Free Cash-Flow, Issuance Costs and Stock Volatility[*]

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First Draft: July 2006
This draft: July 2007.

Abstract

We study the issuance and payout policy that maximizes the value of a firm facing both agency costs of free cash-flow and external financing costs. We find that firms have target cash levels and optimally issue equity when they run out of cash. We characterize the process modelling the number of outstanding shares and the dynamics of the stock prices. In line with the leverage effect identified by Black (1976), we show that both the volatility of stock returns and the dollar volatility of stock prices increase after a negative shock on stock prices.

Keywords: Issuance and Dividend Policies, Optimal Cash Management, Stock Volatility
JEL Classification: G12; G35.

1. Introduction

Following Easterbrook (1984) and Jensen (1986), an influential branch of the theoretical corporate finance literature has emphasized the role of payout policies in addressing agency conflicts between corporate insiders and outside shareholders. The basic idea is that profits that are not paid out to shareholders can be diverted by insiders at their own advantage. For instance, firms' managers may commit these funds to inefficient projects that generate private benefits for them to the detriment of outside shareholders.[1] As pointed out by Jensen (1986), this conflict over payout policies is particularly severe when firms hold substantial amounts of free cash-flow, that is, when cash inflows significantly exceed efficient reinvestment needs or opportunities.

In a world of perfectly functioning capital markets, this conflict would easily be solved with no adverse impact on shareholder value. Indeed, firms could finance their investments by issuing shares and commit to distribute all their profits to outside shareholders, leaving no cash to be diverted or wasted by corporate insiders. Operating costs and new investment expenditures would simply be met by issuing new shares. In practice, however, publicly traded companies often issue new shares by organizing a seasoned equity offering (SEO), which is far from being a costless operation. Smith (1977) estimates direct underwriting costs for U.S. corporations from 1971 to 1975 to be 6.17% on average, rising to 13.74% for smaller issues.[2] These costs include the fees paid to the investment banks as well as other direct expenses such as legal and auditing costs. Lee, Lochhead, Ritter, and Zhao (1996) report the average costs of raising capital for U.S. corporations from 1990 to 1994, and find that the direct costs of SEOs vary from 3.1% of the proceeds of the issuing (for large issues), to 13% (for small issues), with an average of 7.1%. As a result of this, new issues of equity are relatively unfrequent, and typically involve substantial amounts. In face of these costs, the issue of the optimal cash management policy of firms becomes non trivial.

To address this issue, this paper proposes a stylized continuous-time model of a firm facing both internal agency costs and external financing costs. On the one hand, frictions within the corporation reduce the rate of return on cash held internally below the cost of capital. This creates a pressure from outside investors to disgorge cash from the firm. On the other hand, capital market imperfections make issues of new shares costly. This in turn generates a precautionary demand for cash to meet short-term obligations without resorting to the market. The firm's optimal cash management policy is the result of this trade-off. The problem is to determine how many new shares should be issued, and how much of the firm's earnings should be paid out.

Our contribution is twofold. First, we characterize the issuance and payout policy that maximizes the value of the firm. By endogenizing the payout policy, our paper therefore contributes to the large literature on the optimal design of securities and the optimal capital structure of firms.[3] A noticeable feature of our analysis is that while most of the literature

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[1] See La Porta, Lopez-De-Silanes, Shleifer and Vishny (2000) for an overview of these theories and for empirical tests that support them.

[2] As pointed out by Smith (1977), rights offering are significantly less costly than underwritten offerings. Yet underwriters are employed in more than 93% of the offerings in his sample. A possible explanation is that corporate insiders receive private benefits from the use of underwriters, for instance through a preferred customer status in case the issue is oversubscribed. The cost to shareholders of monitoring the choice of financing may turn out to be greater than the cost of using a more expensive financing method.

[3] See for instance Harris and Raviv (1989) and Myers (2001) for surveys of this literature.

emphasizes debt as the optimal claim held by outsiders, we show instead that issuing equity is optimal. In line with the empirical findings of DeAngelo, DeAngelo and Stulz (2006), equity in our model distributes dividends when the cumulative performance of the firm has been high enough and the cash reserves of the firm hit a threshold. By contrast, new equity is issued as the firm runs out of cash. Hence equity trades off in an optimal way the shareholders' desire to obtain cash from the firm and thereby mitigate the free cash-flow problem, against the costs of issuing new shares to maintain the firm's operations when its cash reserves are depleted. When issuing activity involves both fixed and proportional costs, equity adjustments take place in lumpy and unfrequent issues, as documented by Bazdresch (2005) and Leary and Roberts (2005). A key insight of our analysis is that the value of the firm is an increasing and concave function of the level of its cash reserves, so that it reacts less to changes in the latter when past performance has been high.

Second, we spell out the asset pricing implications of agency and financing costs. We show that when both types of costs are taken into account, stock prices naturally exhibit heteroskedasticity, even when the volatility of earnings is constant. In particular, our model predicts that when the price of a stock falls, the volatility of its return should increase. This follows from the concavity of the value of the firm with respect to cash reserves. Indeed, issuance costs imply that the marginal value of cash within the firm increases after a fall in the price of the stock. This reflects the fact that one more unit of cash within the firm decreases the risk of having to incur issuance costs in the near future, an event that becomes more likely after a fall in the stock price. By contrast, after an increase in the stock price, due for instance to unexpected operating profits, the marginal value of cash within the firm decreases. Further shocks on profitability have therefore a larger impact on stock price following a negative initial shock than following a positive initial shock.

These findings are in line with the leverage effect first identified by Black (1976), according to which the volatility of stock returns tend to raise following bad news and ensuing falls in stock prices. As pointed out by Black (1976), this effect remains significant even if the firm holds no debt, as long as it has operating leverage, that is, a potential need to finance future operating costs by external funds. In fact, our model yields more than the standard leverage effect, which is usually stated in terms of stock returns. Indeed, it predicts that the dollar volatility of stock prices increases after a negative shock on stock prices. This feature is also documented by Black (1976), who provides no explanation for it. In our model, it is an immediate consequence of the fact that stock prices are a concave function of the level of cash reserves within the firm. The same shock on earnings has thus a greater impact on stock prices, and not only on stock returns, when the stock price is initially low than when it is high.

Interestingly, in the benchmark case with no issuance costs, the value of the firm is linear with respect to cash reserves. As a result of this, the volatility of stock returns is constant. This highlights a new, and somewhat unexpected connection between the Black and Scholes (1973) and Merton (1973) option pricing model, that precisely assumes constant volatility of stock returns, and the absence of transaction costs on financial markets. It is indeed often pointed out that the arbitrage pricing methods that underlie the Black and Scholes (1973) and Merton (1973) formula are only valid in the absence of transaction costs on secondary markets. In our model, the volatility of stock returns is constant only in the absence of transaction costs on primary markets.

Our paper is related to the debate on the relationship between transaction costs and

volatility on financial markets. However, most contributions focus on secondary markets. Going back to the controversy on the (de)stabilizing role of speculation, two opposing strands of the literature have coexisted. Some, following Keynes (1936) and Tobin (1978), argue that speculation may have destabilizing effect and thus that increasing transaction costs might have beneficial effects by decreasing volatility. Others, following Friedman (1953) and Miller (1991), claim on the contrary that, at least in the long run, higher transaction costs increase volatility on financial markets. Using recent data from a natural experiment on the French stock market, Hau (2006) finds indeed that transaction costs and volatility are positively related. Our model predicts a similar feature in the context of primary markets. We find that when issuance costs for new securities are high, the survival of profitable firms may be jeopardized by liquidity problems. This is because the continuation value of a firm, even when it is profitable, may be insufficient to outweigh the costs of raising new funds. When issuance costs are lower and the survival of the firm is not at stake, a further decrease in issuance costs will still make the value of the firm less sensitive to current liquidity problems, and its stock price less volatile.[4]

Our approach complements the continuous-time corporate finance literature initiated by Black and Cox (1976) and Leland (1994). A key assumption in this literature is that the shareholders of a company can at no cost inject new liquidity whenever they wish, so that there is no need for cash reserves. This allows one to determine the value of corporate debt in a context where default only occurs when shareholders find it optimal to exercise their limited liability option. Close to the liquidation threshold, this strategy implies that shareholders constantly inject new liquidity up to the point where current operating losses outweigh expected future profitability. Because of the limited liability option, stock prices are convex with respect to the value of the firm's assets. This implies that the dollar volatility of stock prices should decrease after a negative shock, and that risk management activities might decrease shareholder value. In our model, by contrast, the concavity of stock prices with respect to cash reserves provides a natural explanation of why risk management activities might increase shareholder value.

Finally, our paper is technically related to the mathematical finance literature on optimal dividend and liquidity management policies. In particular, it imbeds the pure dividend distribution models of Jeanblanc-Picqué and Shiryaev (1995) and Radner and Shepp (1996), who do not consider the possibility of raising of new funds. More recently, Sethi and Taksar (2002) and Løkka and Zervos (2005) have studied optimal issuance and dividend policies in models with proportional issuance costs. Compared to these papers, a distinctive feature of our analysis is that we explicitly spell out the implications of the firm's optimal issuance and dividend policy for the dynamics of stock prices. Our model is also more realistic in that it allows for fixed issuance costs.

The paper is organized as follows. Section 2 presents the model. Section 3 studies the first-best benchmark that would obtain in the absence of issuance costs. Section 4 characterizes the optimal issuance and dividend policies. Section 5 draws their asset pricing implications. Section 6 concludes.

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[4]One limitation of our model is that the technology is exogenous, as the size of physical assets and the profitability of the firm are fixed. With an endogenous technology choice, a decrease in transaction costs might encourage firms to choose larger or riskier investment.

2. The Model

The following notation will be maintained throughout the paper. Time is continuous, and labelled by $t \geq 0$. Uncertainty is modelled by a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ over which is defined a standard Wiener process $W = \{W_t; t \geq 0\}$. We let $\{\mathcal{F}_t; t \geq 0\}$ be the $\mathbb{P}$–augmentation of the filtration $\{\sigma(W_s; s \leq t); t \geq 0\}$ generated by W, and $\mathbb{E}$ be the expectation operator associated to $\mathbb{P}$.

A firm has a single investment project of fixed size that generates random cash-flows over time. The cumulative cash-flow process $R = \{R_t; t \geq 0\}$ is an arithmetic Brownian motion with strictly positive drift μ and volatility σ,

$$R_t = \mu t + \sigma W_t \tag{1}$$

for all $t \geq 0$. Hence the volatility of cash-flows is constant in our model. In the absence of financial frictions, this implies a constant volatility of stock returns, see Section 3 below. Our analysis could be extended to more general diffusion processes for cash-flows, along the lines of Shreve, Lehoczky and Gaver (1984). The main point of the paper, which is that issuance costs are a source of heteroskedasticity in stock returns, is however most manifest if we take as primitive a homoskedastic cash-flow process such as (1). Note from (1) that the project can involve operating losses as well as operating profits.

At each date, the project can be continued or liquidated. For simplicity, the liquidation value of the project is set equal to 0. The firm is held by a diffuse basis of risk-neutral security holders. We impose no a priori restrictions on the securities issued by the firm and held by these agents, besides that they are claims with limited liability. Thus the firm could for instance issue bonds distributing a constant coupon, or stocks distributing a more irregular stream of dividends. One of the goal of the analysis is precisely to characterize the optimal security issued by the firm, that is, the optimal flow of payments that security holders are entitled to. Our focus on securities rules out long-term commitments to inject cash into the firm such as credit lines. To meet operating costs, the cash reserves of the firm must therefore always remain non-negative. Security holders discount future payments at the risk-free interest rate $r > 0$.

