \gamma V(k', \mathbf{w}')] \tag{18}$$

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Notice that in addition to the familiar probability objects, $\pi(\tau, k', \mathbf{w}'|\mathbf{w}_m, k)$ to solve the problem computationally we now need to add more choice variables - $v(k, \hat{k}, k', \tau)$. These variables, which we call *utility bounds*, specify the maximum expected utility that an agent of type $\hat{k}$ could obtain by not announcing his capital stock truthfully (report k instead) who receives transfer τ and an investment recommendation k' (see Prescott, 2003 for details). This translates into the constraint:

$$\sum_{\mathbf{W}'} \pi(\tau, k', \mathbf{w}'|\mathbf{w}_m, k)[u(\tau + (1 - \delta)\hat{k} - \hat{k}') + \beta w'(\hat{k}')] \leq v(k, \hat{k}, k', \tau) \tag{19}$$

for all possible combinations $\tau, k', \hat{k}', \hat{k} \neq k$, and $\hat{k}' \neq k'$. To obtain the total utility, $w_m(\hat{k})$ that an agent can obtain in this second sub-period by reporting k when the true state is $\hat{k}$, we add up the bounds $v(k, \hat{k}, k', \tau)$ over all possible $\tau, k' \in T \times K'$ resulting in the constraint:

$$\sum_{T \times K'} v(k, \hat{k}, k', \tau) \leq w_m(\hat{k}) \tag{20}$$

The two sets of constraints in (19) and (20) rule out any joint deviations in the report k and the action k'. By definition, the interim utility must satisfy:

$$w_m(k) = \sum_{T \times K' \times \mathbf{W}'} \pi(\tau, k', \mathbf{w}'|\mathbf{w}_m, k)[u(\tau + (1 - \delta)k - k') + \beta w'(k')] \tag{21}$$

Finally, the probabilities π must satisfy non-negativity and adding up as usual.

4 Numerical Implementation

In this section we discuss the numerical computation of studied regimes. As already mentioned, we employ the linear programming (LP) approach pioneered by Phelan and Townsend (1991) to solve for the optimal financial contracts. The linear programming approach has the advantage to be virtually always valid[19] by construction as it convexifies the original problem by allowing for any possible lotteries over allocations. Unfortunately, this parsimony does not come without a cost - the LP approach requires using discrete grids for the state and control variables and the number of unknowns and constraints can increase rapidly with the grid size (see more on this below) demanding high memory requirements and computing time especially for the UC regime. Still, we show that by judicious formulation of the linear programs one can minimize these deficiencies.

4.1 Recursive Techniques

To speed-up the computation, the recursive problems for each regime defined in the previous section are (whenever possible) solved using *policy function iteration* (see Judd, 1998). We start with an initial guess for V[20] and iterate until convergence on the Bellman operators defined in the recursive problems above. At each iteration, for the given current value function V, we solve a linear program in the unknowns π. In the unobserved capital regime there are actually two layers of such linear programs corresponding to the two sub-problems. The actual coding of the linear programs was done in Matlab and the specialized LP software CPLEX[21].

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[19]That is, for any preferences and production functions given as functions or in table form.

[20]Typically we use a vector/matrix of zeros or the value function obtained from a "nearby" parametrization. The LP algorithms we use are very robust, so the initial guess does not affect the results only the time to convergence.

[21]All matrices of coefficients on the objective and constraints are created in the comparatively simple and highly efficient matrix language Matlab using built-in functions for Kronecker products and other matrix operations while the actual solving of the linear programs is done using the CPLEX compiled C++ library. All computations were performed on a 2.2Ghz dual core machine with 2GB RAM running Windows XP.

Remember that in the UC regime the promised utilities set, $\mathbf{W}$ is endogenously determined and has to be solved for as well together with V. Using theoretical considerations coming from incentive compatibility, we restrict attention to only non-decreasing promise vectors $\mathbf{w}(k)$. More specifically, we "discretize" the functional set $\mathbf{W}$ by starting with a broad dense set $\mathbf{W}_0$ consisting of linear functions $\mathbf{w}(k)$ with intercepts that take values from the grid $W = \{w_{\min}, w_2, ... w_{\max}\}$ defined above and varying slopes. We initially proceed iterating on the UC dynamic programming problem using value function iteration and iterate towards the feasible promise set $\mathbf{W}^* \subset \mathbf{W}$ together with the value function iteration by dropping all infeasible vectors $\mathbf{w}$ at each iteration thereby "shrinking" the set $\mathbf{W}$ based on the theory developed in Abreu, Pierce and Stachetti (1990). Once we have successively eliminated all vectors in $\mathbf{W}_0$ for which the linear program has no feasible solution, i.e. we have converged to the feasible promises set $\mathbf{W}^*$, we switch to policy function iteration[22] and continue iterating on the Bellman equation till convergence in V.

4.2 Functional Forms, Grids and Parameters

Below we describe the functional forms used in the numerical computations. Agent preferences are of the CES type:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} - \kappa z^\theta$$

As discussed above, our LP-based numerical methodology does not require separable preferences in consumption, c and effort, z but separability has been commonly used in the existing literature and it simplifies the analysis in the unobserved k case. The production function, interpreted as the probability of obtaining output level $q \in \{q_0, q_1, .. q_{\#Q}\}$, given effort z and capital, k is:

$$p(q = q_0|z, k) = \min\{0.99, \max\{0.01, 1 - (\frac{k^\rho + z^\rho}{2})^{1/\rho}\}\}$$

$$p(q = q_i|z, k) = (\frac{1-\lambda}{1-\lambda^{\#Q-1}})\lambda^{i-1} \min\{0.01, \max\{0.01, (\frac{k^\rho + z^\rho}{2})^{1/\rho}\}\} \text{ for } i = 1, .. \#Q$$

where the lowest output case $q = q_0$ will be interpreted as "project failure". Notice that the probability of obtaining any output level is bounded away from zero independently of the levels of k and z - the standard "full support" assumption needed to make the moral hazard problem non-trivial by ensuring that the principal cannot learn the input levels with certainty observing q only.

The above formulation allows for different specifications of the production technology through the parameter ρ. When $\rho = 1$ we have essentially a perfect substitutes technology, when $\rho \to 0$ we obtain a Cobb-Douglas specification and when $\rho \to -\infty$ the inputs are perfect complements. The weighting parameter $\lambda \in (0, 1)$ ensures that the probabilities defined above sum up to 1.

The grids used in the computations are defined in Table 1 below. For simplicity and to allow easier interpretation of the results we assume two output levels (low and high), q_0 and q_1 with $q_0 < q_1$. Effort takes 3 values[23]. The grid on capital is of size $\#K = 11$.

In the simulations and empirical exercises below we use two different parametrizations for the grid bounds corresponding for the cases of full or incomplete depreciation. The reason for this is to avoid corner solutions like zero investment which arise when using the same grid bounds.

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[22]We have also checked our results against proceeding with value function iteration all the way and they are identical.

[23]The lowest value is set to be slightly higher than zero for technical reasons.

Table 1 - Grids

	Grid Size	**Range $(\delta = 1)$**	**Range $(\delta < 1)$**
Q	2	$\{0.1, 3\}$	$\{0.1, 0.5\}$
K	11	$[0,1]$	$[0,1]$
Z	3	$[0.01,1]$	$[0.01,1]$
B	21	S: $[-5,0]$, BL: $[-5,5]$	S: $[-1,0]$, BL: $[-1,1]$
T	31 (86 if $\delta < 1$)	$[0,3]$	$[0,1]$
W	21 (plus 20 slopes for UC)	$[w_{\min}, w_{\max}]$	$[w_{\min}, w_{\max}]$
W_m	31 (630 if $\delta < 1$)	$[w_{\min}, \beta w_{\max} + u(t_{\max})]$	$[w_{\min}, \beta w_{\max} + u(t_{\max})]$

To get a better idea of the size of the resulting problems and the amount of computation required, we report in Table 2 below the number of linear programs computed at each iteration and the number of variables and constraints per linear program that need to be solved for in the various regimes. The number of programs is closely related to the number of state variables while the number of variables and constraints is related to the product of the grid dimensions of some combination of k', z, q, τ, b', w' depending on the regime. For example, there are only $\#Z \times \#Q + 1$ constraints in the exogenously incomplete regimes but many more in the private information ones. Remember also that the total number of linear programs computed in the UC regime equals the number in stage 1 plus $\#K$ times the number in stage 2 (e.g. 7,161 programs for the full depreciation case).

Table 2 - Problem Dimensionality

	# lin. programs	# variables (π)	#constraints
Autarky	11	66	7
Saving / Borrowing	231	1386	7
Full Information	231	42,966	8
Moral Hazard, $\delta = 1$	231	42,966	14
Moral Hazard, $\delta < 1$	231	119,196	14
Unobserved k, $\delta = 1$, **stage 1**	6,930	186	284
Unobserved k, $\delta = 1$, **stage 2**	21	410,130	188
Unobserved k, $\delta < 1$, **stage 1**	6,930	3,780	284
Unobserved k, $\delta < 1$, **stage 2**	6,930	1,137,780	95,548

The biggest computational difficulties arise from increasing $\#K$ and $\#Z$ because of the exponential increase in the number of variables or constraints which this leads to. That is why we keep these dimensions relatively low while increasing #T would be relatively "cheap" computationally. In practice, the number of variables that we solve for is slightly lower than the numbers reported in the table above because we drop from the computation any allocations that result in negative consumption, i.e. we assign probability zero to their corresponding probabilities, π. Because of the huge dimensionality and computational time requirements for the unobserved capital case with incomplete depreciation we only report results for the full depreciation baseline at the moment.

The following table lists the baseline parameters used in the various simulations, estimation and testing exercises described in the next sections. We have also performed various robustness checks for the preference and technology parameter values (shown in the parentheses) the results of which are not reported here for lack of space but available upon request.

<div align="center">**Table 3 - Baseline Parameter Values**</div>

Parameter	Value(s)
depreciation rate, δ	1 (full depreciation), .05 (inc. depreciation)
agent's discount factor, β	0.95
principal's discount factor, γ	0.95
risk aversion, σ	0.5 *(0, 2)*
effort curvature, θ	2
preference parameter, κ	1
technology parameter, ρ	0 *(-1, 1)*
probability scaling factor, λ	0.5

5 Computational Results and Examples

This section describes a representative set of computational results obtained from the different regimes. Our numerical approach allows us to look at the model implications from many possible angles in both static and dynamic sense. Given the goals of the paper, we concentrate on the behavior of investment, capital stock (firm size) and consumption smoothing. We also study several financial aspects of the optimal contracts - firm size growth, cash flow sensitivity and growth variance.

We begin by looking at the implications of the various regimes with regards to five "stylized facts" about firm dynamics as listed in Cooley and Quadrini (1999). We show that our dynamic model is capable of matching most of these facts but we also find that the restrictions they put on the model implications seem in general too weak to be able to distinguish between financial regimes. Because of this, we extend our analysis to search for other testable predictions of the model regimes that could be used as a basis of an empirical methodology to differentiate the regimes and hence give insights as to identifying the major sources of financial constraints in given data. Specifically, we characterize the optimal allocations and contracts in the studied information regimes focusing on their implications for consumption smoothing.

Next we turn to the dynamics implied by the model. We show how the numerically derived policy and value functions can be used to compute the distributions (histograms) of consumption, output, capital stock, investment, effort, and financial flows (saving, borrowing or transfers) over time. The expected time paths of those variables are generated and compared. We show that the model regimes differ with regards to the speed with which these time paths evolve in a way that can be potentially used to distinguish among them in the data.

5.1 Computation

To compute the per-period and dynamic implications of the model we use the policy functions, $\pi^*(.)$ which solve the dynamic programs from section 3. Using these policy functions, and the discrete and recursive structure of our model we construct the Markov state transition matrices corresponding to each regime. Formally, denote by $s \in S$ the current state ($s = k$ in the autarky regime, $s = (k, b)$ in S, BL, $s = (k, w)$ in the MH regime, etc.). The next period state, s' is determined by summing over the optimal contract allocation probabilities π integrating out all non-state variables such as τ, z, q. Take the MH regime for example. The transition probability of going from state (w, k) to state (w', k') next period is:

$$\Pr{}_t(w', k'|w, k) = \sum_{Q \times Z \times T} \pi^*(\tau, q, z, k', w'|w, k)$$

We can use the above state transition probabilities to form the Markov state transition matrix for each regime, M of dimension $\#S \times \#S$ that can be used to completely characterize the dynamics of the model. In particular, we can use this matrix to compute cross-sectional distributions over states starting from an arbitrary initial state distribution[24] represented by the vector $\mathbf{v}^0 = \{v_1^0, v_2^0, ...v_{\#S}^0\}$. The state distribution at time t is then:

$$\mathbf{v}^t = (M')^t \mathbf{v}^0 \tag{22}$$

where $t = \infty$ gives the long run distribution over states for a given regime.

Next, the probability distribution over states from (22) can be used in conjunction with the policy function, π^* to compute *cross-sectional distributions* (histograms), $\mathbf{H}_t(x)$ for any contract element (e.g. $x = c, z, \tau, q$, etc.) or functions thereof at any given time period t. For example, in the MH regime, the distribution of k' in period t over the possible values $k_i' \in K'$ can be computed as:

$$\Pr{}_t(k' = k_i | \mathbf{v}^0) = \sum_{j=1..\#S} v_j^t \sum_{Q \times T \times Z \times W'} \pi(k' = k_i, \tau, q, z, w' | s_j)$$

We also use the state distribution, $\mathbf{v}$ and the Markov matrix, M to compute the matrix of *transition probabilities,* $\mathbf{P}_t(x, x')$ between any contract element values, x e.g. the probability that firm's size changes from k_i to k_j for any $i, j \in \{1, ..\#K\}$ at any time period (see Lehnert, Ligon and Townsend, 2003). Assuming a large number of agents we can interpret, for instance, $\mathbf{H}_t(k_j')$ as the fraction of firms observed with next period capital stock equal to $k_j' \in K'$ at time t and correspondingly, $\mathbf{P}_t(k_i', k_j')$ as the number of firms transitioning from capital stock k_i to k_j. Using cross-sectional or panel (transitions) data on any observable dimension of the model we can then estimate the various regimes and test whether we can statistically distinguish across them (see more on this in the next section).

Due to space constraints we are able to present visually in the figures and discuss in the text only a relatively small fraction of the whole set of results we have computed. Admittedly, because of the computational nature of the paper, it is impossible to exhaust all possibilities and we are well aware of the limitations of this approach and the fact that the computed examples we present do not constitute proofs. We should stress, however, that we have tried our best to check robustness of the reported results by using numerous parameter specifications and initial conditions. The full set of numerical computations is readily available from the authors.

5.2 Matching Stylized Facts on Firm Growth and Finance

In this section we compare the implications of the information regimes we study with several stylized empirical facts and regularities about firm size, growth and finance. The list of facts presented below is taken from Cooley and Quadrini (1999) who find the following empirical regularities of firm dynamics and financial behavior[25]:

      Fact 1 - Firm growth decreases with firm age and size

      Fact 2 - The variability of firm growth decreases with firm age and size

      Fact 3 - Small firms invest more.

      Fact 4 - Small firms take on more debt.

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[24]The initial distribution may come from suitable data on k, b if available or can be parametrized and estimated. The promise variable w could also be mapped out from the expected consumption policy function which is monotonic in w (see fig. 7).

[25]We omit some of the reported regularities which our model currently does not allow to match or verify.

Fact 5 - The investment of small firms is more sensitive to cash flows even after controlling for their future profitability.

In order to compare the predictions of our model with the above facts[26] we map the model variables into their real world counterparts. We interpret agent's capital, k as the firm size. This implies that firm growth is given by the ratio k'/k while investment is $i = k' - (1 - \delta)k$. Cash flow in our setup is naturally matched to output q. Sensitivity of investment to cash flows will be measured by the *difference* in expected investment when q is high, $E(i(q_1))$ and expected investment when output is low, $E(i(q_0))$. Notice that this is equivalent to looking at the difference $E(k'(q_1)) - E(k'(q_0))$ which is what we actually use in the reported results on fact 5. Also, note that studying firm growth k'/k as a function of firm size is the same (up to a constant) as studying relative investment i/k as a function of firm size since $i/k = k'/k - (1 - \delta)$ so Facts 1 and 3 can be looked at together for the purposes of our model. Firm growth variability is measured by the growth variance, $var(\frac{k'}{k})$.

Figures 1-4 display the behavior of the expected values[27] of the following financial indicators defined above: firm growth, sensitivity to cash flows, growth variance, and for the BL regime only, relative indebtedness, $E(b'/k)$. The results were computed for both the cases of full and incomplete depreciation ($\delta = 1$ and $\delta = 0.05$) using the policy functions and transition matrices as described above. We compute the financial variables listed above for each of the regimes and plot them as function of firm size, k.

Figure 1 shows that virtually all regimes, from autarky to full information, match *qualitatively* facts 1 and 3. This suggests that using these facts as a basis to distinguish between our alternative models of firm financing may be of limited usefulness. However, there are noticeable quantitative differences between the predicted lines for the different regimes that could be potentially used for testing the regimes against each other. First, notice that, as expected, autarky is farthest from the first best, followed by S, BL, etc. The MH regime produces lines that are very close to those for full information as well as to those of the UC regime. While all lines on fig. 1 are downward sloping they differ in their slopes with the autarky line being flattest and that for full information steepest with the rest of the regimes falling in between. Intuitively, the less constrained the regime, the higher its ability to adjust in an incentive feasible way the firm size. This is a testable implication of our model that can be used on real data.

