Problem Set 10
Due in Lecture Tuesday Nov. 18

1 Romer 8.10

2 Romer 9.6

3 The Local Dynamics of the q model

This question is a former final exam question. Special ground rules: (1) First try it under exam conditions - no readings, no notes, no help. (2) then you may look at your notes and readings but not consult with others; (3) you may then consult with others; (4) as always, you may not look at pas years' problem set solutions (or past years midterm and final exam solutions).

Consider the two equations of the q model we derived in class, for the special case where $\delta = 0$ and $C(I, K) = C(I)$ (i.e. adjustment costs only depend on the level of investment). In that case, $I = \phi(q)$ instead of $I/K = \phi(q)$ and the differential system is given by:

$$\dot{K}_t = \phi(q_t)$$
$$\dot{q}_t = rq_t - \Pi_K(K_t)$$

1. Define the steady state of the model, $(\bar{q}, \bar{K})$. Show that the model's linear (Taylor) approximation in the neighborhood of the steady state takes the form:

$$\left(\begin{array}{c} \dot{q} \\ \dot{K} \end{array} \right) = G \left(\begin{array}{c} q - \bar{q} \\ K - \bar{K} \end{array} \right)$$

where

$$G = \left(\begin{array}{cc} A & B \\ C & D \end{array} \right)$$

is a 2×2 matrix. Be sure to express the elements of G in terms of the exogenous parameters and steady state values of q and K and the properties of $\Pi_K(.)$ and $\phi(.)$.

2. Show that the characteristic roots of the G matrix are given by

$$\lambda_1, \lambda_2 = \frac{r \pm \sqrt{r^2 - 4\phi'(1)\Pi_{KK}(\bar{K})}}{2}$$

where $\lambda_1 > 0$ and $\lambda_2 < 0$. Please indicate why the second condition holds.

3. Show that the eigenvectors of the matrix G are proportional to the matrix

$$X = \begin{pmatrix} \lambda_1/\phi'(1) & \lambda_2/\phi'(1) \\ 1 & 1 \end{pmatrix}$$

4. Define

$$\begin{pmatrix} \tilde{q} \\ \tilde{K} \end{pmatrix} = X^{-1} \begin{pmatrix} q - \bar{q} \\ K - \bar{K} \end{pmatrix}$$

and note that this implies that

$$\begin{pmatrix} \dot{\tilde{q}} \\ \dot{\tilde{K}} \end{pmatrix} = X^{-1} \begin{pmatrix} \dot{q} \\ \dot{K} \end{pmatrix}$$

Explain how the change of variables enables us to write the solution to our differential equation system in the form

$$\begin{pmatrix} \tilde{q} \\ \tilde{K} \end{pmatrix} = \begin{pmatrix} \tilde{q}(0)e^{\lambda_1 t} \\ \tilde{K}(0)e^{\lambda_2 t} \end{pmatrix}$$

for arbitrary initial conditions $\tilde{q}0$ and $\tilde{K}0$.

5. From this last relationship, deduce that:

$$q(t) - \bar{q} = \tilde{q}(0)\frac{\lambda_1}{\phi'(1)}e^{\lambda_1 t} + \tilde{K}(0)\frac{\lambda_2}{\phi'(1)}e^{\lambda_2 t}$$
$$K(t) - \bar{K} = \tilde{q}0e^{\lambda_1 t} + \tilde{K}(0)e^{\lambda_2 t}$$

6. Recalling that $\lambda_1 > 0$ and $\lambda_2 < 0$, identify the initial condition that will ensure the economy is on the convergent saddle path in the usual phase diagram with K on the horizontal axis and q on the vertical axis.

7. For our linear approximation above, express the (linear) equation for the saddle path in the form:
$$q(t) - \bar{q} = \Omega(K(t) - \bar{K})$$

for an appropriate slope Ω. Check that $\Omega < 0$ and discuss how Ω depends on the model's parameters, the steady state value $\bar{q}$ and $\bar{K}$ and/or the properties of $\Pi_K(.)$ and $\phi(.)$.

8. Recall that
$$\lambda_2 = \frac{r - \sqrt{r^2 - 4\phi'(1)\Pi_{KK}(\bar{K})}}{2}$$

Discuss which parameters and steady state values affect the slope of the saddle-path. How do they impact the slope? Why?

4 External Adjustment Costs

This question consider what happens when firms faces *external* adjustment costs. Assume that there is continuum of firms of mass 1, each solving an identical problem. We will denote firm level variable with lower cases: k_t for capital and i_t for investment at time t. Aggregate variables will be denoted with uppercase variables: K_t and I_t for aggregate capital stock and aggregate investment at time t.

Assume that there are no internal adjustment costs (i.e. $C(i,k) = 0$) but that the price of investment goods is a function of aggregate investment:$p_K(I)$ with $p'_K(I) > 0$. Denote $\Pi(k_t)$ the profits of a firm operating with capital stock k_t at time t, $\delta > 0$ the depreciation rate and $r > 0$ the real risk-free rate.

- Write the problem that the firm solves.

- Write the present value Hamiltonian and the first-order conditions for the firm problem, taking as given aggregate variables (K and I). What does this imply for Tobin's marginal q defined as the marginal value of an extra unit of capital relative to the price of investment goods? Does it vary over time? Does the firm satisfy the neoclassical 'user-cost' rule for investment we derived in class?

- Recognizing that in equilibrium aggregate capital and aggregate investment satisfy $K = k$ and $I = i$, characterize the dynamics of the economy in a (I, K) space. Describe whether the economy exhibits saddle-path dynamics.

- Assume now that there is only one large firm in the economy, acting as a monopsonist on the market for investment goods: it recognizes that if it invests I, it faces a price schedule $p_K(I)$. Assume that the price of investment goods has a constant elasticity, that is $p_K(I) = pI^\eta$ for $\eta > 0$ and $p > 0$. Solve the firm's problem and explain how aggregate dynamics differ from the case of perfect competition.

- Goolsbee (1998) considers the effect of a an investment tax credit. This can be modeled as a reduction in the price of investment goods from $p_K(I)$ to $(1-\tau)p_K(I)$ where $\tau > 0$ is the investment tax credit. Suppose the tax credit is unanticipated and permanent. Describe what happens to investment, capital and the price of investment goods over time in the competitive case when $p_K(I) = pI^\eta$. Explain how your answer may vary with the price elasticity of investment goods η.