Consider now the issuance policy of the firm. At each date, the firm can retain part of its earnings, or issue new shares. As discussed in the introduction, this issuance activity typically involves substantial costs. Because we aim at analyzing the firm's optimal security design, we do not want to a priori tilt the balance in favor of any given type of claims by assuming that issuance costs differ across securities. Rather, we will assume that the issuance cost depends only on the absolute amount of funds raised by the firm.

What is the form of these costs? Using detailed evidence on the flotation costs associated to various methods of raising new equity, Smith (1977) documents significant economies of scale in the issuance process: equity issues are very costly for small operations, and this is reflected in a declining average cost of financing. To capture these economies of scale in issuance activity, we posit a combination of variable and fixed issuance costs. First, as in Gomes (2001), Sethi and Taksar (2002), Hennessy and Whited (2005) or Løkka and Zervos (2005), share issues have a constant marginal cost, which can for instance result from a proportional brokerage commission. Thus, for each dollar of new shares issued, the firm receives $1/p$ dollars in cash, where $p > 1$ measures the proportional transaction cost. In addition, a distinctive feature of our model is that each issue of shares involves a fixed

transaction cost $f > 0$. The presence of this fixed cost implies that the firm will raise new funds through lumpy and infrequent share issues. To obtain an order of magnitude for these issuance costs in the case of equity, one can fit a linear cost function to Smith's (1977) data, which cover equity issues in the U.S. between 1971 and 1975. A conservative estimate yields $p = 1.028$ and $f = 80,000$ dollars, see Gomes (2001).

Because of the fixed issuance cost, the firm's issuance policy can without loss of generality be described by an increasing sequence $(\tau_n)_{n \geq 1}$ of $\{\mathcal{F}_t; t \geq 0\}$–adapted stopping times that represent the successive dates at which new shares are issued, along with a sequence $(i_n)_{n \geq 1}$ of $(\mathcal{F}_{\tau_n})_{n \geq 1}$–adapted non-negative random variables representing the total issuance proceeds at these dates. It should be noted that we allow the stopping times $(\tau_n)_{n \geq 1}$ to be infinite, which may for instance happen when the issuance costs are so high that the firm finds it optimal to never raise new funds.

At any date $t \geq 0$,

$$I_t = \sum_{n \geq 1} i_n 1_{\{\tau_n \leq t\}} \tag{2}$$

corresponds to the total issuance proceeds up to and including date t, while

$$F_t = \sum_{n \geq 1} f 1_{\{\tau_n \leq t\}} \tag{3}$$

corresponds to the total fixed issuance costs incurred up to and including date t. We denote by $I = \{I_t; t \geq 0\}$ and $F = \{F_t; t \geq 0\}$ the processes defined by (2)–(3), which are $\{\mathcal{F}_t; t \geq 0\}$–adapted by construction.

What is not retained from earnings is paid out to security holders. Let $L = \{L_t; t \geq 0\}$ be the cumulative payout process. We put no restriction on the process L, besides assuming that it is $\{\mathcal{F}_t; t \geq 0\}$–adapted and right-continuous, and that it is non-decreasing, reflecting the security holders' limited liability. Thus the firm can make payments at no cost, but has to incur issuance costs if it wants to raise new funds from new or old security holders. The net cash inflow from or to security holders is thus $dI_t/p - dF_t - dL_t$, while it would just be $dI_t - dL_t$ in the absence of issuance costs, that is when $p = 1$ and $f = 0$.

In addition to these issuance costs, we also introduce a second type of friction, in the spirit of the free cash-flow theory of Jensen (1986). Specifically, we assume that the managers of the firm engage in wasteful activities when the firm holds cash or other liquid assets. For instance, managers may waste cash on inefficient projects or unjustified expenses that generate private benefits for them. We do not model explicitly the resulting agency costs, and simply assume that they effectively reduce the rate of return on cash held internally from r to $r - \lambda$, where $\lambda \in (0, r]$.[5] The optimal level of cash reserves then results from a trade-off between the agency costs within the firm, which create a cost of holding cash, and the issuance costs on primary markets, which create a precautionary demand for cash as in the inventory models of Baumol (1952) and Tobin (1956). The liquidity management problem would become trivial in the absence of either type of costs. Without agency costs, the firm would fully retain earnings, and make no payments to security holders.[6] Without issuance costs, the firm would hoard

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[5]Instead of cash, the firm could hold other liquid assets. We assume that agency costs also reduce the expected rate of return of these assets, as perceived by the firm, from r to $r - \lambda$.

[6]In fact, if $\lambda = 0$, postponing payments increases the value of the firm at any level of cash reserves, and no optimal dividend policy exists.

no cash at all, and distribute all net earnings using dividends together with share issues or repurchases.[7]

When agency and issuance costs coexist, the cash reserves $M = \{M_t; t \geq 0\}$ of the firm evolve according to

$$M_{0-} = m, \quad dM_t = (r - \lambda)M_t dt + dR_t + \frac{1}{p} dI_t - dF_t - dL_t \tag{4}$$

for all $t \geq 0$. The processes R, I and F are defined by (1)–(3), and $m \geq 0$ represents the initial cash holdings of the firm. The firm goes bankrupt if it cannot meet its short-term operating costs by drawing cash from its reserves or by issuing new shares. Thus (4) represents the dynamics of the cash reserves up to the time

$$\tau_B = \inf\{t \geq 0 \,|\, M_t < 0\} \tag{5}$$

at which the firm goes eventually bankrupt.

It should be noted that we allow the stopping time τ_B to be infinite, which may for instance happen when the issuance costs are so low that the firm finds it optimal to always issue new shares when it runs out of cash, thereby avoiding bankruptcy.

Given an issuance policy $((\tau_n)_{n\geq 1}, (i_n)_{n\geq 1})$, a payout policy L, and initial cash reserves m, the value of the firm is given by

$$v(m; (\tau_n)_{n\geq 1}, (i_n)_{n\geq 1}, L) = \mathbb{E}^m \left[\int_0^{\tau_B} e^{-rt} \left(dL_t - dI_t \right) \right], \tag{6}$$

where $(\tau_n)_{n\geq 1}$, $(i_n)_{n\geq 1}$, I, L and τ_B are related by (1)–(5), and $\mathbb{E}^m$ is the expectation operator induced by the process M starting at m. Note that, by construction,

$$\mathbb{E}^m \left[\int_0^{\tau_B} e^{-rt} dI_t \right] = \mathbb{E}^m \left[\sum_{n\geq 1} e^{-r\tau_n} i_n 1_{\{\tau_n \leq \tau_B\}} \right].$$

We define the corresponding value function as

$$V^*(m) = \sup_{(\tau_n)_{n\geq 1}, (i_n)_{n\geq 1}, L} \{v(m; (\tau_n)_{n\geq 1}, (i_n)_{n\geq 1}, L)\} \tag{7}$$

for all $m \geq 0$, where the supremum in (7) is taken over all admissible issuance and payout policies. It is technically convenient to extend the value function V^* to $(-\infty, 0)$ by setting $V^*(m) = 0$ for all $m < 0$. This allows us to put no restrictions on the issuance proceeds $(i_n)_{n\geq 1}$ besides that they remain non-negative.

Remark. Our setup embeds the pure dividend distribution models of Jeanblanc-Picqué and Shiryaev (1995) or Radner and Shepp (1996) as special cases, in which the proportional cost p or the fixed cost f are very high. In this situation, issuing new shares is not a profitable option for the firm, which is then liquidated as soon as it runs out of cash. As we will see in Section 4, the case where new share issues are not feasible actually plays a key role in the analysis. Our setup also embeds the model of Løkka and Zervos (2005) as a limit case, in which share issues involve a proportional cost but no fixed cost. It should be noted that, in

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[7]The case where $p = 1$, $f = 0$ and $\lambda > 0$ is studied in detail in Section 3.

these papers, the cash-flow generated by the firm cannot be reinvested. In our agency cost interpretation, this corresponds to the extreme case in which cash-flows can be reinvested but the resulting proceeds are entirely wasted on organization inefficiencies, that is $\lambda = r$. Our analysis covers the case $\lambda \in (0, r)$ as well.

Our objective in the remainder of the paper is twofold. First, we characterize the optimal value function, as well as the optimal issuance and payout policies that maximize the value of the firm. It turns out that the optimal security can be interpreted as stocks. Second, we show how, in the presence of issuance costs, these optimal policies translate into a uniquely determined stock price process whose dynamics we fully characterize, allowing us to derive several testable asset pricing implications.

3. The First-Best Benchmark

Before considering how issuance costs affect the firm's issuance and payout policies, as well as the dynamics of security prices, we examine a benchmark case in which such costs are absent, that is $p = 1$ and $f = 0$. Agency costs of free cash-flow are still present, that is $\lambda > 0$, but as we will see, they do not affect the value of the firm. Indeed, in this first-best environment, the firm is never liquidated, and its value at date 0 is simply the sum of initial cash reserves and of the present value of future cash-flows:

$$\hat{V}(m) = m + \mathbb{E}^m \left[\int_0^\infty e^{-rt} \left(\mu dt + \sigma dW_t \right) \right] = m + \frac{\mu}{r}. \tag{8}$$

In the absence of issuance costs, hoarding cash reserves is of no value to shareholders, while agency costs reduce the rate of return on cash held internally. It is therefore optimal for the firm to distribute all its initial cash reserves m as a special dividend at date 0, and to hold no cash beyond that date. In the absence of other financial frictions, the Modigliani and Miller (1958) logic applies, so that we have many degrees of freedom in designing issuance and payout processes $\hat{I} = \{\hat{I}_t; t \geq 0\}$ and $\hat{L} = \{\hat{L}_t; t \geq 0\}$ that deliver the value (8). Indeed, as can be seen from (6), the only variable that matters is the difference $\hat{L} - \hat{I}$. To illustrate this point, suppose for instance that, after date 0, the flow of payments stays constant per unit of time. Since the firm distributes all its cash reserves m at date 0, this means that the payout process can be written as

$$\hat{L}_t = m 1_{\{t=0\}} + lt \tag{9}$$

for all $t \geq 0$, where $l > 0$ is some arbitrary constant. Allowing for share repurchases, that is, for a non-monotonic issuance process $\hat{I}$, and taking advantage of (4) with $p = 1$ and $f = 0$, the requirement that cash reserves be constant and equal to 0 after date 0 yields

$$\hat{I}_t = (l - \mu)t - \sigma W_t \tag{10}$$

for all $t \geq 0$. This formula just means that new shares are issued or repurchased to exactly offset the difference between payments $l dt$ and earnings $\mu dt + \sigma dW_t$, so that cash reserves are maintained at 0. Applying formula (6) to $(\hat{L}, \hat{I})$ as defined by (9) and (10), and observing that the integral on the right-hand side of (6) includes $\hat{L}_0 = m$, it is immediate to check that the pair $(\hat{L}, \hat{I})$ defined by (9)–(10) delivers the first-best value (8), independently of the payment flow l.