Figure 2 shows a similar picture to fig. 1 - all regimes exhibit decreasing variance of firm size growth as a function of k matching fact 2. Once again, however, there are potentially large quantitative differences across the various financial environments with the exogenously incomplete regimes displaying markedly higher growth variability especially at lower capital levels. The intuition is that, being more constrained, these regimes have a harder time smoothing out the output fluctuations compared to the FI, UC and MH regimes where the output contingent insurance transfers from the bank help serve this role.

The empirical fact from the above list that produces distinctly different results across the regimes is the diminishing cash flow sensitivity of investment (fact 5) exhibited on fig. 3a. We find this regularity to hold to some extent for the exogenously incomplete regimes when $\delta < 1$ (see also fig.

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[26]Currently we explore only the facts with regards to firm size and not firm age. In principle we can account for firm age as well by assuming that one of the capital states (e.g. $k = 0$) is interpreted as "being out of business". This would enable us to study firm entry, aging and exit. This research is still to be completed.

[27]In these figures the expectation is taken over an underlying normal distribution of w or b over their corresponding grids in addition to over any lotteries inherent in the optimal contracts. Using uniform distribution over w and b or putting all mass at a particular value of the state variable (done as robustness checks) produces qualitatively indistiguishable results.

16a where the lines are plotted for different parametrization) but it does not hold in the MH or FI environments and also when capital depreciation is full (fig. 3b). Within the model's framework, this finding is consistent with the hypothesis that, on average, firms in the real world are more likely to behave as if facing constrained means for insurance limited to saving and/or borrowing. Comparing the regimes, notice that the *level* of cash flow sensitivity decreases with relaxing the exogenous or incentive constraints as intuition suggests. This is another testable implication of our structural model. Finally, fig. 4 shows that fact 4 is matched as well by the BL regime, namely small firms in the model do take relatively more debt as measured by the debt-asset ratio, $E(b'/k)$.

Overall, the model is shown to be successful in matching the firm growth and finance empirical regularities from the Cooley and Quadrini list that we can map into our setting. However, this success should be taken with a some caution. The finding that most of the facts are matched (at least qualitatively) by financial environments filling the whole spectrum from autarky (no access to financial markets) to complete markets suggests that perhaps this list of facts is not too demanding to model theoretically and unfortunately it seems not to provide us with much insight into what types of theoretical models of firm finance under incomplete markets one has to use to match real data. Still, there are significant quantitative differences across the regimes and in section 7 we test whether those can be used as a basis to statistically test and distinguish between the competing models of incomplete financial markets.

5.3 Solution Properties - Consumption and Investment

Figures 7a-7d depict the expected values of consumption, effort and investment for the different regimes computed at initial capital level of 0.5 for both the full and incomplete depreciation baseline parametrizations. The discussion below outlines the differences across the compared financial environments and the underlying optimal allocations with the goal of providing reference points in terms of the following empirical testing section.

Consumption Smoothing and Insurance

Compare first the degree of consumption smoothing (insurance) achieved in the different regimes (fig. 7a-b). Look at the spread in expected consumption between the high and low output states plotted as a function of b or w. We see a lot of variation in consumption in the autarky regime since the only channel available to smooth consumption is investment (k'). In the saving only regime the consumption differential is reduced at least in half with the agent employing saving as a second smoothing instrument Moving to the BL regime the maximum consumption differential is reduced even further on average as the agent can borrow (the smoothing is more limited if the agent starts with a lot of debt, i.e. high b). The consumption smoothing is almost perfect in the UC and MH regimes. The latter occurs because we found that most of incentive provision occurs through promised utility i.e. high output is rewarded and low output is punished via next period's promise, w' rather than consumption. The consumption smoothing dimension suggests that the various regimes provide household firms with varying ability to smooth output shocks across states of the world and time.

The differential degree of consumption smoothing across the studied financial regimes provides a potential instrument that can be used to empirically distinguish them in cross-sectional data on consumption, c and cash flow (output), q where we would expect to distinguish between the exogenously incomplete vs. the endogenously constrained regimes. We perform such an exercise in section 7 below.

Investment and Effort

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Now look at the graphs of next period's capital, k' and effort, z as function of the states, b or w (fig. 7c-d). Notice that under FI the principal can implement high effort and investment even at the lowest promise (which implies low c as well) but this is impossible in the UC and MH cases because of incentive reasons. Fig. 8 shows that investment is depressed in the UC and MH regimes relative to the first best due to the information asymmetry. The exogenously incomplete regimes achieve lower levels of effort and investment on average compared to the full information case although this is not so clear in the figure since there is no natural mapping between b and w. As economic intuition suggest, effort is decreasing with higher w but increasing with higher b (less savings and more debt). The next period's capital stock and hence investment (given we hold k constant along the lines) is generally (weakly) decreasing in both w and b apart at the lower bound as resources are used for either higher consumption (in the case of w) or paying back the debt (in the case of b).

5.4 Solution Properties - Welfare

Notice that the studied information regimes can be ranked in terms of Pareto efficiency. The autarky regime is clearly characterized with lowest welfare, followed by the saving only and borrowing regimes which allow for non-contingent intertemporal consumption smoothing but no additional insurance across states apart from self-insurance. In turn, if $\beta = 1/R$ as in our baseline, the moral hazard regime with unobserved capital (UC) Pareto dominates the BL regime as the optimal allocation achieved by the latter is incentive and truth-telling compatible since it can be achieved by simply allowing the agent to borrow and lend through the principal i.e. implementing the values of b through the transfer τ.

In general, given that in our model the agent has incomplete control over his income (output is stochastic) the bank can provide higher utility for the firm by providing extra insurance. The results of Allen (1985) and Cole and Kocherlakota (2001) stating that no additional insurance on top of self-insurance can be provided by the principal do not apply in our setting because of the incomplete control assumption (see Abraham and Pavoni, 2005). The moral hazard regime with observed k (MH) is in turn Pareto superior to the UC regime as the truth-telling constraints are relaxed by the observability of capital and investment. Finally, the full information regime dominates all others as it achieves the first best, complete markets allocation.

These Pareto rankings are illustrated on fig. 5 (for the incomplete depreciation baseline) and fig. 6 (for the full depreciation baseline). We plot the agent's and principal's utility levels and resulting Pareto frontiers for the studied regimes. Note that quantitatively there is a relatively large gap between the A and S exogenously incomplete regimes and the endogenous ones[28] while relatively small welfare gaps among the rest of the regimes. Even at high promised utility for the agent there are perceptible differences in the regimes which can look small on the graph because of its scale but can be significant in terms of consumption equivalents.

The increasing portion of the utility possibility frontier for the MH and UC regimes (familiar from Phelan and Townsend, 1991) is due to the fact that at very low promised utility the agent's incentives to exert effort and invest are limited which results in low value for the principal. Notice that no such feature is present under full information. The BL and S frontiers are downward sloping as higher initial savings (lower debt) with the principal (which map onto the vertical axis since the bank is risk neutral) correspond to higher agent's present value utility.

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[28]The relatively small gap between the FI and MH regimes is due to the low risk aversion level in our parametrization and widens when using higher σ.

5.5   Dynamics

In this section we present some of the dynamic features of the financial environments we study. As in the per-period results above, we focus on the behavior of consumption and investment as two dimensions of the model most likely to be present in data and also capturing well the theoretical differences across the financial regimes.

We start by looking at the evolution of the cross-sectional distributions of consumption, c and next period capital, k' over time (100 periods) for the various regimes we study under the incomplete depreciation baseline parametrization (fig. 8-11)[29]. The figures are plotted for uniform initial distribution over the states but we have verified that our discussion below is not sensitive to this. We see that in the more constrained regimes (A and S) consumption and investment tend to converge faster to what looks like a stationary distribution. Autarky seems to converge within 10 periods while S takes around 50. The less constrained regimes like BL and MH do not exhibit convergence within the first 100 periods reported on the figures.

Notice that in our simulations the A and S exogenously incomplete regimes converge quickly to non-degenerate stationary distributions. That is, while there is a lot of mobility inside the distribution by agents going from one consumption realization to another, the overall cross-sectional consumption distribution and hence its variance may be observed not to vary over time. This observation is relevant in view of the literature on testing the full vs. partial insurance hypotheses (e.g. Blundell, Pistaferri and Preston, 2002). Observing time varying variance in the consumption distribution is certainly a sufficient condition to reject the full insurance hypothesis but, as our results show, the opposite is not true: observing zero time variation in the cross-sectional consumption variance is perfectly consistent with partial insurance as well as observing non-zero time variation is.

We investigate further the dynamics generated by the different regimes by looking at the population-weighted *expected time paths* (empirical means) of c and k' starting from uniform initial state distribution. For example, to compute the time path of investment, k' we sum, for each t, over all possible values of $k' \in K'$ weighted by the corresponding fraction of agents with next period capital at this value:

$$E_t(k') = \sum_{j=1..\#K'} k'_j \mathbf{H}_t(k'_j)$$

The results are exhibited on fig. 12 and 13 for the incomplete and full depreciation baselines. All regimes were initialized at uniform initial distribution over the states. Once again we see the paths for the highly constrained autarky (A) and saving only (S) regimes converging relatively fast while the rest of the regimes still exhibit drift in the empirical mean after 100 periods confirming our results from the cross-sectional distributions (fig. 8-11).

We use the above findings as yet another possible basis to distinguish between the financial environments we study. In section 7 we take two cross-sectional distributions of consumption (conditional on output) lying either one period apart (i.e. corresponding to short run dynamics) or 50 periods apart (corresponding to longer run dynamics). The simulations results from fig. 8-13 suggest that the ability to distinguish across the regimes should differ with the length of the time period from which data is drawn.

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[29]To save space we omit the FI regime where the distribution is stationary after time 0 at our baseline $\beta = 1/R$ and looks similar to the MH results.

6 Empirical Implementation

In this section we use our solution methodology to generate simulated data and perform estimation and model selection tests of the different information regimes.

6.1 Methodological Issues

Initial Conditions

When estimating dynamic models an important issue is initial conditions. In terms of our model these are the initial values for the state variables, some of which can be unobservable to the econometrician. Specifically to our setting, to initialize the model and generate transitional dynamics we need to know (or estimate) the initial values of the state variables (capital, k for autarky, (k, b) for savings or borrowing, and (k, w) for the endogenous regimes). While k or b are likely to be observable so they can be taken from the data, the initial promise values, w being a purely mathematical object summarizing history, are clearly not observable. Thus, to initialize the model they can be estimated as if drawn from some known distribution. In section 7 below we adopt this approach and show how to estimate the parameters of the distribution. Notice, however, that if we are interested in the long-run distributions or transitions generated by the information regimes we do not need to estimate the initial distribution over states but only the preference, technology, etc. parameters.

The possibility of unobservable initial conditions is related to the more general issue of heterogeneity in unobservables. As written, our structural models have exact predictions about the distributions and transition probabilities for consumption, investment, cashflow, etc. at any point of time. However, we are well-aware that these models are at best crude, simplified versions of reality and stand no chance of perfectly matching any real dataset. Hence, to fit real data we must either introduce a source of heterogeneity in unobservables into the model, e.g. heterogeneity in the initial promises and/or treat the data as if there is measurement error in the observables. For example, we can think of initial promises as originating from a known parametric distribution (properly discretized and normalized to fit our grid) the parameters of which we will structurally estimate together with the rest of the model parameters. Estimation strategies based both on unobserved heterogeneity and measurement error are presented below.

Identification

Unfortunately, due to the analytical complexity of our setting, we are unable to provide theoretical identification proofs. In fact, as Honore and Tamer (2005) find, point identification could fail in structural models like ours. We are well aware of this limitation of our numerical approach. To address it we propose a "second best" solution in the form of *numerical identification*. Specifically, what we do is as follows:

    (1) take a baseline model regime parametrized by a vector of parameters, ϕ^{base}

    (2) generate simulated data from the baseline model from (1)

    (3) estimate the baseline model using the data in (2) using maximum likelihood and obtain estimates, $\hat{\phi}^{base}$

    (4) if the estimates from (3) are the same or close in numerical sense (e.g. norm) to the baseline ϕ^{base}, report success, else report failure.

Steady States vs. Transitions

All of the model regimes described in section 3 have implications for both transitional dynamics and long-run distributions. This needs to be reflected in the estimation strategy. If a stationary distribution exists and is independent of initial conditions and if we believe that the actual data

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reflects a steady state we can estimate the models to match the simulated stationary distribution with the actual empirical distribution disregarding transitional dynamics and initial conditions. If, however, we are dealing with actual data that is more likely to correspond to a transition process (as would be natural to assume if the data comes from a developing economy) then the model regimes need to be estimated using both their cross-sectional and intertemporal implications. We ficus on the latter case which we find both empirically more plausible in our developing economy setting and also technically more challenging.

6.2 Maximum Likelihood Estimation

In Paulson, Townsend and Karaivanov (2006) we have successfully employed maximum likelihood estimation (MLE) in a static environment of occupational choice under financial constraints. In the current dynamic context we use MLE to fit data on cross-sectional frequencies of c, q, k and/or state transition probabilities (e.g. k to k' or k to k' conditional on q) with data simulated from the studied regimes[30]. The baseline "data" used in the estimation results in this paper (see next section) is drawn from one of our models (moral hazard) but in future research we plan to apply the methods developed here to real data from one of the datasets mentioned in the introduction.

More specifically, denote the data used in the MLE (e.g., the joint empirical distribution of c, q) by $\hat{m}_{ij}$ where $\sum_{j=1}^{J} \hat{m}_{ij} = 1$ for all $i = 1..I$. Here, the subscript j refers to data frequencies in mutually exclusive cells (e.g. if there are $\#C$ grid points[31] for c and $\#Q$ grid points for q we have $J = \#C \times \#Q$) while the subscript i refers to different *sets* of data frequencies (e.g. $i = 1, ..\#K$ could denote the frequency distributions of going from a given firm size k_i today to firm size k'_j, $j = 1, ..\#K$ tomorrow).

Suppose our structural model is parametrized by the parameter vector ϕ that we are interested in estimating. The elements of ϕ can include preference or technology parameters, or distributional parameters (e.g. mean, variance) for initial promises or measurement error (see section 7.1). Let the counterparts of the data $\hat{m}_{ij}$ in the estimated model be $m_{ij}(\phi)$, where once again, $\sum_{j=1}^{J} m_{ij}(\phi) = 1$ for all i.

The fact that we have frequencies over data cells as our objects of interest allows us to write the log-likelihood function in explicit form. The maximum likelihood estimator, ϕ^{MLE} is given by:

$$\hat{\phi}^{MLE} = \arg \max_{\phi} \ n \sum_{i=1}^{I} \left[\sum_{j=1}^{J-1} \hat{m}_{ij} \ln m_{ij}(\phi) + (1 - \sum_{j=1}^{J-1} \hat{m}_{ij}) \ln(1 - \sum_{j=1}^{J-1} m_{ij}(\phi)) \right] \quad (23)$$

where n is the overall sample size. The minimization above can be done by any optimization algorithm robust to local minima, e.g. pattern search, genetic algorithms, or simulated annealing (Goffe, Ferrier and Rogers, 1994).

6.3 Testing and Distinguishing Between Competing Models

The estimation method proposed above can be used to construct a formal statistical test to evaluate the relative goodness of fit of our competing information regimes relative to the baseline data. It

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[30]In general, the MLE approach proposed here can be used on any data in the form of probabilities/ fractions - that is any one-time or repeated cross-sectional joint distibutions of model variables or transitional probabilities. Any real data can be put in this form by sorting the observations in appropriately chosen "bins" and then using the frequencies over those bins to perform the MLE.

[31]Notice that the requirement to use grids in our linear programming computational algorithm turns out to be well-suited for our estimation approach.

is crucial to point out that the different models we study are non-nested in statistical sense. For our purposes we say that model A nests model B, if, for any possible allocation that can arise in model B, there exist parameter values such that this is the allocation in model A (see Paulson et al., 2006 for more details).

We use the results of Vuong (1989) to construct asymptotic test statistics to distinguish between competing non-nested models. Vuong (1989) proposes a test based on the maximum likelihood method described above. A very attractive feature of this test is that it does not require that either of the compared models be correctly specified. This property is appealing given the fact that we are studying models that are clearly simpler than reality[32].

Suppose the values of the estimation criterion function being minimized (i.e. minus the log-likelihood) for the two models are given by $L_n^1(\hat{\phi}^1)$ and $L_n^2(\hat{\phi}^2)$ where n is the sample size and $\hat{\phi}^1$ and $\hat{\phi}^2$ are the parameter estimates for two competing non-nested models (model 1 and model 2 respectively). The null hypothesis, H_0 of the Vuong test is that the two models are "asymptotically equivalent" relative to the true data generating process. Define the "difference in lack-of fit" statistic:

$$T_n = n^{-1/2} \frac{L_n^1(\hat{\phi}^1) - L_n^2(\hat{\phi}^2)}{\hat{\sigma}_n}$$

where $\hat{\sigma}_n$ is a consistent estimate of the asymptotic variance[33], σ_n of $L_n^1(\hat{\phi}^1) - L_n^2(\hat{\phi}^2)$ (the likelihood ratio). The main result of Vuong (1989) is:

Proposition E1 (Vuong, 1989)

Under some regularity assumptions and if the models are non-nested, then T_n is distributed $N(0, 1)$ under H_0.

7   Estimation and Model Selection Results

In this section we use the MLE and testing methods from the previous section to test across the competing theoretical regimes on basis of the dimensions suggested by the simulation results: (1) firm growth and finance; (2) consumption smoothing and (3) short vs. long run dynamics. We emphasize that the results exhibited below should be treated just as examples illustrating the way to use the proposed estimation and testing methodology and are not calibrated (for the moment) to fit real data.

We take as a baseline the implications of the moral hazard model with observed investment from section 3.2. (MH). The first step in our empirical strategy is to generate "data" from the model that we will use in the estimation. Remember that in our MLE methodology we need to use probabilities over cells as a basis for the estimation[34]. Thus, regarding firm dynamics (1), we generate data on the joint distribution of (k, k') and (k, k', q). The idea is that using the joint distributions of capital today and tomorrow (conditional or unconditional on output) should help

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[32]A further advantage of this test (not utilized here but very useful for numerical work) is that the method used to estimate the competing models need not optimize the selection criteria used for model selection (see Rivers and Vuong, 2002).

[33]In the case of MLE, a consistent estimate of σ_n is given by the sample analogue of the variance of the LR statistic (see Vuong, 1989, p. 314).

[34]In principle we could employ a GMM or minimum χ^2 techniques to estimate on the basis of any moments on the data instead of proxying them with frequency distributions but then testing across the non-nested regimes becomes computationally infeasible (see Rivers and Vuong, 2002), so we choose to stick to the (admittedly somewhat limited) MLE approach.

us distinguish the regimes based on the stylized facts from section 5.1. which involve only k, k' (and in the case of fact 5, q). Similarly, to test the consumption smoothing dimension (2) as a basis of distinguishing between the regimes, we use the cross-sectional joint distribution of (c, q). Finally, our results above imply that the endogenous regimes have different implications about consumption and investment dynamics compared to the exogenous regimes. We test this dimension based on data on the (c, q) cross-sectional distribution in two consecutive periods (for short run behavior) versus (c, q) cross-sections 50 periods apart (for longer run behavior).

7.1 Generating Baseline Data

We use the two baseline parametrizations exhibited in table 3. However, because of the high computational time requirements of the MLE procedure (we need to compute and iterate on the linear programs at each parameter vector during the grid search and estimation) we reduce some of the grid sizes to $\#K = 5$, $\#T = 19$, $\#W = 5$, $\#B = 5$ (for S) and $\#B = 9$ (for BL)[35]. Consumption is also gridded up, using $\#C = 10$ values on $[0, 1.2]$ or $[0, 3.5]$ respectively for the $\delta = .05$ and $\delta = 1$ cases.

To initialize the baseline MH model from which we draw the data used in the estimations, we assume that the initial distribution over states (k, w) is uniform in firm size and normal in w, i.e. $w\tilde{}N(\mu_w, \gamma_w^2)$ for each value of k. To generate the baseline data we pick μ_w equal to the average of the grid W i.e. we set the baseline distributional parameters as $(\mu_w, \gamma_w) = (19.99, 8)$ for the case of incomplete depreciation and $(34.64, 15)$ for the case $\delta = 1$. We then draw $n = 1,000$ random observations from $N(\mu_w, \gamma_w^2)$ (200 for each k) and generate the distribution over the initial state space, $D_0(k, w)$. We then compute the MH regime at the baseline parameter values and use its Markov transition matrix to simulate n values for c, q, k'.

At this stage we also allow for additive measurement error in the observables - consumption c and/or firm size k and k'. Output, q is assumed to be observed without error since it can take only 2 values. The measurement errors ε_i, $i = 1, ..n$ applied to all observables are assumed to be drawn from the normal distribution $N(0, \gamma_{me}^2)$ where the standard deviation, γ_{me} is estimated. For example, if the true value of consumption (i.e. the value simulated from the MH model at the initial state (k_i, w_i) is c_i, the "observed" data value we use in the estimation is $\tilde{c}_i = c_i + \varepsilon_i$ and similarly for k, k'[36]. Finally, we use those values to create the distributions used in the estimation, i.e. $H_0(\tilde{c}, q)$ - the observed distribution over (c, q), $H_0(\tilde{k}, \tilde{k}')$ - the observed distribution of transitions from firm size k at $t = 0$ to size k' at $t = 1$, etc.

7.2 Results

This section contains the results from the estimation and model selection tests of the studied regimes using the methods outlined above. The parameters we estimate are the three distributional parameters for promises and measurement error (μ_w, γ_w and γ_{me}) as well as three structural parameters: the preference curvature parameters, σ and θ and the technology parameter, ρ. In the case of the saving only and borrowing regimes instead of the promise distribution parameters we estimate the mean, μ_b and variance, γ_b^2 of the distribution of the state variable b (assumed unobserved but drawn from a normal distribution). We perform all estimation and testing exercises for two cases of measurement error (ME) used to generate the baseline data ("low ME" i.e. $\gamma_{me} = .1$ or "high

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[35]We can handle much larger problems at the cost of additional computational time.

[36]Applying the measurement error can in some times lead to "truncation" if the resulting value falls outside our grid so we lose some information in the estimation procedure.

ME" i.e. $\gamma_{me} = .5$). As we will see, the magnitude of measurement error affects significantly our ability to distinguish between the studied information regimes.

For each information regime we start at some initial parameter vector and then follow the same procedure described above to simulate data from the model: draw initial states, generate the observables, apply measurement error, generate the frequency distributions of interest. We then use the likelihood function in (23) to compute the criterion value and use an optimization routine[37] to solve for the estimates $\hat{\phi}$ maximizing the likelihood between the baseline data and the currently estimated model regime. The first of the estimated regimes is always the MH regime itself to see if we can recover the actual parameters used to generate the data (see our "numerical identification" discussion above). Finally, we use the Vuong test to see if we can distinguish between the data generating and the estimated model regimes as well as between all possible pairs of estimated regimes.

7.2.1 Firm Growth and Finance - Incomplete Depreciation

In this first set of results, we focus on estimation and testing the implications of the models related to firm growth and financing. That is, the data we generate and use to estimate the model consist of the distributions of capital today and tomorrow, k, k' potentially conditional on cashflow, q, i.e. the joint distributions (k, k') and (k, k', q) in the initial period as explained in the discussion at the beginning of section 6. The tables below contain the estimation and model selection results. We first report the case of incomplete depreciation ($\delta = 0.05$) where the regimes compared are MH, FI, A, S and BL and then the case of complete depreciation where the moral hazard with unobserved capital regime (UC) is also included.

Table 2a - Estimation results based on the (k,k') distribution (incomplete depreciation)

Model	$\hat{\mu}_w$	$\hat{\gamma}_w$	$\hat{\gamma}_{me}$	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$	LL Value
Low Measurement Error							
MH ("identification")	20.0625	7.9375	0.1020	0.5000	2.0625	-0.0625	**-2727.4**
FI	20.0000	9.0000	0.1050	0.5000	2.1875	0.0000	**-2749.3**
Autarky (A)	n.a.	n.a.	0.5948	0.0000	0.0433	3.0050	**-3029.1**
Saving Only (S)	-0.4000$^{\#}$	0.1917$^{\#}$	0.6410	0.0000	0.0766	3.1875	**-2977.4**
BL	-0.0024$^{\#}$	0.2581$^{\#}$	0.4480	1.1000	4.2000	2.8750	**-2947.4**
baseline values	*19.999*	*8*	*0.1*	*0.5*	*2*	*0*	
High Measurement Error							
MH ("identification")	20.0000	8.0000	0.6182	0.5005	2.2500	-0.1250	**-3114.9**
FI	20.0625	7.9375	0.6289	0.5000	1.9844	0.2188	**-3115.7**
Autarky (A)	n.a.	n.a.	0.6538	0.5035	0.8487	3.4661	**-3143.8**
Saving Only (S)	-0.1595$^{\#}$	0.0926$^{\#}$	0.6566	0.0777	0.1014	2.7349	**-3133.4**
BL	-0.1501$^{\#}$	0.2000$^{\#}$	0.6268	0.9904	1.8107	-2.9959	**-3117.9**
baseline values	*19.999*	*8*	*0.5*	*0.5*	*2*	*0*	

Note: # - for the S and BL regimes the first two columns report $\hat{\mu}_b$ and $\hat{\gamma}_b$. This applies to all tables below.

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[37]We first perform a grid search over the parameter values to prevent getting stuck at local extrema and then use the Matlab routines `patternsearch` and `fminsearch` to optimize the likelihood.

Table 2b - Vuong Test p-values based on the (k,k') distribution

$\gamma_{me}{=}.5\Downarrow \setminus \gamma_{me}{=}.1\Rightarrow$	MH	FI	A	S	BL
MH	n.a.	.0022***(MH)	.0000***(MH)	.0000***(MH)	.0000***(MH)
FI	.7800 (draw)	n.a.	.0000***(FI)	.0000***(FI)	.0000***(FI)
A	.0005***(MH)	.0007***(FI)	n.a.	.0000***(S)	.0000***(BL)
S	.0126**(MH)	.0097***(FI)	.1321 (draw)	n.a.	.0017***(BL)
BL	.5049 (draw)	.6133 (draw)	.0031***(BL)	.0197** (BL)	n.a.

Note: The abbreviations in the parentheses denote the better fitting regime. *** - significant at 1%; ** - significant at 5%; * - significant at 10%; "draw" denotes the tested regimes cannot be statistically distinguished from each other relative to the data. The p-values in the part of the table above the main diagonal read by rows correspond to the case $\gamma_{me} = .1$ while those in the part of the table below the main diagonal read by columns correspond to the case $\gamma_{me} = .5$[38]. The same comments apply to all similar tables below.

We see from table 2a that the MH baseline parameters are identified quite well in the low measurement error (ME) case validating our numerical procedure although, as expected, not so well in the high ME case. In terms of the log-likelihood values the regimes follow a natural ordering: the MH as the data generating one is highest, followed by the FI, then BL, S and finally A. Interestingly, the same ordering is preserved using other types of data e.g. the (c,q) distribution, although with high measurement error the likelihood values sometimes get very close to each other. The parameter estimates differ across the estimated regimes as the MLE is trying to fit the data as well as possible but they are generally similar between the FI and MH regimes which appear to be "close" to each other in terms of their investment implications. In terms of testing, we find that the low ME specification distinguishes among all regimes on basis of (k,k') very strongly, while with higher measurement error we cannot distinguish between MH, FI and BL (and separately, between A and S). This suggests that in terms of firm size and growth implications (facts 1-3 from Cooley and Quadrini, 1999) the moral hazard regime seems to be close to the full information or borrowing and lending but distinguishable in data from the saving only or autarky models.

Below we re-do the estimation and testing exercises using the additional information on cashflow, q to try address stylized fact 5 from our list.

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[38]For example, the value in the row labeled "FI" and column "A" (above the main diagonal) is the p-value of comparing the full information regime with autarky for $\gamma_{me} = .1$ while the value in the row labeled "A" and column "FI" (below the diagonal) is the p-value of comparing autarky to full information under $\gamma_{me} = .5$. Each comparison is bilateral and symmetric which is why every pair of regimes is tested only once.

Table 3a - Estimation results based on the (k,q,k') distribution (incomplete depreciation)

Model	$\hat{\mu}_w$	$\hat{\gamma}_w$	$\hat{\gamma}_{me}$	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$	LL Value
			Low Measurement Error				
MH ("identification")	20.1875	8.2500	0.1000	0.5000	2.0313	0.0000	**-3165.1**
FI	19.9766	8.7188	0.1366	0.5039	1.9961	-0.0547	**-3221.4**
Autarky	n.a.	n.a.	0.7039	0.1000	8.1000	0.2188	**-3617.2**
Saving Only	-0.4000#	0.2000#	0.6374	0.1313	1.2000	2.3984	**-3480.3**
BL	-0.1521#	0.2161#	0.5276	0.6000	1.3750	-0.9063	**-3441.7**
baseline values	*19.9999*	*8*	*0.1*	*0.5*	*2*	*0*	
			High Measurement Error				
MH ("identification")	19.0000	9.0000	0.6414	0.5000	2.0156	-0.0469	**-3581.3**
FI	19.9688	8.0625	0.5684	0.5000	1.9844	-0.0625	**-3595.2**
Autarky	n.a.	n.a.	0.6845	1.1125	1.4917	-3.4563	**-3741.2**
Saving Only	-0.1651#	0.0000#	0.6619	0.4537	2.0753	-2.7498	**-3679.6**
BL	0.0000#	0.1377#	0.6559	0.5000	2.0000	-3.8203	**-3630.3**
baseline values	*19.9999*	*8*	*0.5*	*0.5*	*2*	*0*	

Table 3b - Vuong Test p-values based on the (k,q,k') distribution

γ_w=.5⇓ \ γ_w=.1⇒	**MH**	**FI**	**A**	**S**	**BL**
MH	n.a.	.0000***(MH)	.0000***(MH)	.0000***(MH)	.0000***(MH)
FI	.2392 (draw)	n.a.	.0000***(FI)	.0000***(FI)	.0000***(FI)
A	.0000***(MH)	.0000***(FI)	n.a.	.0000***(S)	.0000***(BL)
S	.0000***(MH)	.0000***(FI)	.0005***(S)	n.a.	.0854* (BL)
BL	.0000***(MH)	.0138** (FI)	.0000***(BL)	.0015***(BL)	n.a.

The above results are qualitatively similar in terms of likelihood ordering and estimated to those in tables 2a-b. Notice the worse likelihood values as we now use more information to estimate (more probability terms in (23)). Compared to the (k, k')-based results, when including more information (q) the Vuong test statistics become stronger and we can distinguish most regimes but we still cannot distinguish MH from FI when ME is high.

We visualize the above results on fig. 14-16 which plot firm size growth, growth variance and cash flow sensitivity as functions of firm size at the parameter estimates reported above for both the low and high measurement error parametrizations. We see that the MH and FI regimes are very close in terms of their financial implications and also how the higher measurement error blurs the distinction between the regimes.

7.2.2 Firm Size and Finance - Full Depreciation

We now look at the case of full depreciation which allows us to compute, estimate and test the UC regime together with the others. Because of computational speed this is not feasible for the more realistic incomplete depreciation scenario discussed above.

Table 4a - Estimation results based on the (k,k') distribution ($\delta = 1$ case)

Model	$\hat{\mu}_w$	$\hat{\gamma}_w$	$\hat{\gamma}_m$	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$	LL Value
Low Measurement Error ($\gamma_m = .1$)							
MH ("identification")	34.8900	15.0156	0.0969	0.5000	2.0000	0.0938	**-2526.4**
FI	17.1875	30.0000	0.1176	0.5000	2.0000	-2.0000	**-2554.9**
Autarky	n.a.	n.a.	0.1961	1.1000	0.7000	4.9375	**-2601.6**
Saving Only	0.0000#	0.2500#	0.2300	1.1000	1.2000	5.1250	**-2591.4**
BL	0.3594#	0.8781#	0.2361	1.6000	3.7500	7.0000	**-2583.3**
Unobserved k (UC)	n.a.	n.a.	0.1088	0.5156	1.7578	-2.8438	**-2542.5**
baseline values	*34.6401*	*15*	*0.1*	*0.5*	*2*	*0*	
High Measurement Error ($\gamma_m = .5$)							
MH ("identification")	30.6153	11.8672	0.6169	0.5017	2.1137	-0.0312	**-2982.8**
FI	33.2247	15.8548	0.6208	0.5171	1.2070	0.4999	**-2982.6**
Autarky	n.a.	n.a.	0.6167	1.1000	0.7000	10.000	**-2990.0**
Saving Only	-0.5317#	0.1855#	0.6182	2.8500	9.2000	9.6875	**-2983.3**
BL	0.4683#	0.1855#	0.6182	2.3500	9.2000	9.6563	**-2983.8**
Unobserved k (UC)	n.a.	n.a.	0.6224	0.1352	5.0156	-5.5938	**-2982.3**
baseline values	*34.6401*	*15*	*0.5*	*0.5*	*2*	*0*	

Table 4b - Vuong Test p-values based on the (k,k') distribution ($\delta = 1$ case)

$\gamma_m=.5\Downarrow \setminus \gamma_m=.1\Rightarrow$	**MH**	**FI**	**A**	**S**	**BL**	**UC**
MH	n.a.	.000***(MH)	.000***(MH)	.000***(MH)	.000***(MH)	.011**(MH)
FI	.972 (draw)	n.a.	.000***(FI)	.000***(FI)	.002***(FI)	.062* (UC)
A	.149 (draw)	.028**(FI)	n.a.	.035**(S)	.002***(BL)	.000***(UC)
S	.918 (draw)	.809 (draw)	.061*(S)	n.a.	.101 (draw)	.000***(UC)
BL	.832 (draw)	.660 (draw)	.064*(BL)	.564 (draw)	n.a.	.000***(UC)
UC	.911 (draw)	.935 (draw)	.142 (draw)	.827 (draw)	.734 (draw)	n.a.

As we see from table 4a, the unobserved capital regime (UC) fits in terms of likelihood very close to the MH regime, and close to the full information regime. The likelihood ordering of the remaining regimes is the same as in the $\delta = 0.05$ case. The estimates it produces are also similar to the baseline values used to generating the data indicating that it may be difficult in practice to distinguish between the moral hazard regimes with observed and unobserved k. This is confirmed to some extent in table 4b where we see that, for low measurement error, we can distinguish but only at 5% or 10% between MH and UC and UC and FI respectively. For high measurement error it seems that the distinction between the regimes is so blurred that we can only distinguish autarky from the other exogenously incomplete regimes.