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Now turn to the dynamics of security prices in this frictionless market. Let $\hat{S} = \{\hat{S}_t; t \geq 0\}$ be the process describing the ex-payment price of a share of the security issued by the firm, and $\hat{N} = \{\hat{N}_t; t \geq 0\}$ the process modelling the number of outstanding shares. Without loss of generality, one can set $\hat{N}_0 = 1$. After date 0, the market capitalization $\hat{N}_t \hat{S}_t$ of the firm stays constant at a level μ/r. Hence, at any date $t > 0$, the following relation holds:

$$d\hat{I}_t = d(\hat{N}_t \hat{S}_t) - \hat{N}_t d\hat{S}_t = -\hat{N}_t d\hat{S}_t = -\frac{\mu}{r} \frac{d\hat{S}_t}{\hat{S}_t}, \tag{11}$$

where the first and second equalities reflect that the flow of funds into the firm, whether positive or negative, is entirely absorbed by existing shareholders, as issuing new shares has no impact on the value of the firm.

Assuming as above a constant payment flow $l > 0$ per unit of time after date 0, it follows from (10)–(11) that

$$\frac{d\hat{S}_t}{\hat{S}_t} = r\left(1 - \frac{l}{\mu}\right) dt + \frac{\sigma r}{\mu} dW_t \tag{12}$$

for all $t > 0$. Using the fact that, after date 0, the payment per share and per unit of time is given by $l/\hat{N}_t = lr\hat{S}_t/\mu$, which is strictly positive as $l > 0$, the log-normal dynamics (12) for the security price implies that, at any date $t > 0$,

$$\hat{S}_t = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} \frac{lr\hat{S}_s}{\mu} ds \,|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} \frac{l}{\hat{N}_s} ds \,|\, \mathcal{F}_t\right], \tag{13}$$

$\mathbb{P}$–almost surely. That is, the security price is simply the present value of future payments per share, reflecting the fact that security holders are risk-neutral in our model.

Formulas (12)–(13) can be generalized to more general payout processes. Indeed, fix some non-decreasing process $\hat{L}$ such that $\hat{L}_0 = m$, which for simplicity we shall suppose continuous. To obtain the analogue of formula (13), we must ensure that the security price exhibits no bubble, in the sense that it grows at an expected rate strictly lower than r as in (12). This will be the case if the payout process $\hat{L}$ grows at a fast enough rate. One has the following result.

Proposition 1. *Suppose that there are no issuance costs, that is $p = 1$ and $f = 0$. Consider a payout process $\hat{L}$ such that*

$$\lim_{T \to \infty} \mathbb{E}\left[\exp\left(-\frac{1}{2}\left(\frac{\sigma r}{\mu}\right)^2 T + \frac{\sigma r}{\mu} W_T - \frac{r}{\mu}\hat{L}_T\right)\right] = 0. \tag{14}$$

Then, at any date $t > 0$, the market capitalization of the firm is

$$\hat{N}_t \hat{S}_t = \frac{\mu}{r}, \tag{15}$$

the instantaneous return on the security issued by the firm satisfies

$$\frac{d\hat{S}_t + d\hat{L}_t/\hat{N}_t}{\hat{S}_t} = rdt + \frac{\sigma r}{\mu} dW_t, \tag{16}$$

and the security price is the present value of future payments per share,

$$\hat{S}_t = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} \frac{1}{\hat{N}_s} d\hat{L}_s \,|\, \mathcal{F}_t\right], \tag{17}$$

$\mathbb{P}$–*almost surely.*

The security price dynamics (12) and its generalization (16) are consistent with the log-normal specification adopted by Black and Scholes (1973) and Merton (1973) for modelling stock prices. It should be noted that, while the payout process and therefore the security issued by the firm remain indeterminate, the constant volatility of returns in (16) is a direct implication of the fact that the market capitalization of the firm stays constant over time. As we shall see in Section 5, prices no longer exhibit this feature when issuance costs are taken into account.

4. The Optimal Issuance and Payout Policies

In this section, we characterize the optimal issuance and payout policies when there are issuance and agency costs, that is $p > 1$, $f > 0$ and $\lambda > 0$. We first derive heuristically a system of variational inequalities for the value function V^*. We then prove that this system has a solution satisfying appropriate regularity conditions. Finally, a verification argument establishes that this solution coincides with V^*, from which the optimal issuance and payout policies can be inferred.

4.1. A Heuristic Derivation of the Value Function

To derive the system of variational inequalities satisfied by V^*, suppose for the moment that V^* is twice continuously differentiable over $(0, \infty)$, with a uniformly bounded derivative, and that for each $m \geq 0$ there exists an optimal policy that attains the supremum in (7). Fix some $m > 0$. The policy that consists in making a payment $l \in (0, m)$, and then immediately executing the optimal policy associated with cash reserves $m - l$ must yield no more than the optimal policy:

$$V^*(m) \geq V^*(m - l) + l.$$

Subtracting $V^*(m - l)$ from both sides of this inequality, dividing through by l and letting l go to 0 yields

$$V^{*\prime}(m) \geq 1 \tag{18}$$

for all $m > 0$, as is usual in dividend distribution models. Next, the policy that consists in issuing $i > 0$ worth of new shares, and then immediately executing the optimal policy associated with cash reserves $m + i/p - f$ must yield no more than the optimal policy:

$$V^*(m) \geq V^*\left(m + \frac{i}{p} - f\right) - i.$$

Thus, denoting $m + i/p$ by m', one must have

$$V^*(m) \geq \sup_{m' \in [m, \infty)} \left\{ V^*(m' - f) - p(m' - m) \right\} \tag{19}$$

for all $m > 0$. Finally, consider the policy that consists in abstaining from issuing new shares and from making any payments for $t \wedge \tau_B \equiv \min\{t, \tau_B\}$ units of time, where $t > 0$, after which the optimal policy associated to the cash reserves $m + \int_0^{t \wedge \tau_B} [(\mu + (r - \lambda)M_s)ds + \sigma dW_s]$ is

executed. Again, this policy must yield no more than the optimal policy:

$$V^*(m) \geq \mathbb{E}^m \left[e^{-r(t \wedge \tau_B)} V^* \left(m + \int_0^{t \wedge \tau_B} \{ [\mu + (r - \lambda) M_s] ds + \sigma dW_s \} \right) \right]$$

$$= V^*(m) + \mathbb{E}^m \left[\int_0^{t \wedge \tau_B} e^{-rs} \left\{ -r V^*(M_s) + [\mu + (r - \lambda) M_s] V^{*\prime}(M_s) + \frac{\sigma^2}{2} V^{*\prime\prime}(M_s) \right\} ds \right],$$

where the second inequality follows from Itô's formula. Letting t go to 0 results in

$$-r V^*(m) + \mathcal{L} V^*(m) \leq 0 \tag{20}$$

for all $m > 0$, where the infinitesimal generator $\mathcal{L}$ is defined as

$$\mathcal{L} u(m) = [\mu + (r - \lambda) m] u'(m) + \frac{\sigma^2}{2} u''(m). \tag{21}$$

We shall refer to (18)–(20) as the fundamental system of variational inequalities satisfied by V^*. To move forward, we make the following guess about the optimal strategy. Consider first the issuance policy. Because of the fixed transaction cost associated with new share issues, it is natural to expect that these should be delayed as much as possible. This suggests that, if any issuance activity takes place at all, this must be when the cash reserves hit 0 so as to avoid bankruptcy. Because the model is stationary, we postulate that the optimal issuance policy then consists in issuing a constant dollar amount of shares, or in abstaining from issuing new shares altogether, which triggers bankruptcy. As a result of this, the value of the firm when it runs out of cash is given by

$$V^*(0) = \left[\max_{i \in [0, \infty)} \left\{ V^* \left(\frac{i}{p} - f \right) - i \right\} \right]^+, \tag{22}$$

where $x^+ \equiv \max\{x, 0\}$. Denote by i^* a solution to the maximization problem in (22). It will turn out that i^* is uniquely determined at the optimum. It may be that $i^* = 0$, in which case the firm abstains from issuing new shares, and $V^*(0) = 0$.[8] Whenever $i^* > 0$, the firm issues new shares when it runs out of cash, and $V^*(0) = V^*(i^*/p - f) - i^* > 0$. The quantity $m_0^* = i^*/p - f > 0$ then represents the post issuance level of the cash reserves.

Consider now the payout policy. In line with standard dividend distribution models, it is natural to expect payments to be made as soon as cash reserves hit or exceed a payout boundary $m_1^* > 0$. This implies that

$$V^{*\prime}(m) = 1 \tag{23}$$

for all $m \geq m_1^*$. Since V^* is postulated to be twice continuously differentiable over $(0, \infty)$, (23) implies that, in addition, the following super contact condition[9] holds at the payout boundary m_1^*:

$$V^{*\prime\prime}(m_1^*) = 0. \tag{24}$$

When cash reserves lie in $(0, m_1^*)$, no issuance or payout activity take place, and (20) holds as an equality. It then follows from (21) and (23)–(24) that $V^*(m_1^*) = [\mu + (r - \lambda) m_1^*]/r$.

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[8]Remember our convention that $V^*(m) = 0$ when $m < 0$.

[9]See Dumas (1991) for an insightful discussion of the super contact condition as an optimality condition for singular control problems.

We are thus led to the problem of finding a function V, along with a threshold $m_1 > 0$, that solve the following variational system:

$$V(m) = 0; \quad m < 0, \tag{25}$$

$$V(0) = \left[\max_{m \in [-f, \infty)} \{V(m) - p(m + f)\} \right]^+; \tag{26}$$

$$-rV(m) + \mathcal{L}V(m) = 0; \quad 0 < m < m_1, \tag{27}$$

$$V(m) = \frac{\mu + (r - \lambda)m_1}{r} + m - m_1; \quad m \geq m_1. \tag{28}$$

Note that V may be discontinuous, with a positive jump at 0. However, the maximum in (26) is always attained since V is by construction upper semicontinuous. We shall then proceed as follows. First, we prove that there exists a unique solution V to (25)–(28) that is twice continuously differentiable over $(0, \infty)$. It is then easy to check that V satisfies the variational inequalities (18)–(20) over $(0, \infty)$. One can finally infer from this that V coincides with the value function V^* for problem (7).

4.2. Solving the Variational Inequalities

We solve (25)–(28) as follows. First fix some $m_1 > 0$, and consider the following boundary value problem over $[0, m_1]$:

$$-rV(m) + \mathcal{L}V(m) = 0; \quad 0 \leq m \leq m_1, \tag{29}$$

$$V'(m_1) = 1, \tag{30}$$

$$V''(m_1) = 0. \tag{31}$$

Standard existence results for linear second-order differential equations yield that (29)–(31) has a unique solution over $[0, m_1]$, which we denote by V_{m_1}. By construction, this solution satisfies $V_{m_1}(m_1) = [\mu + (r - \lambda)m_1]/r$. Extending linearly V_{m_1} to $[m_1, \infty)$ as in (28), we obtain a twice continuously differentiable function over $[0, \infty)$, which we denote again by V_{m_1}. The following lemma establishes key monotonicity and concavity properties of V_{m_1}.

Lemma 1. $V'_{m_1} > 1$ and $V''_{m_1} < 0$ over $[0, m_1)$.