Table 5a - Estimation results based on the (k,q,k') distribution ($\delta = 1$ case)

Model	$\hat{\mu}_w$	$\hat{\gamma}_w$	$\hat{\gamma}_m$	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$	LL Value
			Low Measurement Error ($\gamma_m = .1$)				
MH ("identification")	36.7338	15.0000	0.1039	0.5000	2.0234	-0.0313	**-2976.3**
FI	34.1713	17.7266	0.1937	0.5068	2.5859	-0.0313	**-3059.4**
Autarky	n.a.	n.a.	0.9031	0.1000	10.000	0.7344	**-3868.4**
Saving Only	-1.0000$^\#$	0.2873$^\#$	0.4402	0.0375	6.0000	0.2402	**-3184.6**
BL	-1.0000$^\#$	0.4367$^\#$	0.5517	0.9750	4.0000	0.0000	**-3165.3**
Unobserved k	n.a.	n.a.	0.1020	0.5469	4.1375	-0.2813	**-3018.1**
baseline values	*34.6401*	*15*	*0.1*	*0.5*	*2*	*0*	
			High Measurement Error ($\gamma_m = .5$)				
MH ("identification")	34.6400	14.5000	0.5898	0.5000	2.2188	0.3125	**-3514.9**
FI	34.6400	14.9688	0.6168	0.5000	2.4375	0.0000	**-3522.9**
Autarky	n.a.	n.a.	0.8521	0.1000	10.000	0.7344	**-3952.3**
Saving Only	-1.0000$^\#$	0.0250$^\#$	0.6550	0.4750	8.1000	0.2344	**-3530.4**
BL	-1.0000$^\#$	0.2414$^\#$	0.6335	0.2188	10.000	-0.3750	**-3528.4**
Unobserved k	n.a.	n.a.	0.6224	0.1352	5.0156	-5.5938	**-3528.7**
baseline values	*34.6401*	*15*	*0.5*	*0.5*	*2*	*0*	

Table 5b - Vuong Test p-values based on the (k,q,k') distribution ($\delta = 1$ case)

$\gamma_m=.5\Downarrow \backslash \gamma_m=.1\Rightarrow$	MH	FI	A	S	BL	UC
MH	n.a.	.000***(MH)	.000***(MH)	.000***(MH)	.000***(MH)	.000***(MH)
FI	.168 (draw)	n.a.	.000***(FI)	.000***(FI)	.000***(FI)	.025** (UC)
A	.000***(MH)	.000***(FI)	n.a.	.000***(S)	.000***(BL)	.000***(UC)
S	.199 (draw)	.475 (draw)	.000***(S)	n.a.	.075* (BL)	.000***(UC)
BL	.173 (draw)	.592 (draw)	.000***(BL)	.885 (draw)	n.a.	.000***(UC)
UC	.156 (draw)	.596 (draw)	.000***(UC)	.906 (draw)	.980 (draw)	n.a.

Once again, adding more data based on which to estimate and test the regimes strengthens the Vuong test results - notice that now we distinguish MH from UC at the 1% level for the low γ_m case. With higher measurement error we still cannot distinguish between the data generating regime (MH) and the other regimes with the exception of autarky. This is in contrast to our findings for the incomplete depreciation case. Intuitively, the full depreciation breaks the link between firm size today and tomorrow, k and k' thus effectively less information is available to base the tests on relative to before.

7.2.3 Consumption Smoothing

We now estimate and try to distinguish between the studied regimes based on data on the degree of consumption smoothing they provide to the households as embedded in the (c, q) cross-sectional distribution at the end of the initial period. The results reported below are based on the incomplete depreciation parametrization.

Table 6a - Estimation Results based on the (c,q) distribution (incomplete depreciation)

Model	$\hat{\mu}_w$	$\hat{\gamma}_w$	$\hat{\gamma}_{me}$	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$	LL Value
			Low Measurement Error				
MH ("identification")	20.1250	8.0625	0.1029	0.5000	1.9844	-0.0002	**-2540.8**
FI	19.9688	7.8438	0.1234	0.5000	1.9893	-5.0313	**-2580.0**
Autarky	n.a.	n.a.	0.2303	0.5104	1.6691	-4.8840	**-2728.4**
Saving Only	-0.3685$^{\#}$	0.3738$^{\#}$	0.2449	0.1625	2.1000	-0.0156	**-2597.6**
BL	-0.3725$^{\#}$	0.3919$^{\#}$	0.2450	0.1625	0.1000	-0.0156	**-2594.1**
baseline values	*19.999*	*8*	*0.1*	*0.5*	*2*	*0*	
			High Measurement Error				
MH ("identification")	20.0000	7.8594	0.4917	0.5000	2.0313	-2.2500	**-2749.5**
FI	20.0000	6.5313	0.5400	0.5039	2.0000	-0.9063	**-2756.1**
Autarky	n.a.	n.a.	0.6182	0.4994	1.9265	-4.5210	**-2795.6**
Saving Only	-0.4000$^{\#}$	0.0000$^{\#}$	0.6235	1.1000	2.0000	-10.0000	**-2772.8**
BL	-0.1500$^{\#}$	0.0000$^{\#}$	0.5963	0.5000	2.0000	-7.0000	**-2762.4**
baseline values	*19.999*	*8*	*0.5*	*0.5*	*2*	*0*	

Table 6b - Vuong Test p-values based on the (c,q) distribution

$\gamma_w{=}.5\Downarrow \setminus \gamma_w{=}.1\Rightarrow$	MH	FI	A	S	BL
MH	n.a.	.0004***(MH)	.0000***(MH)	.0000***(MH)	.0000***(MH)
FI	.1114 (draw)	n.a.	.0000***(FI)	.2105 (draw)	.3148 (draw)
A	.0000***(MH)	.0000***(FI)	n.a.	.0001***(S)	.0000***(BL)
S	.0009***(MH)	.0298** (FI)	.0002***(S)	n.a.	.3468 (draw)
BL	.0140** (MH)	.3037 (draw)	.0000***(BL)	.0247** (BL)	n.a.

As in the results based on firm growth and finance (k, k', q) the likelihood values are ordered MH, FI, BL, S and A from the highest to the lowest indicating that the relative closeness of the regimes to the data generating MH model is robust and not influenced by the particular choice of data to match upon. In terms of parameter estimates we observe similar patterns as before with the baseline parameters recovered very well with low ME and slightly worse with $\gamma_{me} = .5$. The FI and MH regimes produce estimates close to each other. The exogenously incomplete regimes seem to require a higher value for the measurement error variance (higher $\hat{\gamma}_{me}$) than the baseline to fit the data best.

In terms of distinguishing between regimes, the low measurement error results allow the baseline MH regime to be distinguished with high accuracy from all other alternatives based on its consumption smoothing implications but this is not the case for the high ME specification where we cannot distinguish between the moral hazard and the full information regimes. The FI regime also seems to be indistinguishable from the borrowing and lending one in both measurement error specifications and the saving only regime in the low ME case. These results are generally weaker in terms of ability to distinguish than their counterparts when using the (k, k') or (k, k', q) distribution data. Part of the reason is the less data frequencies being fitted (19 in the (c, q)-based results vs. 24 in the (k, k')-based results and 49 in the (k, q, k') case). Once again however, autarky is strongly distinguished from all other regimes even if γ_{me} is high.

7.2.4 Short vs. Longer Run Dynamics

In the following set of results we estimate and test between the regimes on basis of their implications about shorter and longer run behavior of consumption and cashflow over time. Specifically, we now use two cross-sections of (c, q) as the frequency data on basis of which we perform the estimation (i.e. $2(\#Q \times \#C - 1) = 38$ frequencies). In tables 7a-b below the cross-sections are taken at times $t = 0$ and $t = 1$ (i.e. two neighboring periods, corresponding to short run dynamics) while in tables 8a-b the (c, q) cross-sections are taken at $t = 0$ and $t = 50$ corresponding to longer run dynamics. All results below apply to the incomplete depreciation parametrization.

Table 7a - Estimation results based on the (c,q) distributions at $t = 0$ and $t = 1$

Model	$\hat{\mu}_w$	$\hat{\gamma}_w$	$\hat{\gamma}_{me}$	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$	LL Value
Low Measurement Error							
MH ("identification")	20.0000	8.0938	0.1005	0.5000	1.9961	-0.0488	**-5060.9**
FI	20.0000	4.9688	0.2071	0.5000	2.1230	-4.5000	**-5172.1**
Autarky	n.a.	n.a.	0.2561	0.4009	1.8137	-4.5084	**-5636.2**
Saving Only	-0.3980[#]	0.0762[#]	0.3250	1.2500	4.0000	-6.2813	**-5474.1**
BL	-0.3707[#]	0.1103[#]	0.2754	0.6875	2.0000	-7.1563	**-5378.5**
baseline values	*19.9999*	*8*	*0.1*	*0.5*	*2*	*0*	
High Measurement Error							
MH ("identification")	20.0156	8.0938	0.4921	0.5000	1.9824	-0.5000	**-5577.7**
FI	22.0000	2.5469	0.5352	0.5000	2.0898	-4.6797	**-5588.1**
Autarky	n.a.	n.a.	0.6000	0.4766	2.0151	-4.5775	**-5689.5**
Saving Only	-0.1762[#]	0.1966[#]	0.6394	0.4418	1.8207	-3.0389	**-5657.0**
BL	-0.2856[#]	0.0000[#]	0.5754	0.4848	1.6457	-3.3044	**-5616.9**
baseline values	*19.9999*	*8*	*0.5*	*0.5*	*2*	*0*	

Vuong test p-values (c,q at $t = 0$ and $t = 1$)

$\gamma_w{=}.5\Downarrow \setminus \gamma_w{=}.1\Rightarrow$	MH	FI	A	S	BL
MH	n.a.	.0000***(MH)	.0000***(MH)	.0000***(MH)	.0000***(MH)
FI	.0774* (MH)	n.a.	.0000***(FI)	.0000***(FI)	.0000***(FI)
A	.0000***(MH)	.0000***(FI)	n.a.	.0000***(S)	.0000***(BL)
S	.0000***(MH)	.0000***(FI)	.0000***(S)	n.a.	.0000***(BL)
BL	.0000***(MH)	.0032***(FI)	.0000***(BL)	.0000***(BL)	n.a.

As we found above, the regimes different from the data generating baseline (MH) seem to require larger measurement error (the estimate $\hat{\gamma}_{me}$ is higher by about 0.1-0.15) in order to fit the data. We find that adding the time dimension strengthens the ability to distinguish between any pair of regimes relative to the single (c, q) cross-section. Still, MH and FI are only distinguishable at the 10% level for the high measurement error case. The ordering of the regimes in terms of likelihood remains the same as before.

Table 8a - Estimation results based on the (c,q) distributions at $t = 0$ and $t = 50$

Model	$\hat{\mu}_w$	$\hat{\gamma}_w$	$\hat{\gamma}_{me}$	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$	LL Value
			Low Measurement Error				
MH ("identification")	20.1250	8.0000	0.1001	0.5000	2.0022	0.1250	**-5064.8**
FI	19.9375	5.6875	0.2055	0.5000	2.0000	-4.5156	**-5181.0**
Autarky	n.a.	n.a.	0.2374	0.4747	1.8402	-4.9832	**-5795.8**
Saving Only	$-0.4000^{\#}$	$0.2485^{\#}$	0.2817	1.2500	1.9375	-9.6250	**-5458.6**
BL	$-0.4000^{\#}$	$0.2406^{\#}$	0.2900	0.5375	1.8250	-6.0000	**-5433.8**
baseline values	*19.999*	*8*	*0.1*	*0.5*	*2*	*0*	
			High Measurement Error				
MH ("identification")	20.0000	8.1875	0.4998	0.5000	2.0001	-0.1729	**-5586.6**
FI	19.6875	8.5469	0.5317	0.5059	1.9775	-1.1250	**-5602.2**
Autarky	n.a.	n.a.	0.5942	0.4919	1.8088	-2.8045	**-5702.8**
Saving Only	$-0.1534^{\#}$	$0.2117^{\#}$	0.6138	0.4665	2.0187	-2.7014	**-5694.3**
BL	$-0.1552^{\#}$	$0.0000^{\#}$	0.5622	0.4642	1.9501	-3.2052	**-5684.1**
baseline values	*19.999*	*8*	*0.5*	*0.5*	*2*	*0*	

Vuong test p-values (c,q at $t = 0$ and $t = 50$)

γ_w=.5$\Downarrow$ \ γ_w=.1$\Rightarrow$	MH	FI	A	S	BL
MH	n.a.	.0000***(MH)	.0000***(MH)	.0000***(MH)	.0000***(MH)
FI	.0184** (MH)	n.a.	.0000***(FI)	.0000***(FI)	.0000***(FI)
A	.0000***(MH)	.0000***(FI)	n.a.	.0000***(S)	.0000***(BL)
S	.0000***(MH)	.0000***(FI)	.3465 (draw)	n.a.	.0197** (BL)
BL	.0000***(MH)	.0000***(FI)	.0814* (BL)	.1569 (draw)	n.a.

Compare the short run results in table 7 to those where we use data generated further apart in time (table 8). Using the (c, q) cross-section at $t = 50$ instead of that at $t = 1$ helps in distinguishing the MH and FI regimes much better (now at 2%) but, interestingly, seems to blur the distinction between the exogenously incomplete A, S and BL regimes for the high ME specification. Regarding parameter estimates similar patterns to the previous case emerge.

8 Conclusions

We have formulated and solved numerically a range of multi-period financial market regimes with exogenous or endogenous asset structure that allow for moral hazard and unobservable capital/investment. We have characterized the optimal allocations implied by the regimes from both within-period and dynamic perspective. The paper develops methods based on mechanism design and numerical linear programming that can be used to structurally estimate, compare and distinguish between the different information regimes. We have shown that such models can match stylized facts from the empirical firm dynamic literature as listed by Cooley and Quadrini (1999). The compared regimes were also demonstrated to differ significantly with respect to their qualitative and/or quantitative implications for investment, consumption, financial flows, and insurance in cross-section, transitions, and long-run outcomes.

In terms of extensions, notice that we can easily incorporate limited enforcement in our formulation by simply requiring that the minimum possible promise, $w_{\min}$ be equal to the agent's

discounted value under autarky (given k) or, if the agent's saving or borrowing cannot be controlled by the bank, his value under the saving only or borrowing settings exhibited above. We can also look at a regime with observed effort but unobserved investment, k, i.e. such that only the truth-telling constraints are present but not the incentive compatibility ones. Such a setting creates an adverse selection problem (through the unobserved agent type, k) but it is not the typical one studied in the literature since there is also the investment, k' which is an unobserved action linking two neighboring time periods together.

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Fig. 1a − Growth, E(k'/k), inc. depreciation baseline
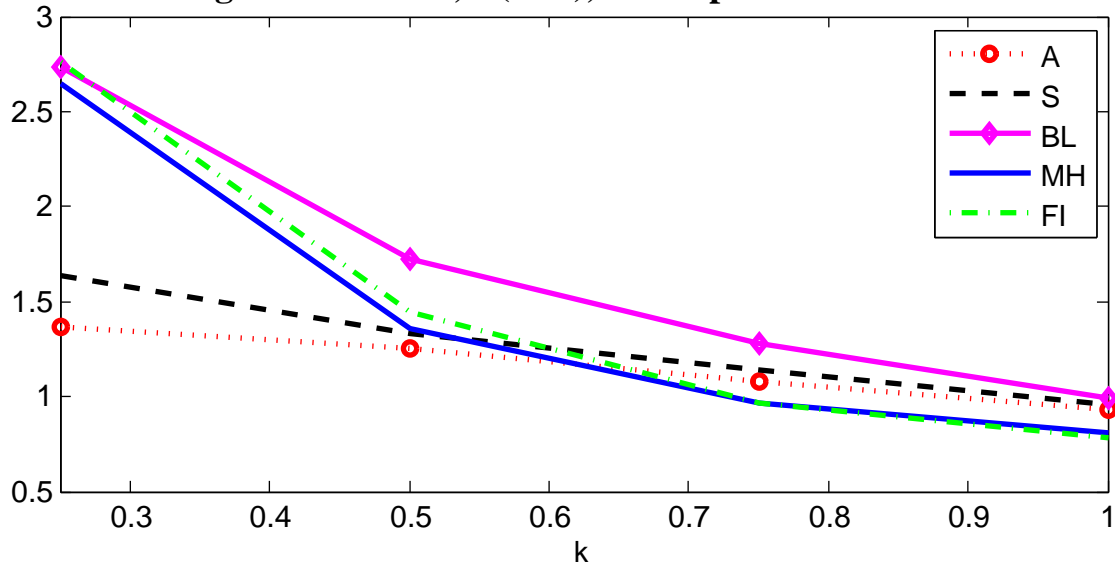
Fig. 1b − Growth, E(k'/k), full depreciation baseline

Fig. 2a – Growth Variance, Var(k'/k), inc. depreciation baseline

Fig. 2b – Growth Variance, Var(k'/k), full depreciation baseline
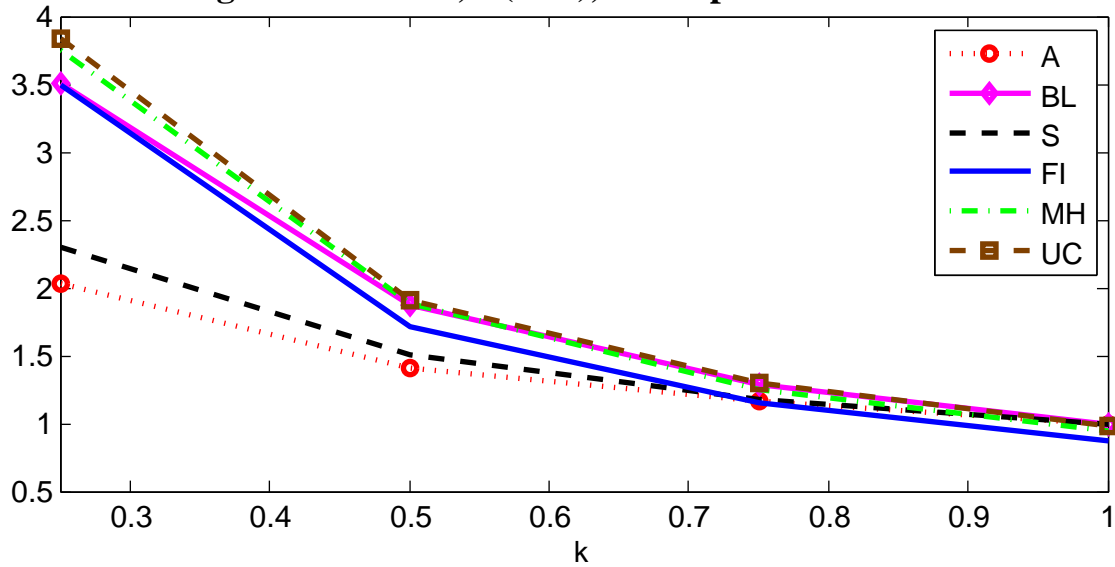
Fig. 3a – Cash Flow Sensitivity, $E(k'_H) - E(k'_L)$, inc. depreciation baseline
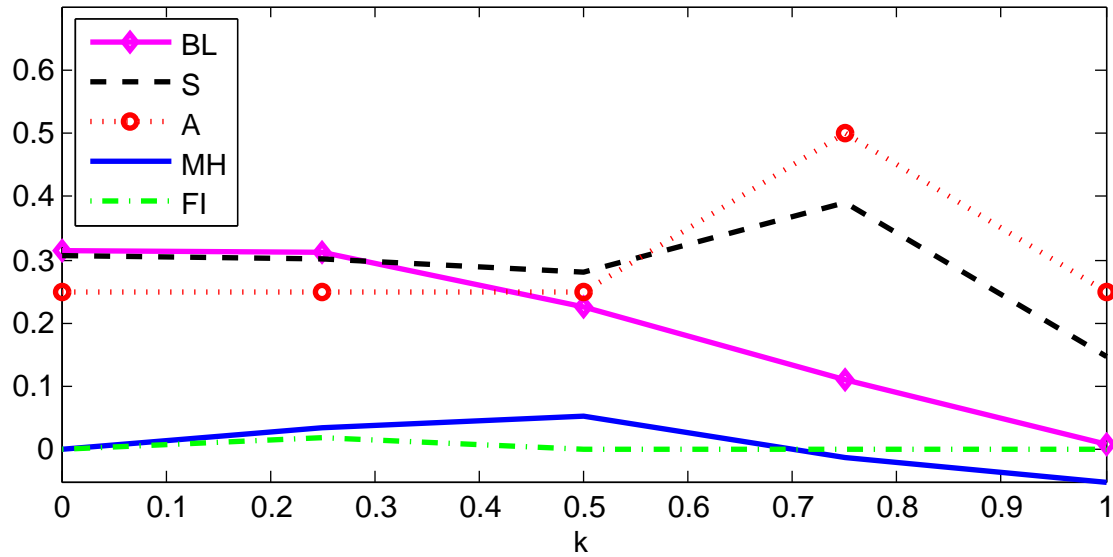
Fig. 3b – Cash Flow Sensitivity, $E(k'_H) - E(k'_L)$, full depreciation baseline
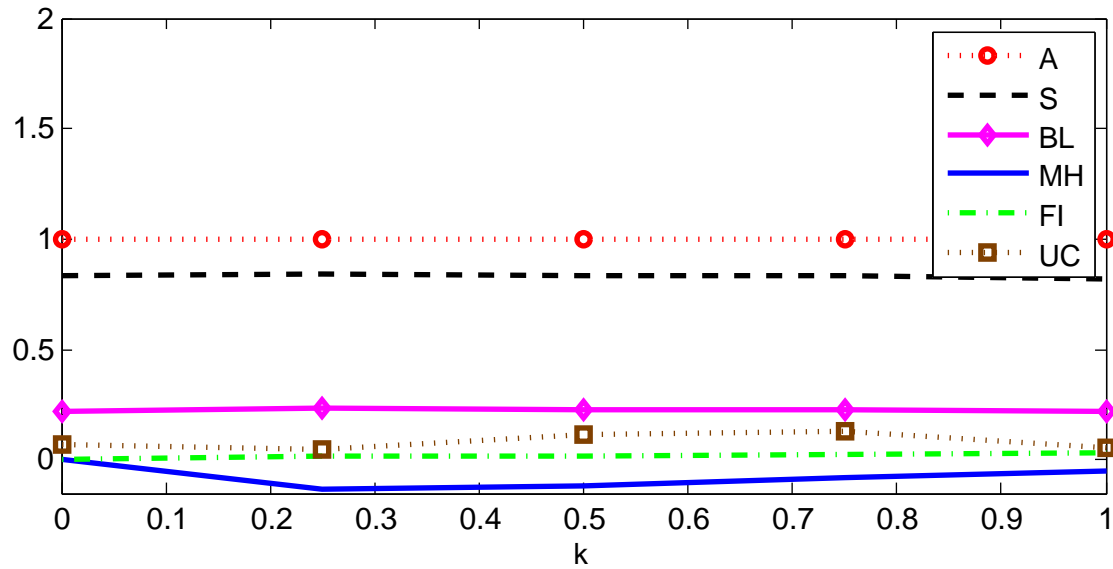
Fig. 4a – Debt, E(b'/k), inc. depreciation baseline
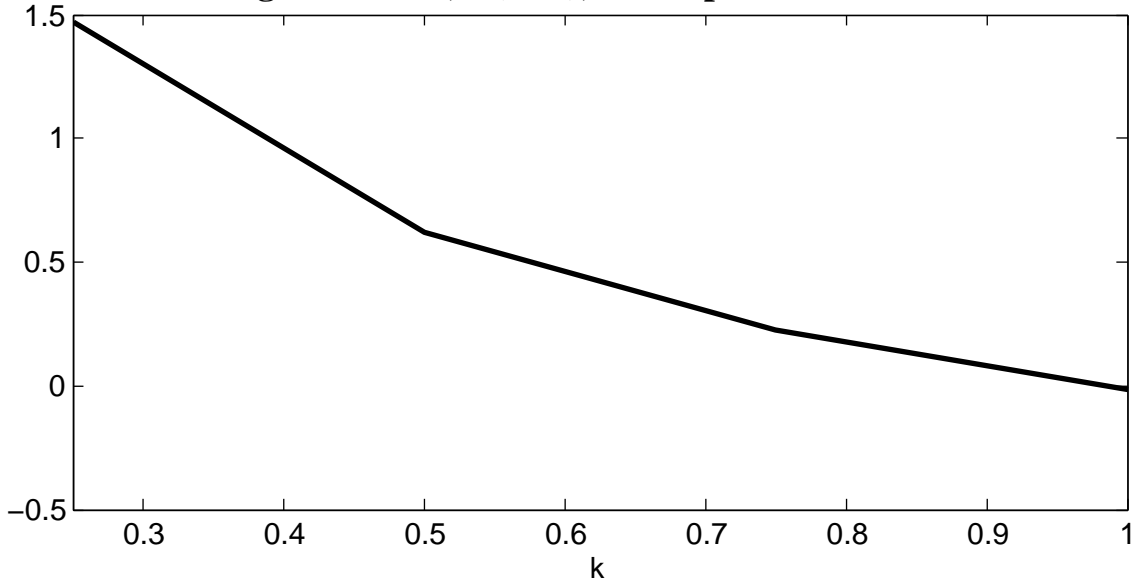
Fig. 4b – Debt, E(b'/k), full depreciation baseline
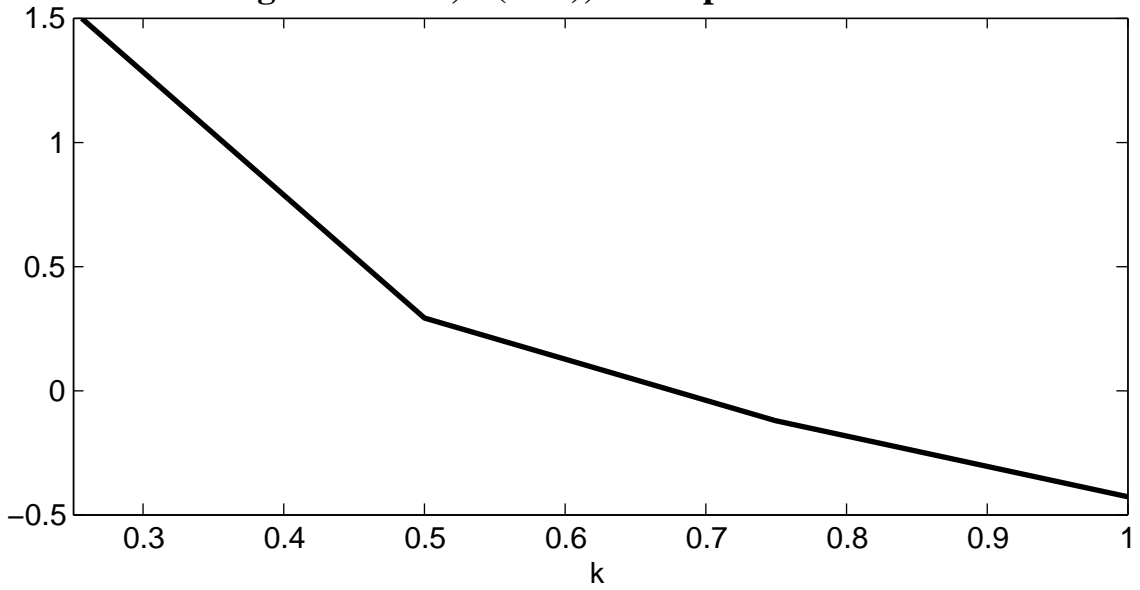
Fig. 5 – Pareto Frontiers, inc. depreciation baseline, $\rho=0$, k=0.5
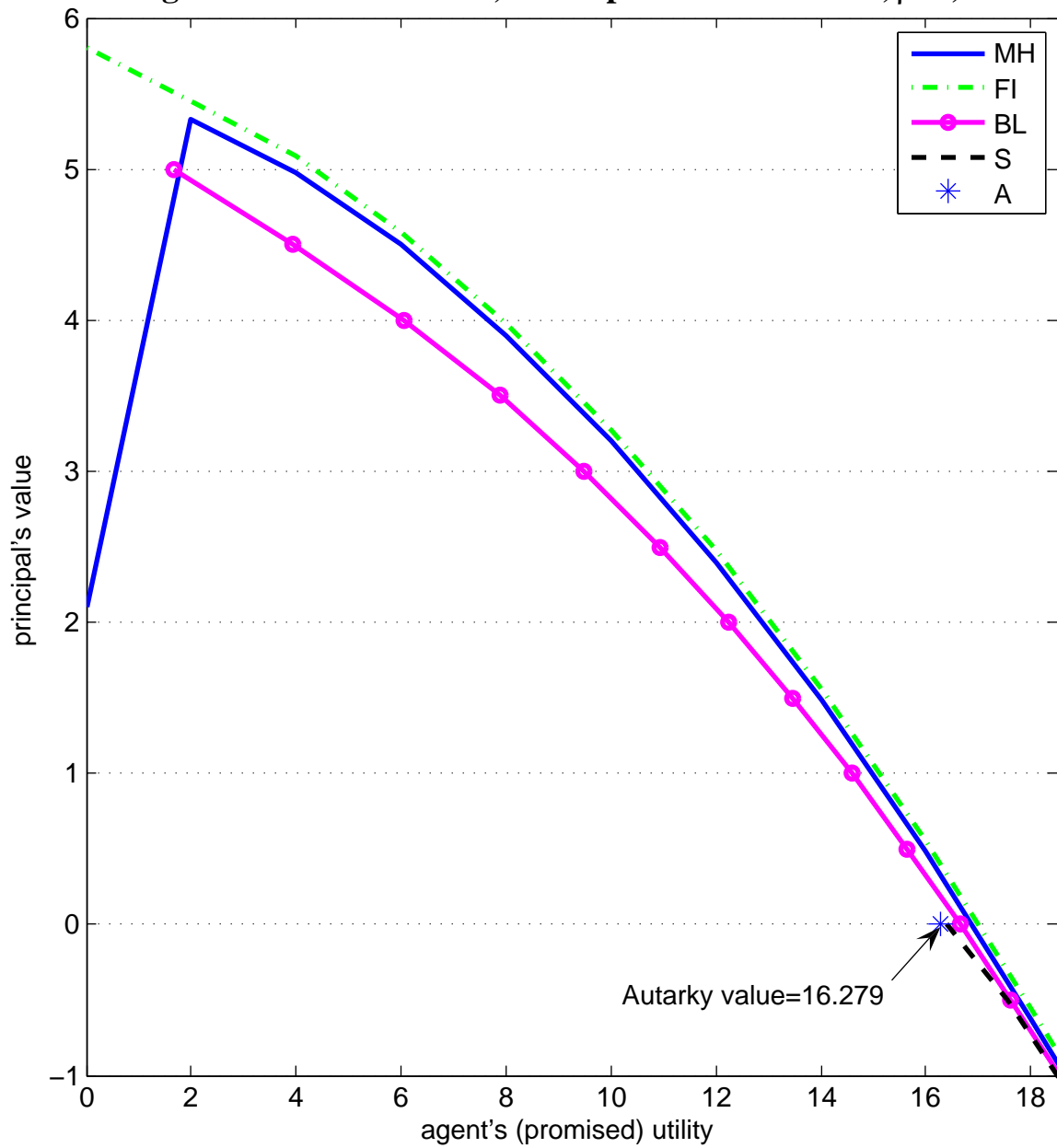
Fig. 6 – Pareto Frontiers, full depreciation baseline, ρ=0, k=0.5

Autarky value=32.817
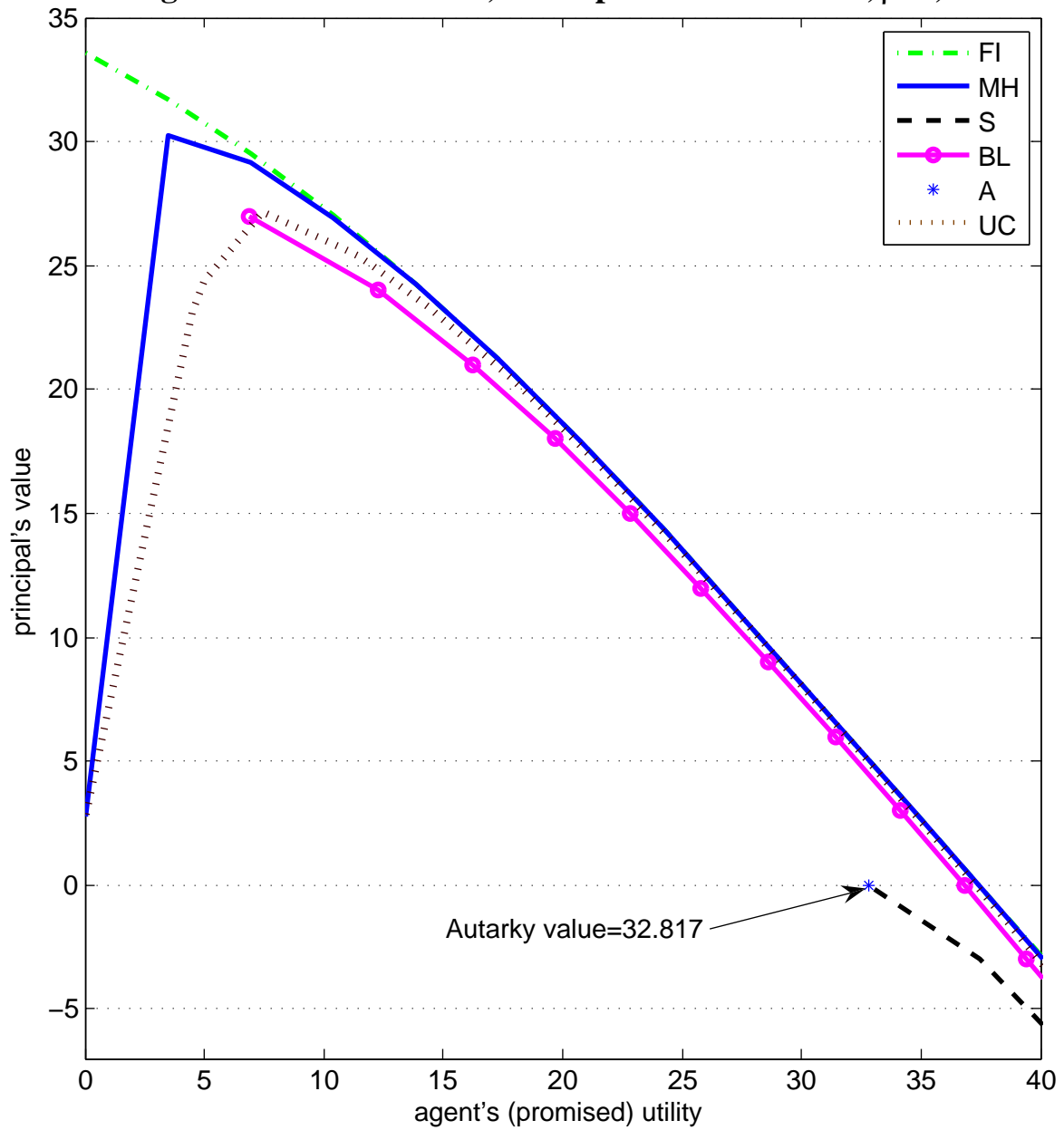
Fig. 7a – Consumption, full depreciation baseline, k=0.5