Now observe that if there exists a solution V to (25)–(28) that is twice continuously differentiable over $(0, \infty)$, then, by construction, V must coincide with some V_{m_1} over $[0, \infty)$ for an appropriate choice of m_1. This choice is in turn dictated by the boundary condition (26) that V must satisfy at 0. It is therefore crucial to examine the behavior of V_{m_1} and V'_{m_1} at 0. One has the following result.

Lemma 2. $V_{m_1}(0)$ is a strictly decreasing and concave function of m_1, and $V'_{m_1}(0)$ is a strictly increasing and convex function of m_1.

11

Since $\lim_{m_1\downarrow 0} V_{m_1}(0) = \mu/r > 0$ and $\lim_{m_1\downarrow 0} V'_{m_1}(0) = 1 < p$, it follows from Lemma 2 that there exists a unique $\hat{m}_1 > 0$ such that $V_{\hat{m}_1}(0) = 0$, and that there exists a unique $\tilde{m}_1 > 0$ such that $V'_{\tilde{m}_1}(0) = p$. It is easy to verify that $\hat{m}_1 > \tilde{m}_1$ if and only if $V'_{\hat{m}_1}(0) > p$. Lemma 1 along with the fact that $V'_{m_1}(m_1) = 1$ further implies that if $m_1 \geq \tilde{m}_1$, there exists a unique $m_p(m_1) \in [0, m_1)$ such that $V'_{m_1}(m_p(m_1)) = p$. This corresponds to the unique maximum over $[0, \infty)$ of the function $m \mapsto V_{m_1}(m) - p(m + f)$. Note that, by construction, $m_p(\tilde{m}_1) = 0$. There are now two cases to consider.

Case 1. Suppose first that

$$\max_{m\in[-f,\infty)} \{V_{\hat{m}_1}(m) - p(m + f)\} = 0. \tag{32}$$

Condition (32) holds if $\hat{m}_1 \leq \tilde{m}_1$, in which case the proportional cost p of issuance is so high that $V'_{\hat{m}_1}(0) \leq p$, or if $\hat{m}_1 > \tilde{m}_1$ and the fixed cost f of issuance is so high that $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] \leq 0$. Define then the function V by

$$V(m) = \begin{cases} 0 & m < 0, \\ V_{\hat{m}_1}(m) & m \geq 0. \end{cases} \tag{33}$$

Note that, by construction, $V(0) = 0$. Furthermore, condition (32) implies that the function $m \mapsto V(m) - p(m + f)$ reaches its maximum over $[-f, \infty)$ at $-f$. Letting $m_1 = \hat{m}_1$, it is then easy to check that V solves the variational system (25)–(28).

Case 2. Suppose next that

$$\max_{m\in[-f,\infty)} \{V_{\hat{m}_1}(m) - p(m + f)\} > 0. \tag{34}$$

Condition (34) holds whenever p and f are low enough, so that $\hat{m}_1 > \tilde{m}_1$ or equivalently $V'_{\hat{m}_1}(0) > p$, and $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] > 0$. One then has the following lemma.

Lemma 3. *If (34) holds, there exists a unique $\bar{m}_1 \in (\tilde{m}_1, \hat{m}_1)$ such that*

$$V_{\bar{m}_1}(0) = V_{\bar{m}_1}(m_p(\bar{m}_1)) - p[m_p(\bar{m}_1) + f]. \tag{35}$$

Define then the function V by

$$V(m) = \begin{cases} 0 & m < 0, \\ V_{\bar{m}_1}(m) & m \geq 0. \end{cases} \tag{36}$$

Note that Lemma 2 along with $\bar{m}_1 < \hat{m}_1$ implies that $V(0) > 0$. Furthermore, since $\bar{m}_1 > \tilde{m}_1$, the function $m \mapsto V(m) - p(m + f)$ reaches its maximum over $[-f, \infty)$ at $m_p(\bar{m}_1)$. Letting $m_1 = \bar{m}_1$, it is then easy to check that V solves the variational system (25)–(28).

Note that, in either case, the function $m \mapsto V(m) - p(m + f)$ reaches its maximum at a single point, m_0. In Case 1, $m_0 = -f$, while in Case 2, $m_0 = m_p(m_1)$. The following proposition summarizes our findings.

Proposition 2. *There exists a unique solution V to the variational system (25)–(28) that is twice continuously differentiable over $(0, \infty)$. Moreover, V satisfies the variational inequalities (18)–(20) over $(0, \infty)$.*

4.3. The Verification Argument

In this subsection, we establish that the solution V to (25)–(28) coincides with the value function V^* for problem (7). Our first result is that V is an upper bound for V^*.

Lemma 4. *For any admissible issuance and payout policy $((\tau_n)_{n\geq 1}, (i_n)_{n\geq 1}, L)$,*

$$V(m) \geq v(m; (\tau_n)_{n\geq 1}, (i_n)_{n\geq 1}, L); \quad m \geq 0.$$

We now construct an admissible policy whose value coincides with V. Given Lemma 4, this establishes that $V^* = V$, and thereby provides the optimal issuance and payout policy. Define $m_0^* = m_0^+$ and $m_1^* = m_1$, where m_0 and m_1 are given by the solution to the variational system (25)–(28). To construct the optimal policy, we rely on the theory of reflected diffusion processes, initiated by Skorokhod (1961). The intuition is that, at the optimum, the cash reserve process can be modelled as a diffusion process that is reflected back each time it hits m_1^*, and that is either absorbed at 0 or jumps to m_0^* each time it hits 0, according to whether Case 1 or 2 holds. Assume without loss of generality that the initial cash reserves m of the firm are below m_1^*.[10] The precise formulation of this process is given by the solution to the following version of Skorokhod's problem:

$$M_t^* = m + \int_0^t [\mu + (r - \lambda)M_s^*]\, ds + \sigma W_t + \sum_{n\geq 1} m_0^* 1_{\{T_n^* \leq t\}} - L_t^*, \tag{37}$$

$$M_t^* \leq m_1^*, \tag{38}$$

$$L_t^* = \int_0^t 1_{\{M_s^* = m_1^*\}}\, dL_s^*, \tag{39}$$

for all $t \in [0, \tau_B^*]$, where $\tau_B^* = \inf\{t \geq 0 \,|\, M_t^* < 0\}$ and the sequence of stopping times $(T_n^*)_{n\geq 1}$ is recursively defined by

$$T_n^* = \inf\{t \geq T_{n-1}^* \,|\, M_{t-}^* = 0\}; \quad n \geq 1, \tag{40}$$

where $T_0^* = 0$. Standard results on Skorokhod's problem along with the strong Markov property imply that there exists a pathwise unique solution $(M^*, L^*) = \{(M_t^*, L_t^*); t \geq 0\}$ to (37)–(40). Condition (39) requires that L^* increases only when M^* hits the boundary m_1^*, while (37)–(38) express that this causes M^* to be reflected back at m_1^*. A key property is that L^* is a continuous process. As for the behavior of M^* at 0, two cases can arise. If (32) holds, $m_0^* = (-f)^+ = 0$, so that $\tau_B^* = T_1^*$ $\mathbb{P}$–almost surely. This corresponds to a situation in which the project is liquidated as soon as the firm runs out of cash. By contrast, if (34) holds, $m_0^* = m_p(m_1^*) > 0$. In that case, the process M^* discontinuously jumps to m_0^* each time it hits 0, so that $\tau_B^* = \infty$ $\mathbb{P}$–almost surely. This corresponds to a situation in which an amount $i^* = p(m_0^* + f)$ of new equity is issued when the firm runs out of cash. One has the following result.

Proposition 3. *The value function V^* for problem (7) coincides with the unique solution V to the variational system (25)–(28) that is twice continuously differentiable over $(0, \infty)$. The*

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[10] If $m > m_1^*$, it is optimal to distribute a special dividend $m - m_1^*$ at date 0, after which the dividend process is continuous.

optimal issuance and payout policy is given by $((\tau_n^*)_{n\geq 1}, (i_n^*)_{n\geq 1}, L^*)$, *where*

$$\tau_n^* = \infty, \quad i_n^* = 0; \quad n \geq 1 \quad \textit{if condition (32) holds,}$$

$$\tau_n^* = T_n^*, \quad i_n^* = i^*; \quad n \geq 1 \quad \textit{if condition (34) holds.}$$

According to Proposition 3, the firm's optimal payout policy consists in retaining all its earnings until accumulated cash reserves exceed the threshold m_1^*. When this arises, the firm pays all the excess over m_1^* as dividends. In particular, payments are effected only when the firm has established a sufficiently high performance record. This suggests interpreting the optimal security in our model as equity. The property that dividends are paid only when the firm holds high enough cash reserves corresponds to the fact that, in practice, contractual clauses often preclude stocks to distribute dividends when firms have insufficient liquid assets. DeAngelo, DeAngelo and Stulz (2006) report a significant positive relationship between dividend distribution and the level of retained earnings. One of the interpretation of this phenomenon is that stockholders may be concerned by the discretion that large cash balances provide managers to make self-serving decisions, as in our model.

Regarding the firm's issuance policy, two situations can arise. If condition (32) holds, which intuitively arises when the issuance costs p and f are high, the firm never resorts to outside financing. The model is then essentially equivalent to that of Jeanblanc-Picqué and Shiryaev (1995) or Radner and Shepp (1996), the only difference being that we allow for cash remuneration. By contrast, if issuance costs are low and condition (34) holds, the firm avoids liquidation by issuing new equity when its cash reserves are depleted. Although the firm is never liquidated, its value $V^*(m)$ falls short of the first-best value $\hat{V}(m) = m + \mu/r$ because of the presence of issuance costs. The concavity of V^* over $[0, \infty)$ reflects that the value of firm reacts less to changes in the level of cash reserves when past performance has been high. This is because high accumulated cash reserves allow the firm to postpone the time at which it will have to raise new equity and incur the corresponding issuance costs. By contrast, following unfavorable cash-flow realizations, cash reserves are low, and the value of the firm reacts strongly to performance and ensuing changes in cash reserves. The value function V^* is illustrated on Figure 1 below.

—Insert Figure 1 here—

Three limiting cases of our analysis are worth mentioning. If $\lambda = 0$, equation (29) leads to the first-best firm value $V(m) = m + \mu/r$, which is not implementable by any issuance and dividend policies. Indeed, when this is the case, issuing equity is costly but earnings can be retained at no cost. Shareholders therefore always agree to postpone dividend distribution and no optimal dividend policy exists. The two other limiting cases concern issuance costs. If $p = 1$, equity issues involve no proportional cost. It is then easy to see that, if f is small enough so as to ensure that condition (34) is fulfilled, the dividend boundary m_1^* coincides with the post issuance level of cash reserves of the firm, $m_0^* = m_1^*$. The intuition is that since issues involve only a fixed cost f, it is optimal for the firm to raise as much equity as possible from the market. In that case, equity issues are tied to dividend distribution: following an equity issue and a favorable cash-flow realization, the firm immediately distributes the excess of cash over m_1^* as dividends. By contrast, if f tends to 0, the lump sum amounts of equity issued tend to 0, and in the limit we have $V^{*\prime}(0) = p$ as in the model of Løkka and Zervos

(2005). In that case, the optimal issuance policy is no longer described by an impulse control as in Proposition 3, generating discontinuous jumps in the cash reserves when the firm runs out of cash, but rather by a singular control similar to the optimal dividend process. Equity issues would then occur in infinitesimal amounts and would typically be highly clustered in time. In practice, equity issues are rarely followed by dividend distributions, and firms undertake equity adjustments in lumpy and infrequent issues (Bazdresch (2005), Leary and Roberts (2005)). This is consistent with a combination of fixed and proportional issuance costs such as the one we have postulated.