Autarky

Saving Only

Borrowing and Lending

Unobserved k

Moral Hazard

Full Information
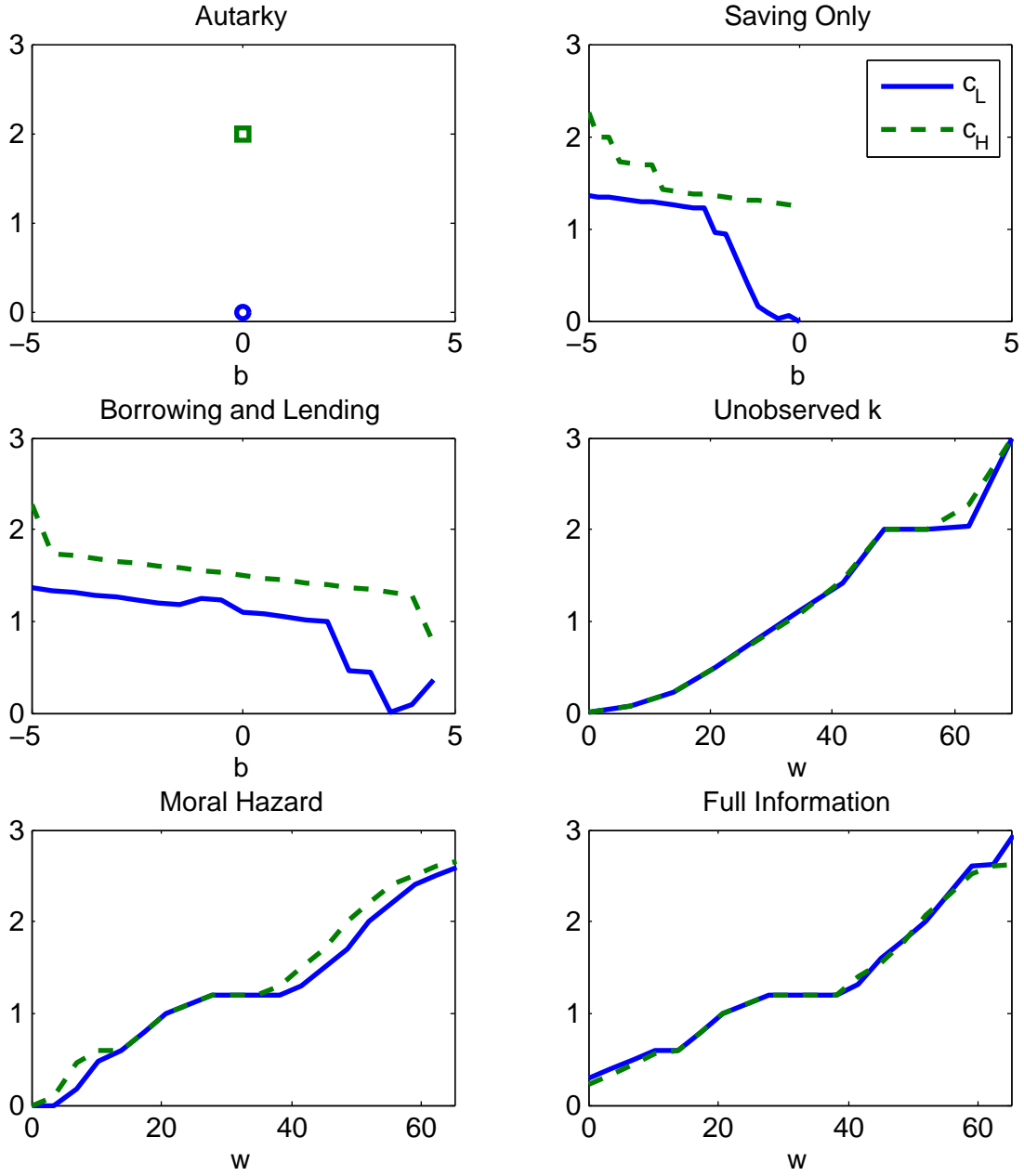
Fig. 7b – Consumption, inc. depreciation baseline, k=0.5