The characterization of the value function V^* provided in Proposition 2 allows us to study the impact of an increase in issuance costs on the sensitivity of the value of the firm to changes in its cash reserves. To do so, let

$$\epsilon^*(m) = \frac{mV^{*\prime}(m)}{V^*(m)}; \quad m \geq 0 \tag{41}$$

denote the elasticity of the value of the firm with respect to its cash reserves. To focus on an interesting case, we assume in the following result that issuance costs remain low enough so as to guarantee that condition (34) holds and hence that the firm does resort to outside financing at the optimum.

Corollary 1. *The elasticity ϵ^* of the value of the firm with respect to its cash reserves is an increasing function of the issuance costs p and f.*

The proof of this result proceeds as follows. An increase in issuance costs obviously results in a fall in the firm's value, which mechanically raises the elasticity (41). This fall in value is tied to an increase in the dividend boundary m_1^*, reflecting the intuitive fact that as issuance costs increase, the firm must accumulate more liquidities before distributing dividends. Using the non-crossing property of the solutions to (29)–(31), it is then easy to establish that this implies that the marginal value of cash increases with issuance costs, which further raises the elasticity (41).

For a given firm, the concavity of V^* guarantees that the semi-elasticity $m\epsilon^*(m) = V^{*\prime}(m)/V^*(m)$ is a decreasing function of the level m of cash reserves. What Corollary 1 establishes is that this effect is magnified by issuance costs. Intuitively, the percentage change in firm value per percentage change in cash reserves is larger when issuance costs are relatively high, because allowing the firm to postpone a costly new equity issuance is more valuable in this situation. Conversely, the holding of liquid assets is less important when the firm has access to cheap outside financing. A testable implication of this is that firms' valuations should be more responsive to changes in their cash reserves on markets with high costs of issuing equity. Alternatively, a reduction in issuance costs triggered for instance by a capital market deregulation should reduce the responsiveness of firms' valuations to changes in their cash reserves.

5. STOCK PRICES

We are now ready to derive the implications of our theory for stock prices. To focus on the case where the firm does resort occasionally to outside financing, we suppose thereafter that condition (34) holds. We denote by $S^* = \{S_t^*; t \geq 0\}$ the process describing the ex-dividend

price of a share in the firm, and by $N^* = \{N_t^*; t \geq 0\}$ the process modelling the number of shares issued by the firm. Thus at any date $t \geq 0$, S_t^* does not include dividends distributed at date t, while N_t^* includes new shares issued at date t. We assume that N^* is a non-decreasing process and we adopt the normalization $N_{0-}^* = 1$. A key observation is that issuance and payout decisions critically depend on the amount of liquidities accumulated by the firm. As a result of this, the stock price and the number of outstanding shares are contingent on the current level of cash reserves. At any date $t \geq 0$, the value of the firm satisfies

$$V^*(M_t^*) = N_t^* S_t^*. \tag{42}$$

Since shareholders are risk-neutral, absence of arbitrage opportunities requires that, at any date $t \geq 0$, the stock price be equal to the present value of future dividends per share,

$$S_t^* = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} \frac{1}{N_s^*} dL_s^* \,\Big|\, \mathcal{F}_t\right], \tag{43}$$

$\mathbb{P}$–almost surely. One has the following result.

Lemma 5. *The stock price process S^* is $\mathbb{P}$–almost surely continuous.*

The proof of Lemma 5 follows from the fact that the discounted stock price process is the difference between a Brownian martingale and the discounted cumulative dividend process, which are both continuous. It follows in particular that the stock price does not jump at the optimal equity issuance dates: for each $n \geq 1$, we have $S_{\tau_n^*}^* = S_{\tau_n^*-}^*$. This result is not surprising since, in our model, the issuance process is predictable: the firm raises new equity as it runs out of cash, an event that is observable by all participants to the market. In particular, equity issuances do not convey bad news about the profitability of the firm, unlike what typically happens when firms have private information about future profitability (Myers and Majluf (1984)). The fact that the stock price does not react to new equity issuances then simply follows from the absence of arbitrage opportunities.

Now turn to the optimal issuance process I^*. At any date $t \geq 0$,

$$dI_t^* = d[V^*(M_t^*)] - N_{t-}^* dS_t^* = d(N_t^* S_t^*) - N_{t-}^* dS_t^* = S_t^* dN_t^*, \tag{44}$$

where the first equality reflects the fact that part of the change in the value of the firm due to new equity issuance is absorbed by existing shareholders, and the third inequality follows from the fact that N^* is an increasing process and S^* a continuous process.[11] At each issuance date τ_n, the value of the firm discontinuously jumps from $V^*(0)$ to $V^*(m_0^*)$. It follows from (44) that, for each $n \geq 1$, we have $(N_{\tau_n}^* - N_{\tau_n-}^*)S_{\tau_n}^* = V^*(m_0^*) - V^*(0)$. From this the dynamics of N^* easily follows.

Proposition 4. *The process N^* modelling the number of outstanding shares is given by*

$$N_t^* = \begin{cases} 1 & 0 \leq t < \tau_1^*, \\[2mm] \left[\dfrac{V^*(m_0^*)}{V^*(0)}\right]^n & \tau_n^* \leq t < \tau_{n+1}^*. \end{cases} \tag{45}$$

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[11]Indeed, the monotonicity of N^* and the continuity of S^* imply that their quadratic covariation, defined by $[N^*, S^*]_t = N_t^* S_t^* - \int_{0+}^t N_{s-}^* dS_s^* - \int_{0+}^t S_s^* dN_s^*$ for all $t \geq 0$, is a constant (Protter (1990, Chapter II, Theorems 26 and 28)).

According to (45), each time new equity is raised, the ratio of new shares to outstanding shares is constant and equal to $[V^*(m_0^*) - V^*(0)]/V^*(0)$, which corresponds to a constant dilution factor. The number of shares is constant between two consecutive issuance dates. Thus, for all $n \geq 0$ and $t \in [\tau_n^*, \tau_{n+1}^*)$, one has $dS_t^* = d[V^*(M_t^*)]/N_{\tau_n^*}^*$. Using Itô's formula along with (27), together with the facts that the dividend process L^* increases only when cash reserves M^* hit m_1^* and that $V^*(m_1^*) = [\mu + (r - \lambda)m_1^*]/r$ and $V^{*\prime}(m_1^*) = 1$, it is easy to derive the following result.

Proposition 5. *Between two consecutive issuance dates τ_n^* and τ_{n+1}^*, the instantaneous return on stocks satisfies*

$$\frac{dS_t^* + dL_t^*/N_t^*}{S_t^*} = rdt + \sigma^*(N_{\tau_n^*}^* S_t^*)dW_t, \tag{46}$$

where the volatility of stock return is given by

$$\sigma^*(v) = \sigma \frac{V^{*\prime}((V^*)^{-1}(v))}{v} \tag{47}$$

for all $v \in [V^*(0), V^*(m_1^*)]$.

Along with the characterization of the value function V^* provided in Section 3, this result implies that the dynamics of the stock price S^* differs in three main important ways from the log-normal specification postulated by Black and Scholes (1973) and Merton (1973), and derived in equation (16) in the first-best benchmark.

First, since the function V^* is strictly increasing and strictly concave over $[0, \infty)$, it follows from (47) that the volatility $\sigma^*(N^* S^*)$ of stock returns is a decreasing function of S^*. Thus changes in the volatility of stock returns are negatively correlated with stock price movements: between two consecutive issuance dates, volatility tends to rise in response to bad news, and to fall in response to good news. Therefore our model predicts heteroscedasticity in stock prices, as documented for instance by Black (1976), Christie (1982) and Nelson (1991). While this leverage effect that ties stock returns and volatility changes cannot be attributed to financial leverage, as our firm is 100% equity financed, one can argue following Black (1976) that the firm has operating leverage, as it must occasionally resort to costly outside financing to continue its activity. When earnings fall, the likelihood that these expenses will have to be incurred in the near future raises. As the value of the firm declines, it becomes more volatile, as small changes in earnings result in large changes in the difference between earnings and anticipated financial costs. Another testable implication of our model is that the value of the firm is always more volatile than the cash-flows, $\sigma^*(N^* S^*) \geq \sigma$, with equality only at the dividend boundary.

Next, the dollar volatility of the stock price also increases after negative shocks, as was observed by Black (1976). This follows again from the fact that the stock price is a concave function of the level of cash reserves, and therefore that the marginal value of cash for shareholders decreases with the level of cash reserves. This in turn creates a need for risk management, whose goal is to reduce the impact on stock prices of negative shock on earnings. Our model thus provides a rationale for why risk management might increase shareholder value. This contrasts with corporate finance models à la Leland (1994), in which liquidity

17

does not matter as deep pocket shareholders can inject new funds in the firm whenever they wish to do so. In this alternative approach, the limited liability option makes stock prices a convex function of the value of the firm's assets, so that the dollar volatility of stock prices decreases after a negative shock.

The last difference between the stock price process (46) and the standard log-normal specification is that the stock price cannot take arbitrarily large values. This is because the stock price is reflected back each time dividends are distributed, which occurs at the stock price threshold $V^*(m_1^*)/N^*$. As a result of this, as more stocks are issued, the number of outstanding shares N^* increases, which modifies both the stock price threshold $V^*(m_1^*)/N^*$ at which dividends are distributed and the volatility $\sigma^*(N^*S^*)$ of stock returns. Thus our model predicts that, as more equity issuances take place, both the stock prices and the volatility on their return tend to fall, making the stock price dynamics path dependent.

It should be noted that, while the stock price processes in the first-best benchmark and in the presence of issuance costs are qualitatively very different, there is nevertheless some formal analogy between (16) and (46). Indeed, in the absence of issuance costs, the value of the firm as a function of its cash reserves has a slope equal to 1, see (8), and the market capitalization of the firm stays constant at μ/r, see (15). Substituting formally in (46), one retrieve exactly formula (16).

It is instructive to compare the stock price process (46) with that arising in the dynamic agency models of DeMarzo and Sannikov (2006), or Biais, Mariotti, Plantin and Rochet (2007). Much like in our framework, these models predict that stock return volatility tends to increase in response to bad performance. However, the mechanism that leads to this result is different. Agency costs in these models typically make it optimal to liquidate the project as soon as the firm runs short of cash. This is what generates a concavity of the firm value and of the stock price in the level of liquidities that the firm has accumulated. In the implementation of the optimal contract, it is never optimal to issue new securities as the firm becomes illiquid. By contrast, time-varying volatility arises in our model precisely because raising new funds from the market is costly.

In line with Corollary 1, it is easy to characterize the impact of an increase in issuance costs on the volatility of stock returns. Again, we assume that condition (34) holds so that the firm does resort to outside financing at the optimum.

Corollary 2. *The volatility σ^* of stock returns as a function of the value of the firm is an increasing function of the issuance costs p and f.*

The proof follows from the fact that V^* is a decreasing function of p and f, while $V^{*\prime}$ is an increasing function of p and f. A testable implication of this result is that, all other things being equal, a reduction in issuance costs triggered by a capital market liberalization should lead to a fall in the volatility of stock returns.