Autarky / Saving Only

Borrowing and Lending

Moral Hazard

Full Information
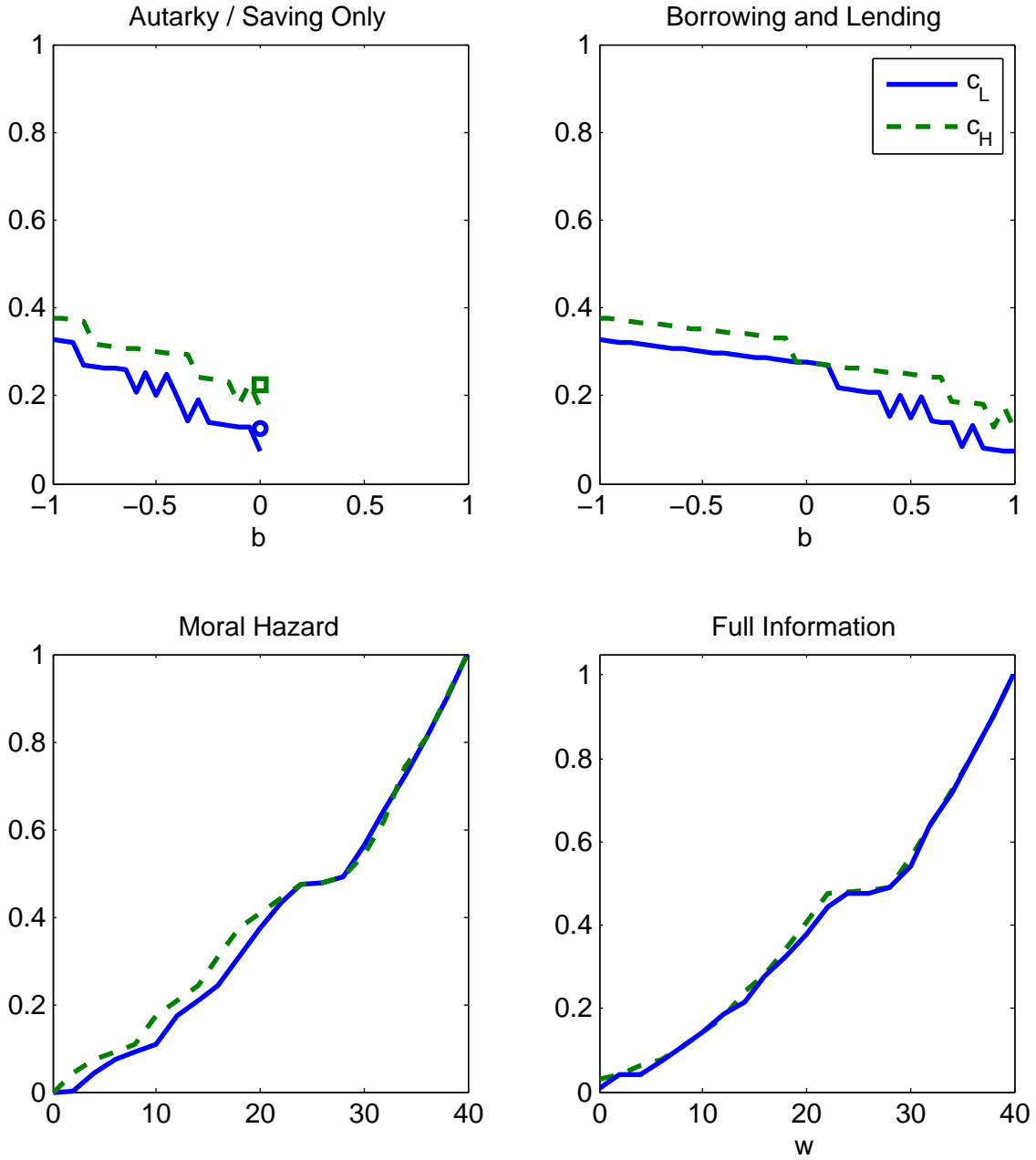
Fig. 7c – Investment and Effort, full depreciation baseline, k=0.5
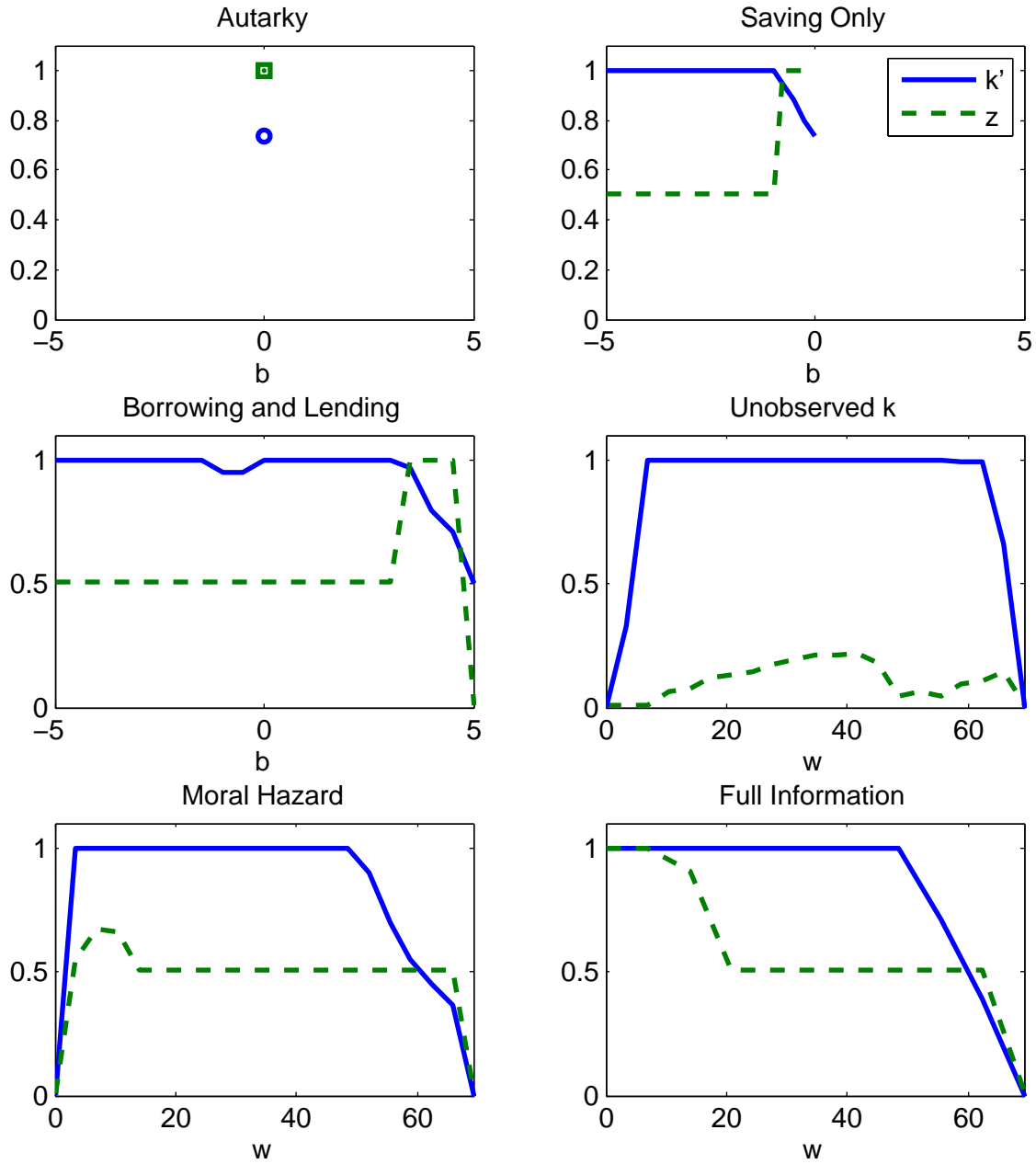
Fig. 7d – Investment and Effort, inc. depreciation baseline, k=0.5
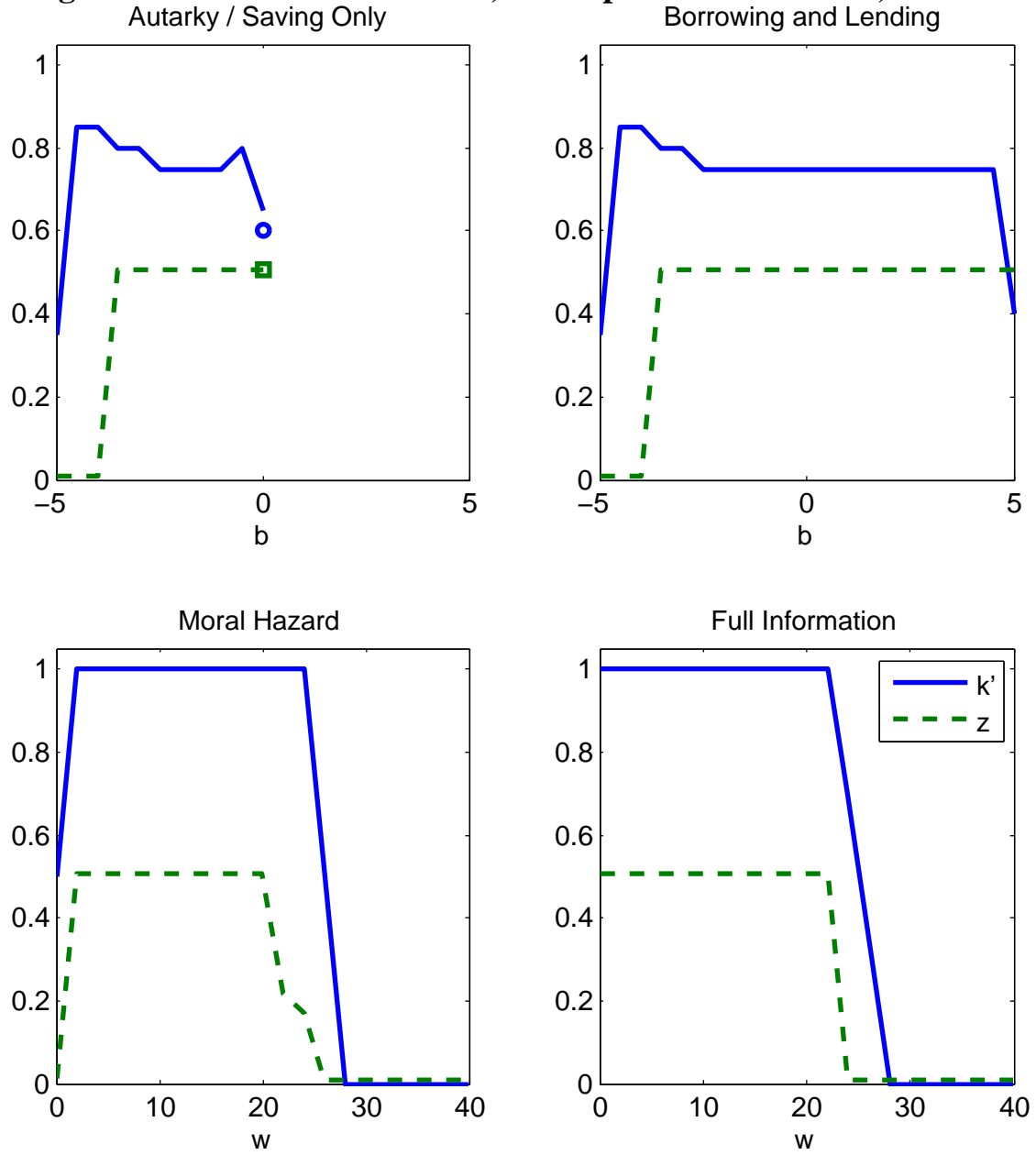
Fig. 8 – Distributions of c and k' over time: A, inc. depreciation baseline
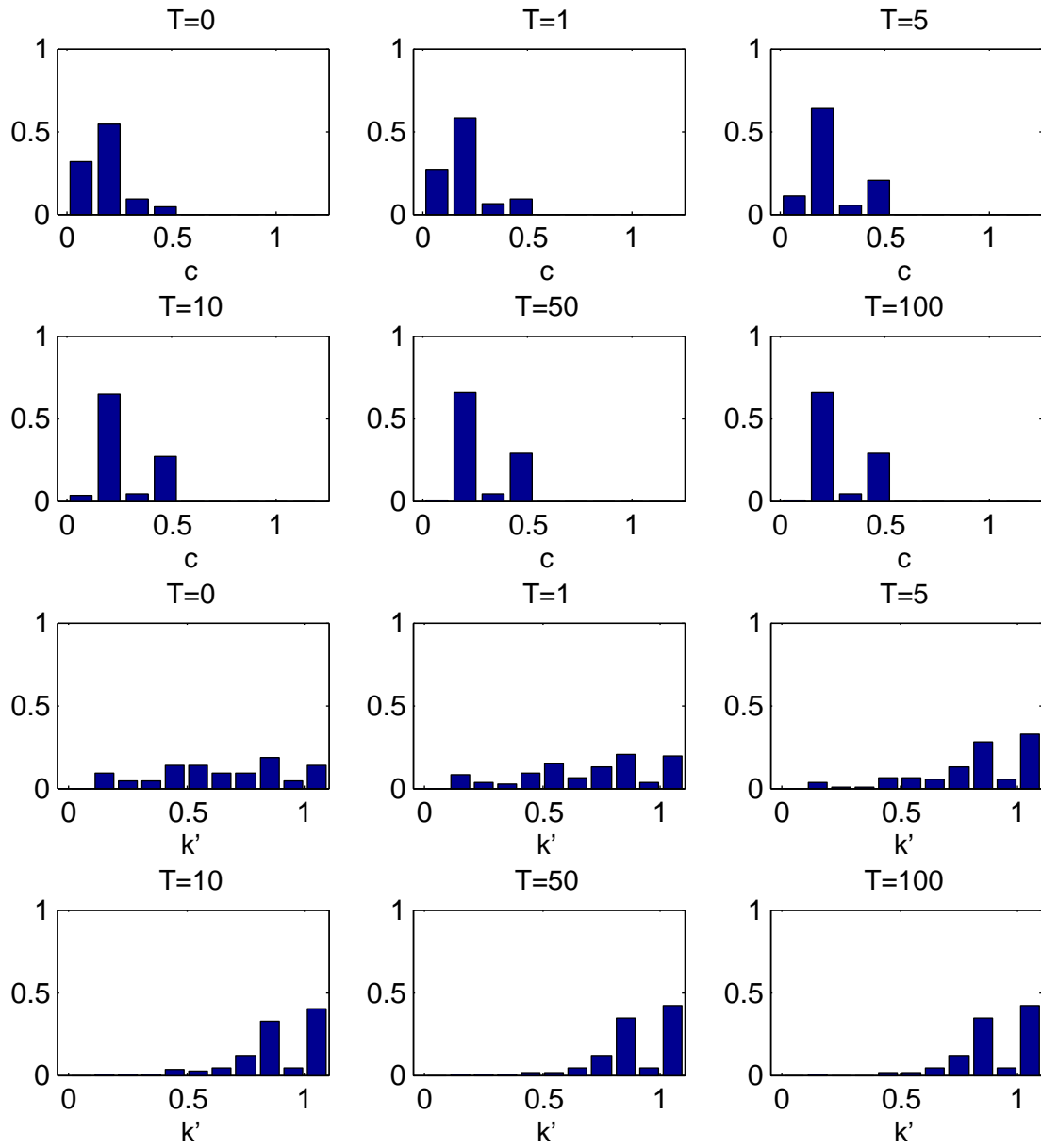
Fig. 9 – Distributions of c and k' over time: S, inc. depreciation baseline
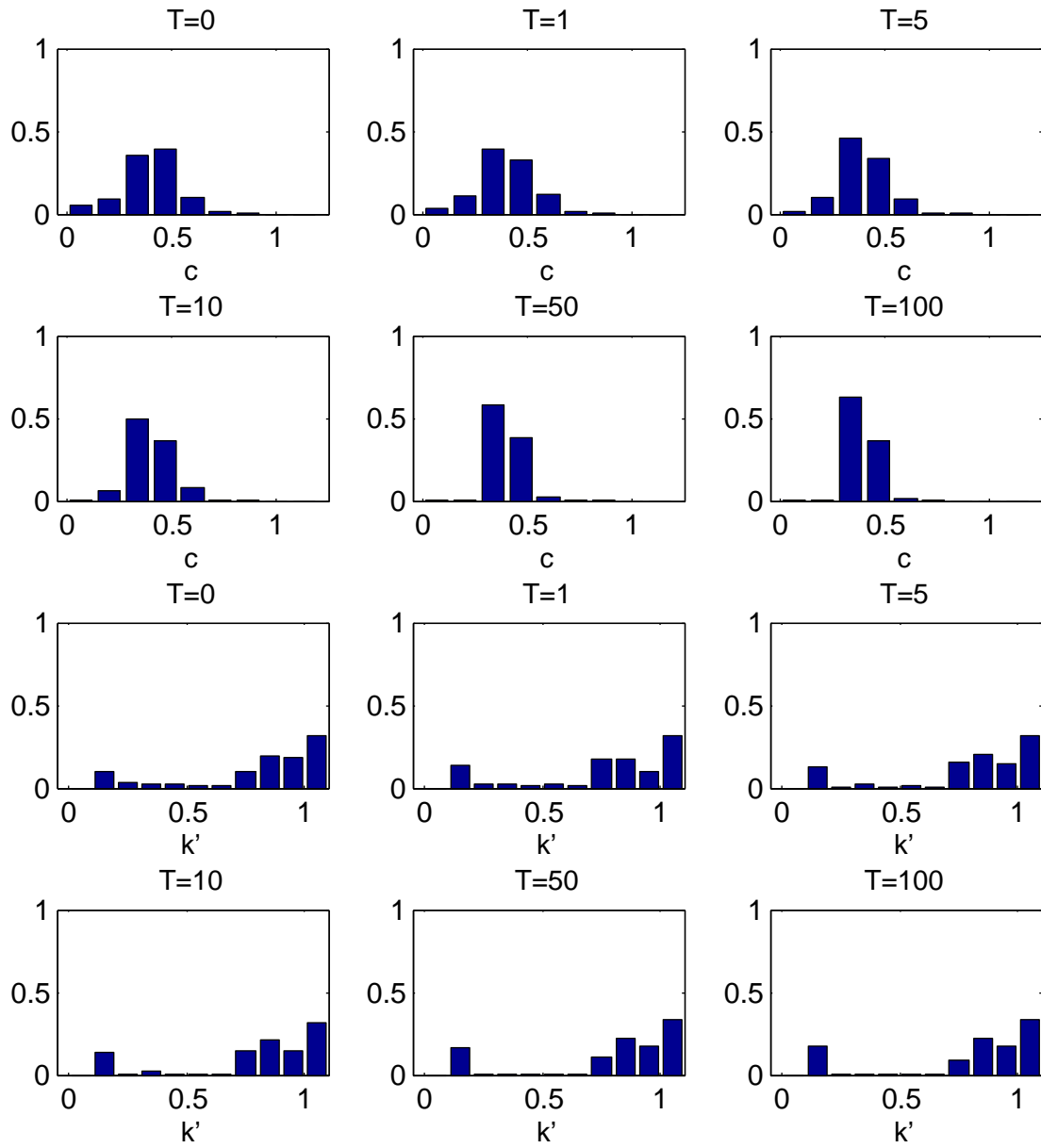
Fig. 10 – Distributions of c and k' over time: BL, inc. depreciation baseline
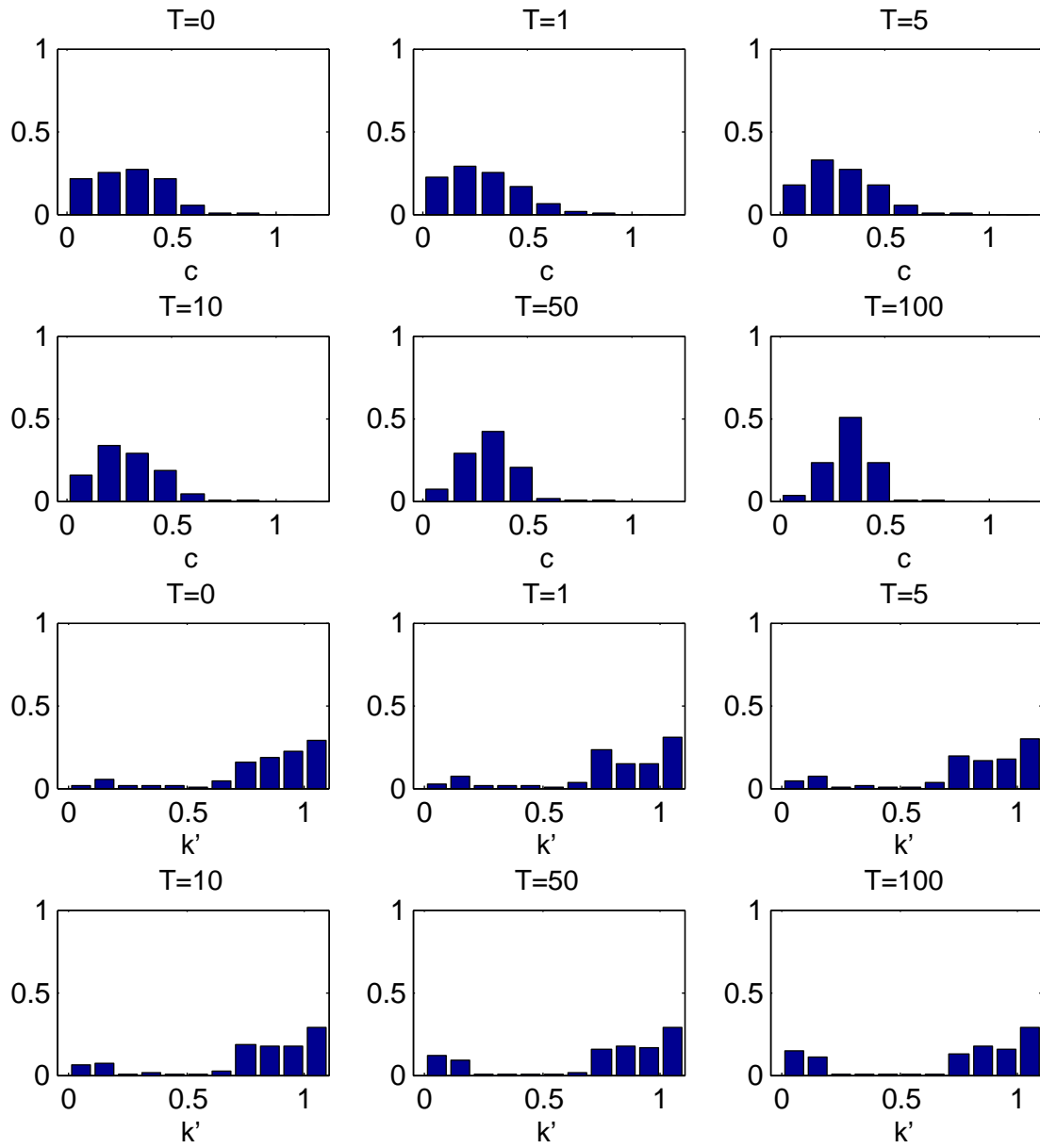
Fig. 11 – Distributions of c and k' over time: MH, inc. depreciation baseline
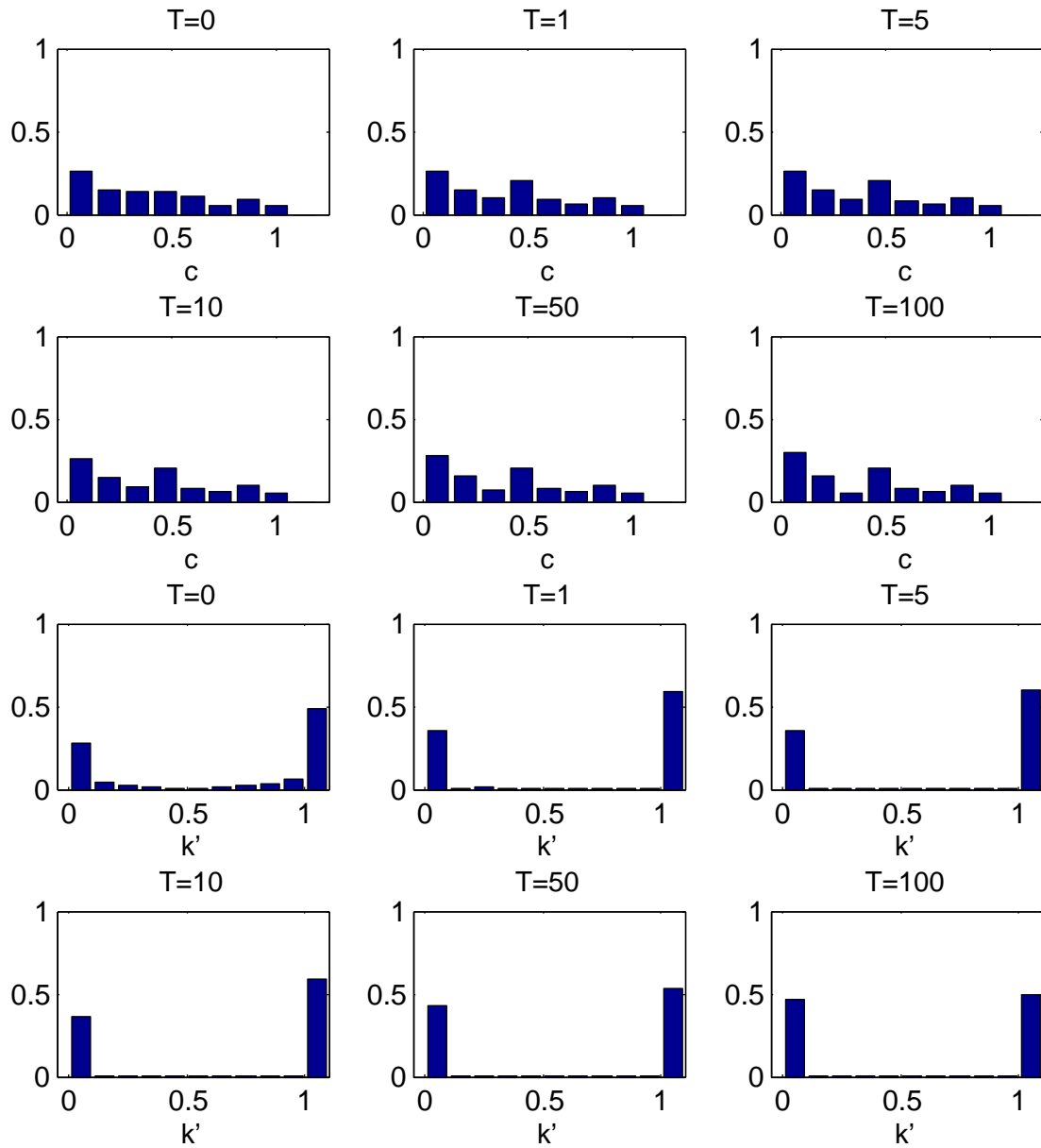
Fig. 12 – Expected Time Paths, incomplete depreciation baseline

Consumption, c

Next period capital, k'
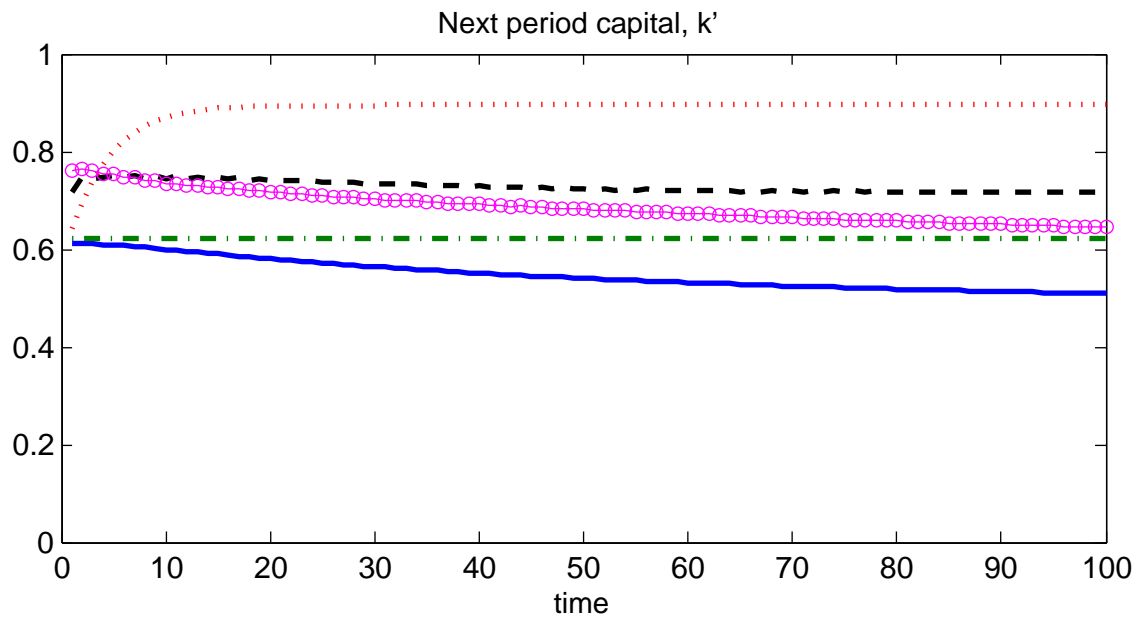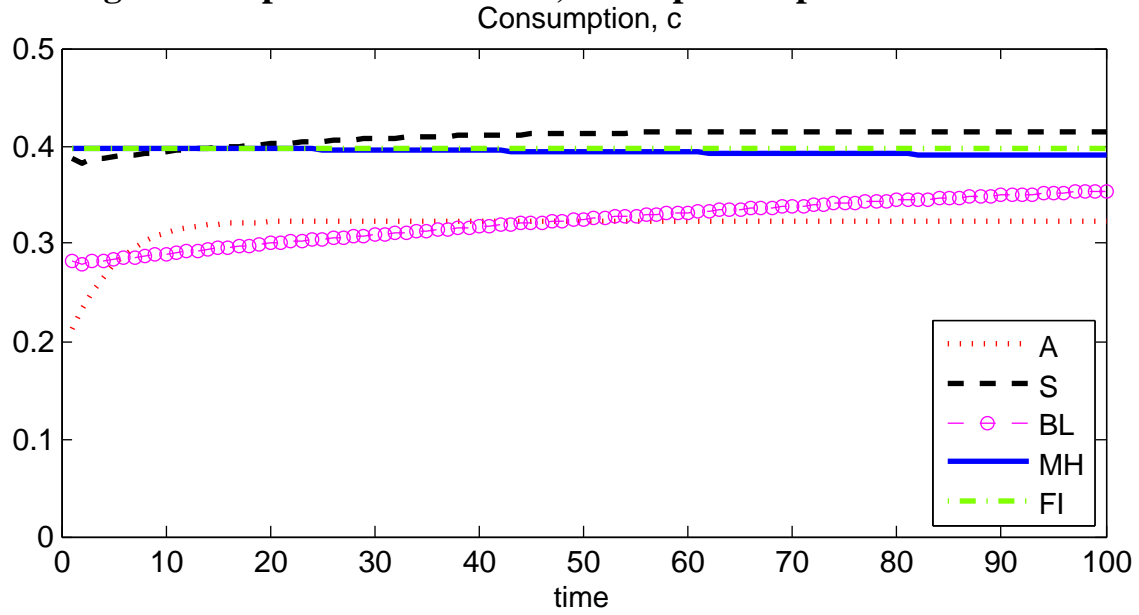
Fig. 13 – Expected Time Paths, full depreciation baseline

Consumption, c

Next period capital, k'
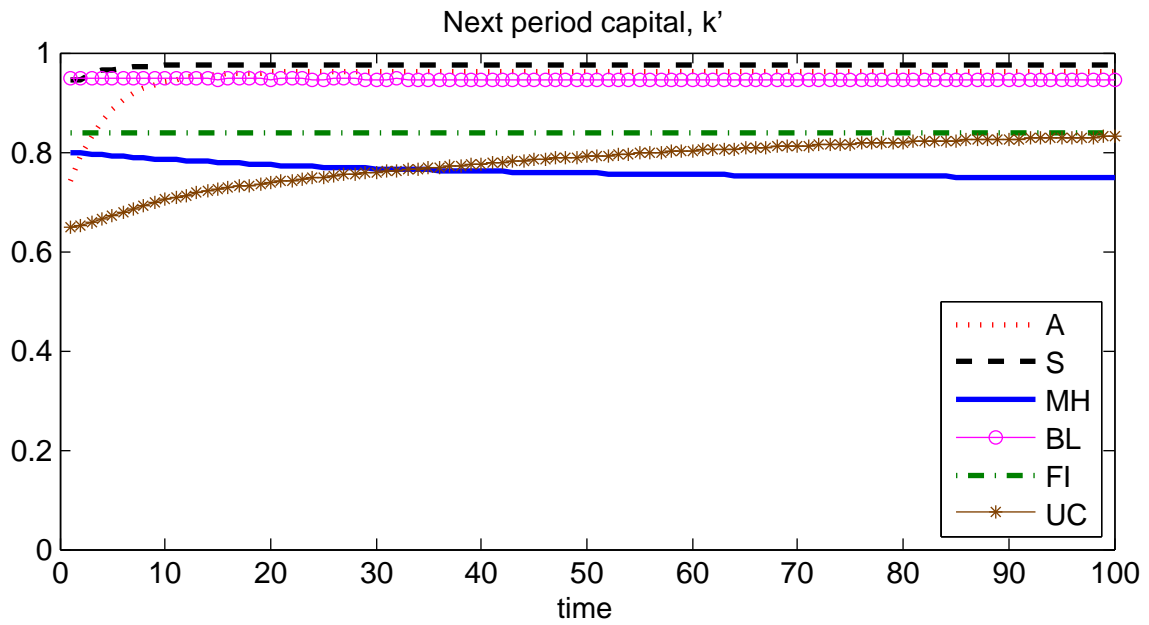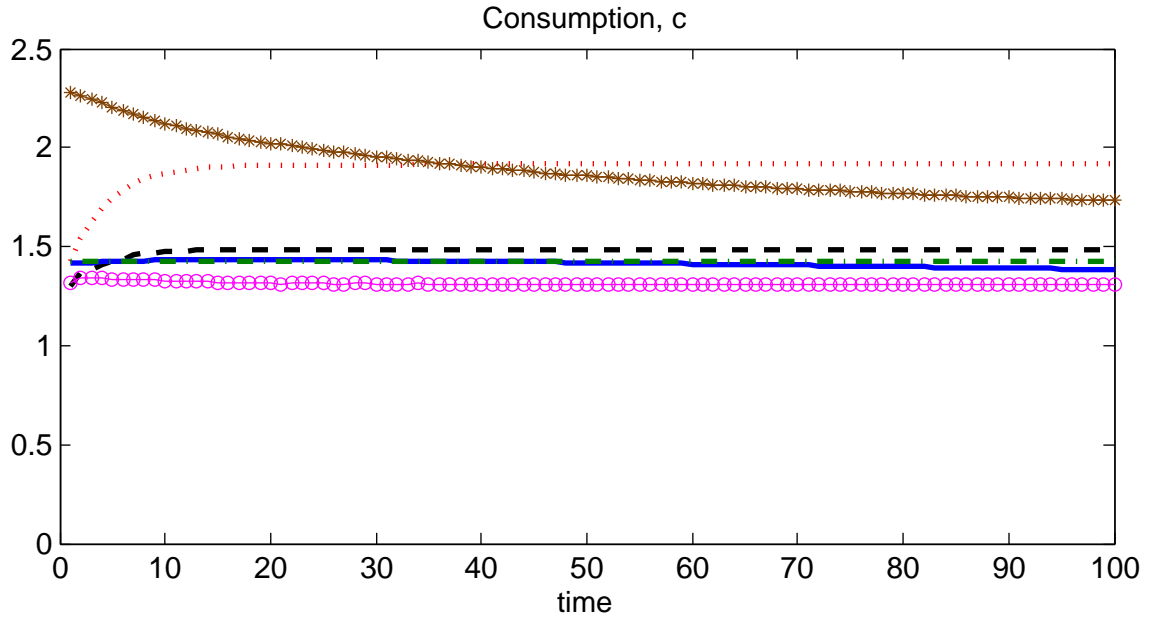
Fig. 14a – Growth, E(k'/k), estimated parameters, low meas. error
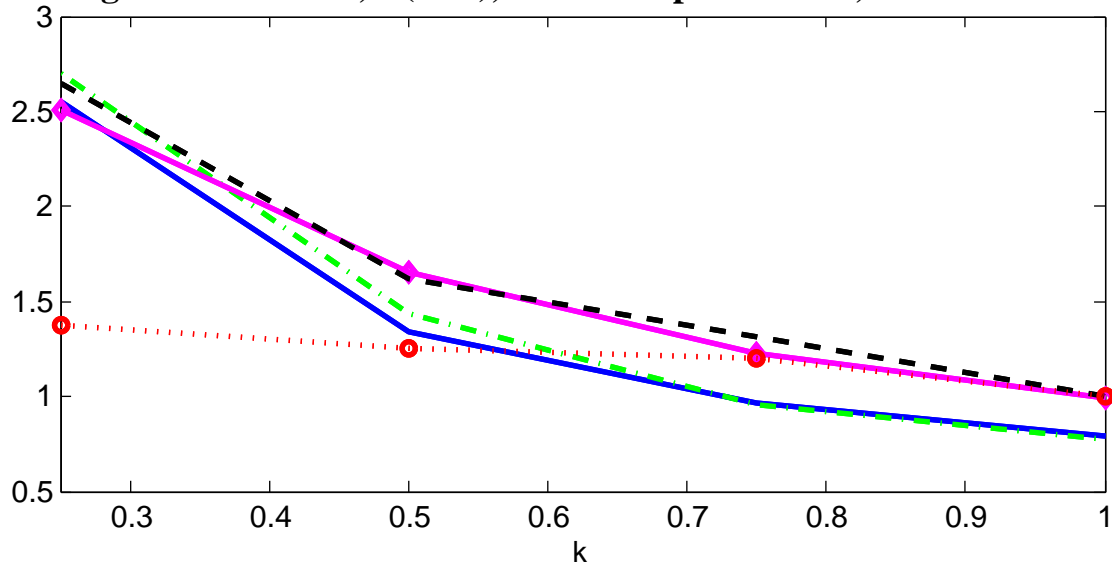
Fig. 14b – Growth, E(k'/k), estimated parameters, high meas. error
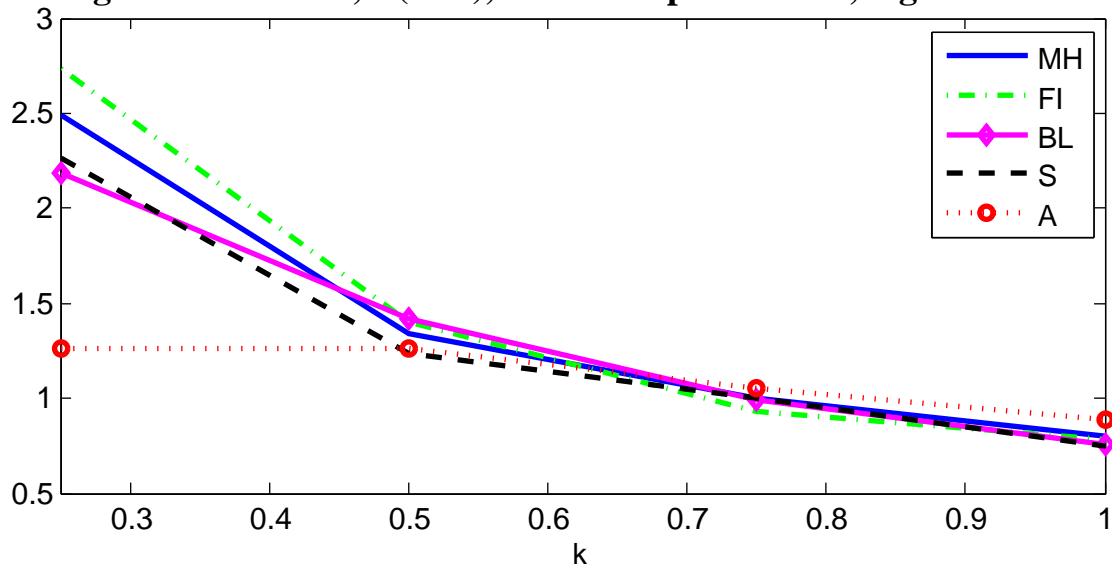
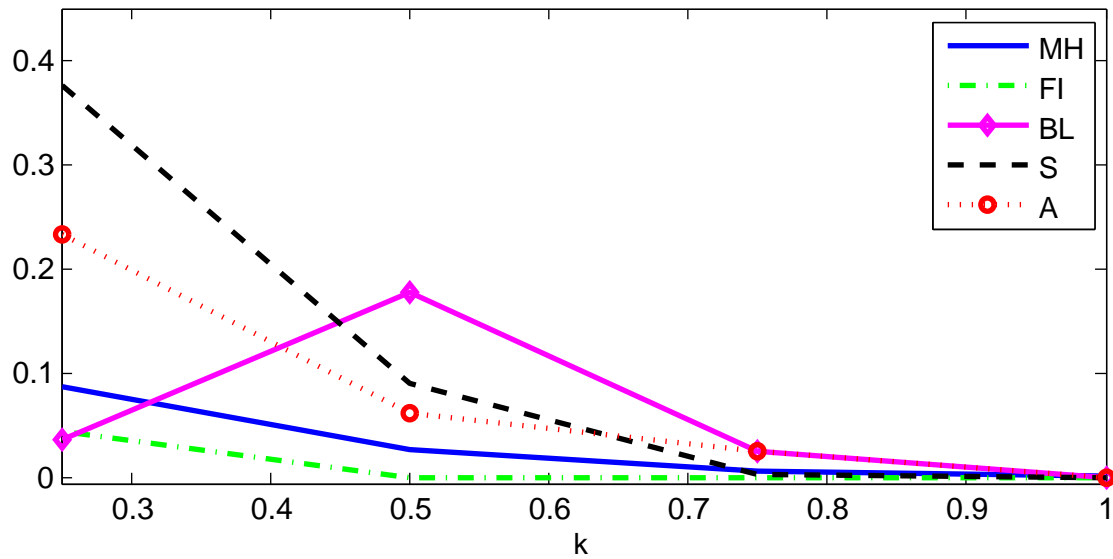**Fig. 15a – Growth Variance, Var(k'/k), estimated parameters, low meas. error**

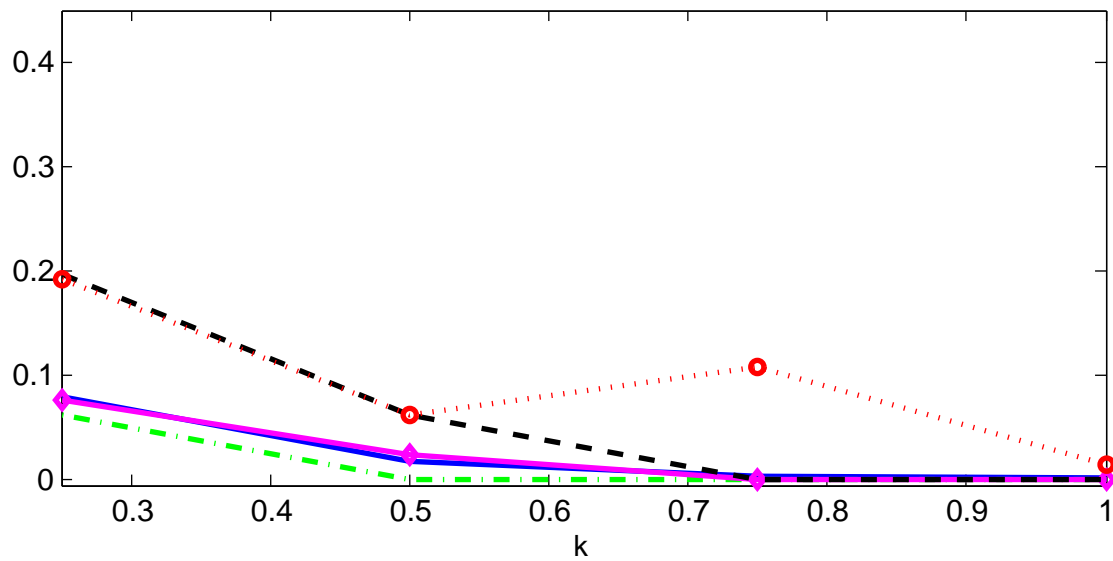**Fig. 15b – Growth Variance, Var(k'/k), estimated parameters, high meas. error**

Fig. 16a – Cash Flow Sensitivity, $E(k_H) - E(k_L)$, estimated parameters, low meas. error
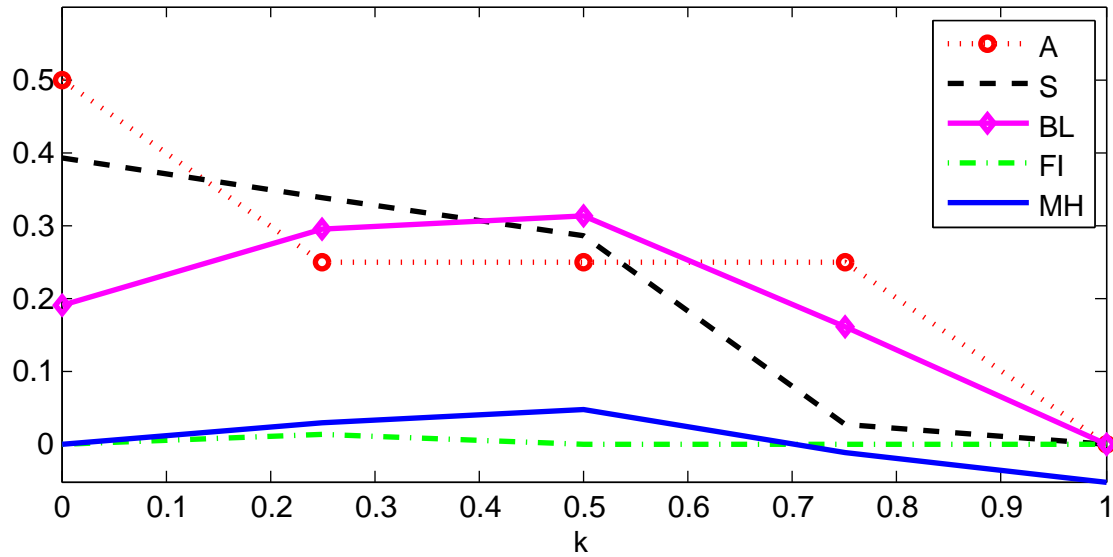

Fig. 16b – Cash Flow Sensitivity, $E(k_H) - E(k_L)$, estimated parameters, high meas. error