6. Concluding Remarks

The main insight of this paper is that the introduction of exogenous issuance costs is enough to generate heteroskedasticity of stock prices, even when earnings are independently and identically distributed. It would be important to extend this result in several directions. One promising extension would be to allow the profitability of the firm to evolve randomly.

This would lead to a more complex Markov model, with two state variables corresponding to the two most important dimensions of the firms' financial policy, liquidity and solvency. Introducing new investment opportunities would also modify the analysis in an important way. Indeed, a decrease in issuance costs is likely to encourage firms to invest more and to choose more risky projects, which may counteract the stabilizing effect identified in this paper. These, and related questions must await for future work.

Proof of Proposition 1. From (4), the requirement that cash reserves be constant and equal to 0 after date 0 yields the following analogue of (10):

$$\hat{I}_t = \hat{L}_t - \mu t - \sigma W_t \tag{A.1}$$

for all $t \geq 0$. Using (11) along with (A.1) then yields

$$\frac{d\hat{S}_t}{\hat{S}_t} = rdt + \frac{\sigma r}{\mu}\, dW_t - \frac{r}{\mu}\, d\hat{L}_t \tag{A.2}$$

for all $t > 0$. The value of the firm at any date $t > 0$ is the present value of future cash-flows, hence (15). This allows in turn to rewrite (A.2) as (16). One has the following lemma.

Lemma A.1. *Given an initial condition $\hat{S}_{0+} > 0$, the stochastic differential equation (A.2) has a unique strong solution, given by*

$$\hat{S}_t = \hat{S}_{0+} \exp\left(\left[r - \frac{1}{2}\left(\frac{\sigma r}{\mu}\right)^2\right] t + \frac{\sigma r}{\mu}\, W_t - \frac{r}{\mu}\, \hat{L}_t\right) \tag{A.3}$$

for all $t > 0$.

Proof. Using Itô's formula (Protter (1990, Chapter II, Theorem 32)), it is easy to check that the process $\hat{S} = \{\hat{S}_t; t > 0\}$ defined by (A.3) solves the stochastic differential equation (A.2). Consider now another solution $\tilde{S} = \{\tilde{S}_t; t > 0\}$ to (A.2) with the same initial condition $\hat{S}_{0+}$ as $\hat{S}$. Applying again Itô's formula, one can verify that, for each $t > 0$,

$$\mathbb{E}\left[(\hat{S}_t - \tilde{S}_t)^2\right] = \left(2r + \frac{\sigma^2 r^2}{2\mu^2}\right)\int_0^t \mathbb{E}\left[(\hat{S}_s - \tilde{S}_s)^2\right] ds - \frac{2r}{\mu}\, \mathbb{E}\left[\int_0^t (\hat{S}_s - \tilde{S}_s)^2\, d\hat{L}_s\right]$$

$$\leq \left(2r + \frac{\sigma^2 r^2}{2\mu^2}\right)\int_0^t \mathbb{E}\left[(\hat{S}_s - \tilde{S}_s)^2\right] ds$$

$$\leq 0,$$

where the first inequality follows from the fact that $\hat{L}$ is a non-decreasing process, and the second from the first and Gronwall's lemma. Thus one has $\hat{S}_t = \tilde{S}_t$ with probability 1 for all $t > 0$. Since the processes $\hat{S}$ and $\tilde{S}$ are $\mathbb{P}$–almost surely continuous, the result follows (Karatzas and Shreve (1991, Chapter 1, Problem 1.5)). $\square$

Note that the initial condition for $\hat{S}$ in Lemma A.1 is stipulated at time 0^+, that is, immediately after the special dividend distribution m at time 0. Letting $\hat{N}_{0+} = 1$ without loss of generality, (15) yields $\hat{S}_{0+} = \mu/r$. To conclude the proof, we need only to check that (17) holds. This requires the following lemma.

Lemma A.2. *Suppose that condition (14) holds. Then, for each $t \geq 0$,*

$$\lim_{T \to \infty} \mathbb{E}\left[\exp\left(-\frac{1}{2}\left(\frac{\sigma r}{\mu}\right)^2 T + \frac{\sigma r}{\mu}\, W_T - \frac{r}{\mu}\, \hat{L}_T\right) \mid \mathcal{F}_t\right] = 0, \tag{A.4}$$

$\mathbb{P}$–*almost surely.*

Proof. Denote by $\{X_T; T \geq 0\}$ the random variables within the expectations in (14), and fix for each $t \geq 0$ and $T \geq t$ a random variable $X_{T,t}$ in the equivalence class of $E[X_T \mid \mathcal{F}_t]$. We first show that the random variables $X_{T,t}$, $T \geq t$, have a $\mathbb{P}$–almost surely well-defined limit as T goes to ∞. For each $t \geq 0$, define

$$Z_t = \exp\left(-\frac{1}{2}\left(\frac{\sigma r}{\mu}\right)^2 t + \frac{\sigma r}{\mu} W_t\right).$$

The process $Z = \{Z_t; t \geq 0\}$ is a martingale, and $\mathbb{E}[Z_t] = 1$ for all $t \geq 0$. Now suppose that $T_2 \geq T_1 \geq t$. Then one has

$$\begin{aligned}
X_{T_2,t} &= \mathbb{E}\left[Z_{T_2} \exp\left(-\frac{r}{\mu}\hat{L}_{T_2}\right) \mid \mathcal{F}_t\right] \\[2mm]
&= \mathbb{E}\left[\mathbb{E}\left[Z_{T_2} \exp\left(-\frac{r}{\mu}\hat{L}_{T_2}\right) \mid \mathcal{F}_{T_1}\right] \mid \mathcal{F}_t\right] \\[2mm]
&\leq \mathbb{E}\left[\mathbb{E}[Z_{T_2} \mid \mathcal{F}_{T_1}] \exp\left(-\frac{r}{\mu}\hat{L}_{T_1}\right) \mid \mathcal{F}_t\right] \\[2mm]
&= \mathbb{E}\left[Z_{T_1} \exp\left(-\frac{r}{\mu}\hat{L}_{T_1}\right) \mid \mathcal{F}_t\right] \\[2mm]
&= X_{T_1,t},
\end{aligned}$$

$\mathbb{P}$–almost surely, where the inequality follows from the fact that $\hat{L}$ is a non-decreasing process, and the third equality from the fact that Z is a martingale. It follows that the random variables $X_{T,t}$, $T \geq t$, $\mathbb{P}$–almost surely decrease as a function of T. Since they are positive, they have a $\mathbb{P}$–almost surely well-defined limit as T goes to ∞, as claimed. We now show that this limit is $\mathbb{P}$–almost surely 0, which concludes the proof. Since the process $\hat{L}$ is non-negative,

$$X_{T,t} \leq \mathbb{E}[Z_T \mid \mathcal{F}_t] \leq Z_t,$$

$\mathbb{P}$–almost surely, where the second inequality follows from the fact that Z is a martingale. Since $\mathbb{E}[Z_t] = 1$, the positive random variables $X_{T,t}$, $T \geq t$, are uniformly bounded above by an integrable random variable. Since they converge $\mathbb{P}$–almost surely to a well defined limit as T goes to ∞,

$$\mathbb{E}\left[\lim_{T\to\infty} X_{T,t}\right] = \lim_{T\to\infty}\mathbb{E}[X_{T,t}] = \lim_{T\to\infty}\mathbb{E}[X_T] = 0,$$

where the first equality follows from Lebesgue's dominated convergence theorem, and the last from (14). Since $\lim_{T\to\infty} X_{T,t}$ is a non-negative random variable, the result follows. $\square$

We are now ready to complete the proof. Using (15) and (A.2), it is easy to verify that, for each $t \geq 0$ and $T \geq t$,

$$e^{-rt}\hat{S}_t = \mathbb{E}[e^{-rT}\hat{S}_T \mid \mathcal{F}_t] + \mathbb{E}\left[\int_t^T e^{-rs}\frac{1}{\hat{N}_s}\,d\hat{L}_s \mid \mathcal{F}_t\right],$$

$\mathbb{P}$–almost surely. It follows from Lemmas A.1 and A.2 that the first term on the right-hand side of this equation goes to 0 as T goes to ∞, $\mathbb{P}$–almost surely. Since $\hat{L}$ is an increasing process and $\hat{N}$ remains strictly positive, the result follows from the monotone convergence theorem. $\blacksquare$

Proof of Lemma 1. Since V_{m_1} is smooth over $[0, m_1)$, differentiating (29) and using the definition (21) of $\mathcal{L}$ yields that $-\lambda V'_{m_1} + \mathcal{L}V'_{m_1} = 0$ over $[0, m_1)$. Using this along with (30)–(31), we obtain

that $V'''_{m_1-}(m_1) = 2\lambda/\sigma^2 > 0$. Since $V''_{m_1}(m_1) = 0$ and $V'_{m_1}(m_1) = 1$, it follows that $V''_{m_1} < 0$ and thus $V'_{m_1} > 1$ over an interval $(m_1 - \varepsilon, m_1)$ for $\varepsilon > 0$. Now suppose by way of contradiction that $V'_{m_1}(m) \le 1$ for some $m \in [0, m_1 - \varepsilon]$, and let $\tilde{m} = \sup\{m \in [0, m_1 - \varepsilon] \,|\, V'_{m_1}(m) \le 1\} < m_1$. Then $V'_{m_1}(\tilde{m}) = 1$ and $V'_{m_1} > 1$ over $(\tilde{m}, m_1)$, so that $V_{m_1}(m_1) - V_{m_1}(m) > m_1 - m$ for all $m \in (\tilde{m}, m_1)$. Since $V_{m_1}(m_1) = [\mu + (r - \lambda)m_1]/r$, this implies that for any such m,

$$V''_{m_1}(m) = \frac{2}{\sigma^2}\left\{rV_{m_1}(m) - [\mu + (r - \lambda)m]V'_{m_1}(m)\right\}$$

$$< \frac{2}{\sigma^2}\left\{r[m - m_1 + V_{m_1}(m_1)] - \mu - (r - \lambda)m\right\}$$

$$= \frac{2}{\sigma^2}\lambda(m - m_1)$$

$$< 0,$$

which contradicts the fact that $V'_{m_1}(\tilde{m}) = V'_{m_1}(m_1) = 1$. Therefore $V'_{m_1} > 1$ over $[0, m_1)$, from which it follows as above that $V''_{m_1} < 0$ over $[0, m_1)$. Hence the result. ∎

Proof of Lemma 2. Consider the solutions H_0 and H_1 to be the linear second-order differential equation $-rH + \mathcal{L}H = 0$ over $[0, \infty)$ that are characterized by the initial conditions $H_0(0) = 1$, $H'_0(0) = 0$, $H_1(0) = 0$ and $H'_1(0) = 1$. We first show that H'_0 and H'_1 are strictly positive over $(0, \infty)$. Consider H'_0. Since $H_0(0) = 1$ and $H'_0(0) = 0$, one has $H''_0(0) = 2r/\sigma^2 > 0$, so that $H'_0 > 0$ over an interval $(0, \varepsilon)$ for $\varepsilon > 0$. Suppose that $\tilde{m} = \inf\{m \ge \varepsilon \,|\, H'_0(m) \le 0\} < \infty$. Then $H'_0(\tilde{m}) = 0$ and $H''_0(\tilde{m}) \le 0$. Since $-rH_0 + \mathcal{L}H_0 = 0$, it follows that $H_0(\tilde{m}) \le 0$, which stands in contradiction with the facts that $H_0(0) = 1$ and that H_0 is strictly increasing over $[0, \tilde{m}]$. Thus $H'_0 > 0$ over $(0, \infty)$, as claimed. The proof for H'_1 is similar, and is therefore omitted. Note that both H_0 and H_1 remain strictly positive over $(0, \infty)$. Next, let $W_{H_0, H_1} = H_0 H'_1 - H_1 H'_0$ be the Wronskian of H_0 and H_1. One has $W_{H_0, H_1}(0) = 1$ and

$$W'_{H_0, H_1}(m) = H_0(m)H''_1(m) - H_1(m)H''_0(m)$$

$$= \frac{2}{\sigma^2}\left(H_0(m)\{rH_1(m) - [\mu + (r - \lambda)m]H'_1(m)\} - H_1(m)\{rH_0(m) - [\mu + (r - \lambda)m]H'_0(m)\}\right)$$

$$= -\frac{2[\mu + (r - \lambda)m]}{\sigma^2}W_{H_0, H_1}(m)$$

for all $m \ge 0$, from which Abel's identity follows by integration:

$$W_{H_0, H_1}(m) = \exp\left(-\frac{2\mu m + (r - \lambda)m^2}{\sigma^2}\right) \tag{A.5}$$

for all $m \ge 0$. Since $W_{H_0, H_1} > 0$, H_0 and H_1 are linearly independent. As a result of this, (H_0, H_1) is a basis of the 2-dimensional space of solutions to the equation $-rH + \mathcal{L}H = 0$. It follows in particular that for any $m_1 > 0$, one can represent V_{m_1} as

$$V_{m_1} = V_{m_1}(0)H_0 + V'_{m_1}(0)H_1$$

over $[0, m_1]$. Using the boundary conditions $V_{m_1}(m_1) = [\mu + (r - \lambda)m_1]/r$ and $V'_{m_1}(m_1) = 1$, one can

solve for $V_{m_1}(0)$ and $V'_{m_1}(0)$ as follows:

$$V_{m_1}(0) = \frac{H'_1(m_1)[\mu + (r - \lambda)m_1]/r - H_1(m_1)}{W_{H_0,H_1}(m_1)}, \tag{A.6}$$

$$V'_{m_1}(0) = \frac{H_0(m_1) - H'_0(m_1)[\mu + (r - \lambda)m_1]/r}{W_{H_0,H_1}(m_1)}. \tag{A.7}$$

Using the explicit expression (A.5) for W_{H_0,H_1} along with the fact that H_0 and H_1 are solutions to $-rH + \mathcal{L}H = 0$, it is easy to verify from (A.6)–(A.7) that

$$\frac{dV_{m_1}(0)}{dm_1} = -\frac{\lambda}{r} \exp\left(\frac{2\mu m_1 + (r - \lambda)m_1^2}{\sigma^2}\right) H'_1(m_1),$$

$$\frac{d^2V_{m_1}(0)}{dm_1^2} = -\frac{2\lambda}{\sigma^2} \exp\left(\frac{2\mu m_1 + (r - \lambda)m_1^2}{\sigma^2}\right) H_1(m_1),$$

$$\frac{dV'_{m_1}(0)}{dm_1} = \frac{\lambda}{r} \exp\left(\frac{2\mu m_1 + (r - \lambda)m_1^2}{\sigma^2}\right) H'_0(m_1).$$

$$\frac{d^2V'_{m_1}(0)}{dm_1^2} = \frac{2\lambda}{\sigma^2} \exp\left(\frac{2\mu m_1 + (r - \lambda)m_1^2}{\sigma^2}\right) H_0(m_1).$$

The result then follows immediately from the fact that $\lambda > 0$ and that H_0, H'_0, H_1 and H'_1 are strictly positive over $\mathbb{R}_{++}$. ∎

Proof of Lemma 3. Equation (35) can be rewritten as $\varphi(\bar{m}_1) = 0$, where

$$\varphi(m_1) = V_{m_1}(m_p(m_1)) - V_{m_1}(0) - p[m_p(m_1) + f].$$

If $\hat{m}_1 > \tilde{m}_1$, the function φ is well defined and continuous over $[\tilde{m}_1, \hat{m}_1]$, while $\varphi(\tilde{m}_1) = -pf < 0$ and $\varphi(\hat{m}_1) = V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] > 0$ if the second half of condition (34) holds. Thus φ has at least a zero over $(\tilde{m}_1, \hat{m}_1)$. To prove that it is unique, we show that φ is strictly increasing over $(\tilde{m}_1, \hat{m}_1)$. Using the Envelope Theorem to evaluate the derivative of φ, this amounts to

$$\frac{\partial W}{\partial m_1}(m_p(m_1), m_1) > \frac{\partial W}{\partial m_1}(0, m_1)$$

for all $m_1 \in (\tilde{m}_1, \hat{m}_1)$, where $W(m, m_1) = V_{m_1}(m)$ for all $(m, m_1) \in [0, \infty) \times (\tilde{m}_1, \hat{m}_1)$. Since $m_p(m_1) \in (0, m_1)$ for all $m_1 \in (\tilde{m}_1, \hat{m}_1)$, all that needs to be established is that for any such m_1, $(\partial W/\partial m_1)(\cdot, m_1)$ is strictly increasing over $[0, m_1]$. From (29)–(31), it is easy to check that $(\partial W/\partial m_1)(\cdot, m_1)$ solves the following boundary value problem over $[0, m_1]$:

$$-r\frac{\partial W}{\partial m_1}(m, m_1) + \mathcal{L}\frac{\partial W}{\partial m_1}(m, m_1) = 0; \quad 0 \le m \le m_1, \tag{A.8}$$

$$\frac{\partial^2 W}{\partial m \partial m_1}(m_1, m_1) = 0, \tag{A.9}$$

$$\frac{\partial^3 W}{\partial^2 m \partial m_1}(m_1, m_1) = -\frac{2\lambda}{\sigma^2}. \tag{A.10}$$

We are interested in the sign of $(\partial^2 W/\partial m \partial m_1)(m, m_1)$ for $m \in [0, m_1]$. As $(\partial^2 W/\partial m \partial m_1)(m_1, m_1) = 0$ and $(\partial^3 W/\partial^2 m \partial m_1)(m_1, m_1) < 0$, $(\partial^2 W/\partial m \partial m_1)(\cdot, m_1) > 0$ over an interval $(m_1 - \varepsilon, m_1)$ for $\varepsilon > 0$.

Now suppose by way of contradiction that $(\partial^2 W/\partial m \partial m_1)(m, m_1) \le 0$ for some $m \in [0, m_1 - \varepsilon]$, and let $\tilde{m} = \inf\{m \in [0, m_1 - \varepsilon] \mid (\partial^2 W/\partial m \partial m_1)(m, m_1) \le 0\}$. Then $(\partial^2 W/\partial m \partial m_1)(\tilde{m}, m_1) = 0$ and $(\partial^2 W/\partial m \partial m_1)(m, m_1) > 0$ for all $m \in (\tilde{m}, m_1)$, so that $(\partial W/\partial m_1)(m, m_1) < 0$ for all $m \in (\tilde{m}, m_1)$ as $(\partial W/\partial m_1)(m_1, m_1) = -\lambda/r < 0$ by (A.8)–(A.10). This implies that for any such m,

$$\frac{\partial^3 W}{\partial^2 m \partial m_1}(m, m_1) = \frac{2}{\sigma^2}\left\{ r\,\frac{\partial W}{\partial m_1}(m, m_1) - [\mu + (r - \lambda)m]\,\frac{\partial^2 W}{\partial m \partial m_1}(m, m_1)\right\} < 0,$$

which contradicts the fact that $(\partial^2 W/\partial m \partial m_1)(\tilde{m}, m_1) = (\partial^2 W/\partial m \partial m_1)(m_1, m_1) = 0$. Therefore $(\partial^2 W/\partial m \partial m_1)(\cdot, m_1) > 0$ over $[0, m_1)$, and the result follows. Note for further reference that the above argument shows that $(\partial W/\partial m_1)(\cdot, m_1) < 0$ over $[0, m_1]$. ∎

Proof of Proposition 2. We first establish uniqueness. As explained in the text, any solution V to (25)–(28) that is twice continuously differentiable over $(0, \infty)$ must coincide with some V_{m_1} over $[0, \infty)$. Since $V(0)$ must be non-negative by (26), one must have $m_1 \le \hat{m}_1$. Suppose first that $\hat{m}_1 \le \tilde{m}_1$, and that $m_1 < \hat{m}_1$. Then $V(0) = V_{m_1}(0) > 0$. But since $m_1 < \tilde{m}_1$, one has $V'_+(0) = V'_{m_1}(0) < p$. It follows that the maximum of the mapping $m \mapsto V(m) - p(m + f)$ over $[f, \infty)$ is either attained at $-f$, for a value of 0, or at 0, for a value of $V(0) - pf$. In either case, this is inconsistent with condition (26). It follows that $m_1 = \hat{m}_1$, and thus V is given by (33). Suppose next that $\hat{m}_1 > \tilde{m}_1$. The above argument can be used to show that necessarily $m_1 > \tilde{m}_1$. Two cases must be distinguished. If $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] > 0$, then Lemma 3 establishes the uniqueness of a value $\bar{m}_1$ of $m_1 \in (\tilde{m}_1, \hat{m}_1)$ consistent with condition (26). It follows that $m_1 = \bar{m}_1$, and thus V is given by (36). Suppose finally that $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] \le 0$. Defining φ as in the proof of Lemma 3, and using the fact that φ is strictly increasing over $(\tilde{m}_1, \hat{m}_1)$, we obtain that φ has no zeros over $(\tilde{m}_1, \hat{m}_1)$. Thus condition (26) cannot be satisfied for $m_1 \in (\tilde{m}_1, \hat{m}_1)$. It follows that the maximum of the mapping $m \mapsto V(m) - p(m + f)$ over $[f, \infty)$ must be attained at $-f$, for a value of 0. The only choice of m_1 that is then consistent with (26) is $m_1 = \hat{m}_1$, and thus V is given by (33).

We now verify that our solution V to (25)–(28) satisfies the variational inequalities (18)–(20) over $(0, \infty)$. Inequality (18) follows from (28) and Lemma 1, while inequality (20) follows from (27)–(28) along with the fact that $\lambda > 0$. As for (19), two cases must be distinguished. Suppose first that $\hat{m}_1 \le \tilde{m}_1$, and hence $V'_+(0) \le p$. For any $m \ge 0$, the mapping $m' \mapsto V(m' - f) - p(m' - m)$ is then strictly decreasing over $[m, \infty)$, and thus (19) holds as $V(m) \ge V(m - f)$ for any such m. Suppose next that $\hat{m}_1 > \tilde{m}_1$, and hence $V'_+(0) > p$. If $m \ge m_p(m_1) + f$, the same reasoning as above applies and (19) holds. If $m_p(m_1) + f > m \ge 0$, the maximum of the mapping $m' \mapsto V(m' - f) - p(m' - m)$ over $[m, \infty)$ is attained at $m_p(m_1) + f$, and we must therefore check that

$$V(m) - pm \ge V(m_p(m_1)) - p[m_p(m_1) + f] \tag{A.11}$$

for any such m. The mapping $m \mapsto V(m) - pm$ is strictly increasing over $[0, m_p(m_1)]$, and strictly decreasing over $[m_p(m_1), m_p(m_1) + f]$. Thus we need only to check that (A.11) holds at $m = 0$ and at $m = m_p(m_1) + f$. The latter point is immediate. For the former, two cases must be distinguished. If (32) holds, then $m_1 = \hat{m}_1$, and (A.11) holds at $m = 0$ since the right-hand side is non-positive while the left-hand side is equal to 0 as $V(0) = 0$. If (34) holds, then $m_1 = \bar{m}_1$, and (A.11) holds as an equality at $m = 0$ as $V(0) = V(m_p(\bar{m}_1)) - p[m_p(\bar{m}_1) + f]$. The result follows. ∎

Proof of Lemma 4. Fix an admissible policy $((\tau_n)_{n \ge 1}, (i_n)_{n \ge 1}, L)$, from which the processes I, F and M and the bankruptcy date τ_B can be obtained as in (2)–(5). Let us decompose the process L as

$L_t = L_t^c + \Delta L_t$ for all $t \geq 0$, where L^c is the pure continuous part of L. Itô's formula yields

$$e^{-r(T \wedge \tau_B)} V(M_{T \wedge \tau_B}) = V(m) + \int_0^{T \wedge \tau_B} e^{-rt} [-rV(M_{t-}) + \mathcal{L}V(M_-)] \, dt$$

$$+ \sigma \int_0^{T \wedge \tau_B} e^{-rt} V'(M_{t-}) \, dW_t - \int_0^{T \wedge \tau_B} e^{-rt} V'(M_{t-}) \, dL_t^c \quad \text{(A.12)}$$

$$+ \sum_{t \in [0, T \wedge \tau_B]} e^{-rt} [V(M_t) - V(M_{t-})]$$

for all $T \geq 0$. Since V satisfies (18) by Proposition 2, it follows that for each $t \in [0, T \wedge \tau_B]$,

$$V(M_t) - V(M_{t-}) = V\left(M_{t-} + \frac{\Delta I_t}{p} - \Delta F_t - \Delta L_t\right) - V(M_{t-})$$

$$\leq V\left(M_{t-} + \frac{\Delta I_t}{p} - \Delta F_t\right) - \Delta L_t - V(M_{t-}).$$

Plugging into (A.12) and using again inequality (18) yields

$$e^{-r(T \wedge \tau_B)} V(M_{T \wedge \tau_B}) \leq V(m) + \int_0^{T \wedge \tau_B} e^{-rt} [-rV(M_{t-}) + \mathcal{L}V(M_{t-})] \, dt$$

$$+ \sigma \int_0^{T \wedge \tau_B} e^{-rt} V'(M_{t-}) \, dW_t - \int_0^{T \wedge \tau_B} e^{-rt} \, dL_t$$

$$\quad \text{(A.13)}$$

$$+ \sum_{n \geq 1} e^{-r\tau_n} i_n \mathbf{1}_{\{\tau_n \leq T \wedge \tau_B\}}$$

$$+ \sum_{n \geq 1} e^{-r\tau_n} \left[V\left(M_{\tau_n-} + \frac{i_n}{p} - f\right) - i_n - V(M_{\tau_n-})\right] \mathbf{1}_{\{\tau_n \leq T \wedge \tau_B\}}.$$

Since V' is bounded over $(0, \infty)$, the third term of the left hand side of (A.13) is a square integrable martingale. Using inequalities (19) and (20) along with the fact that V is non-negative by construction, we can take expectations in (A.13) to obtain

$$V(m) \geq \mathbb{E}^m \left[\int_0^{T \wedge \tau_B} e^{-rt} (dL_t - dI_t)\right], \quad \text{(A.14)}$$

from which the result follows by letting T go to ∞. $\blacksquare$

Proof of Proposition 3. Assume that (34) holds, so that $\tau_B = \infty$ $\mathbb{P}$–almost surely, and suppose without loss of generality that $m \in [0, m_1^*]$. The process M^* has paths that are continuous except at the dates $(\tau_n^*)_{n \geq 1}$ at which new shares are issued, in which case $V(M_{\tau_n^*}^*) - V(M_{\tau_n^*-}^*) = V(m_0^*) - V(0) = i^*$ by construction. Proceeding as in the proof of Lemma 4, we obtain that

$$\mathbb{E}^m \left[e^{-rT} V(M_T^*)\right] = V(m) - \mathbb{E}^m \left[\int_0^T e^{-rt} V'(M_{t-}^*) \, dL_t^*\right] + \mathbb{E}^m \left[\sum_{n \geq 1} e^{-r\tau_n^*} i^* \mathbf{1}_{\{\tau_n^* \leq T\}}\right]$$

$$\quad \text{(A.15)}$$

$$= V(m) - \mathbb{E}^m \left[\int_0^T e^{-rt} (dL_t^* - dI_t^*)\right]$$

25

for all $T \geq 0$, where the process $I^* = \{I_t^*; t \geq 0\}$ is defined as in (2) with $i_n^* = i^*$ for all $n \geq 1$, and the second equality follows from (39) along with the fact that $V'(m_1^*) = 1$. From (7) and Lemma 4, one has $V \geq V^*$. It thus follows from (A.15) that, to show that $V = V^*$ and that the issuance and payout policy $((\tau_n^*)_{n \geq 1}, (i_n^*)_{n \geq 1}, L^*)$ is optimal, one needs only to check that $\lim_{T \to \infty} \mathbb{E}^m[e^{-rT} V(M_T^*)] = 0$. Since V is non-negative with bounded derivatives, one has

$$0 \leq e^{-rT} V(M_T^*) \leq e^{-rT} C(1 + M_T^*) \leq e^{-rT} C(1 + m_1^*) \tag{A.16}$$

for all $T \geq 0$, where C is some positive constant, and the third inequality follows from the fact that the process M^* never leaves the interval $[0, m_1^*]$. Taking expectations in (A.16) and letting T go to ∞ yields the result. The proof for the case in which (32) holds is similar, and therefore omitted. $\blacksquare$

Proof of Corollary 1. To establish this result, we show that V^* is a decreasing function of p and f, and that $V^{*\prime}$ is an increasing function of p and f. To prove the first claim, start without loss of generality from a situation in which p and f are such that condition (34) holds, and consider the impact of a decrease in p or f, $p' \leq p$ and $f' \leq f$ with at least one strict inequality. Then the firm can keep the same dividend policy L^*, while adjusting its issuance policy so as to maintain the same dynamics for cash reserves (37) as when the issuance costs are p and f. Indeed, to do so, it needs only to issue amounts $i' = p'(m_0^* + f')$ worth of equity instead of $i^* = p(m_0^* + f)$, at the same dates $(\tau_n^*)_{n \geq 1}$. That is, the new issuance and dividend policy of the firm is $((\tau_n^*)_{n \geq 1}, (i_n')_{n \geq 1}, L^*)$ with $i_n' = i' < i^*$ for all $n \geq 1$. Since the dividend policy and the dynamics of cash reserves are the same as in the initial situation, while the amounts of equity issued are strictly lower, this policy yields a strictly higher value for the firm than in the initial situation. Thus V^* is a decreasing function of p and f, as claimed. Now, using the notation of the proof of Lemma 3, one has $V^* = W(\cdot, m_1^*)$ over $\mathbb{R}_+$. Since $(\partial W / \partial m_1)(\cdot, m_1) < 0$ over $[0, m_1]$, the above argument implies that an increase in either p or f leads to an increase in m_1^*. Since $(\partial^2 W / \partial m \partial m_1)(\cdot, m_1) > 0$ over $[0, m_1]$, it follows that $V^{*\prime}$ is an increasing function of p and f. Hence the result. $\blacksquare$

Proof of Lemma 5. We show precisely that one can find versions of the conditional expectations in (43) such that the resulting process S^* has $\mathbb{P}$–almost surely continuous paths. From (43) it follows that the stock price process S^* is such that, for each $t \geq 0$,

$$e^{-rt} S_t^* = \mathbb{E}\left[\int_0^\infty e^{-rs} \frac{1}{N_s^*} \, dL_s^* \,\Big|\, \mathcal{F}_t \right] - \int_0^t e^{-rs} \frac{1}{N_s^*} \, dL_s^*, \tag{A.17}$$
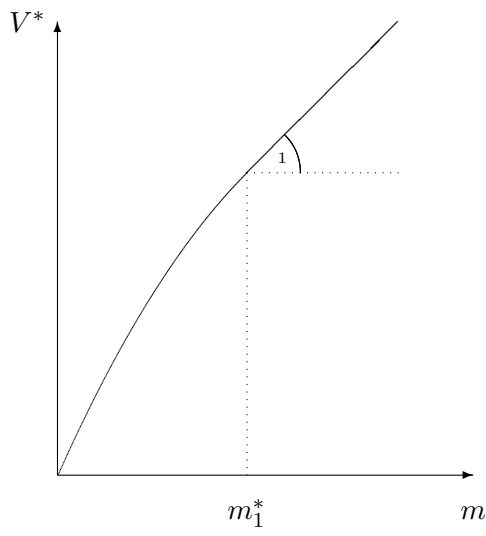
$\mathbb{P}$–almost surely. By choosing for each $t \geq 0$ a random variable Y_t in the equivalence class of $\mathbb{E}\left[\int_0^\infty e^{-rs}(1/N_s^*) \, dL_s^* \,|\, \mathcal{F}_t \right]$, one obtains an $\{\mathcal{F}_t; t \geq 0\}$ adapted martingale $Y = \{Y_t; t \geq 0\}$. Since the filtration $\{\mathcal{F}_t; t \geq 0\}$ is complete and right-continuous, one can choose Y_t for all $t \geq 0$ in such a way that the martingale Y is right-continuous with left-hand limits (Karatzas and Shreve (1991, Chapter 1, Theorem 3.13)). Because $\{\mathcal{F}_t; t \geq 0\}$ is the $\mathbb{P}$–augmentation of the filtration generated by W, Y is in fact $\mathbb{P}$–almost surely continuous (Karatzas and Shreve (1991, Chapter 3, Problem 4.16)). To conclude the proof, observe that because the dividend process L^* is continuous, so is the second term on the right-hand side of (A.17). Hence the result. $\blacksquare$
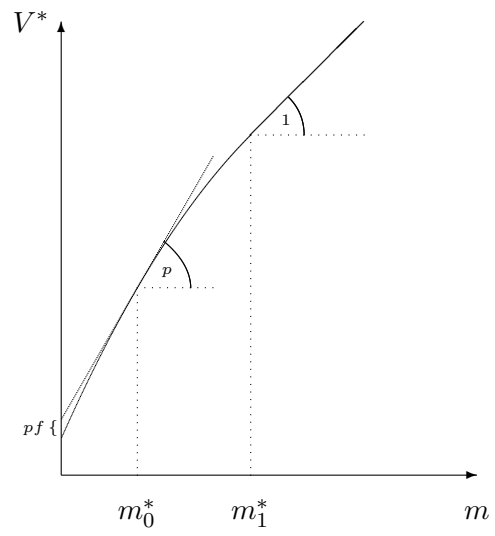
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FIGURE 1. The value function V^*.