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Endogenous Growth

We have already seen one crude endogenous growth model, the so-called "AK" model. It is crude because it does not give a realistic account of the channels through which productivity grows over time – namely, innovation and the creation of new knowledge.

We now turn to a class of models that indeed endogenize the innovative process. The challenge in thinking about these problems is that the creation of knowledge, which has a public-good aspect, is different from the production of other economic goods.

The endogenous growth literature began with contributions of Robert Lucas and especially Paul Romer in the 1980s and 1990s, although the ideas certainly had important precursors in the growth literature of the 1960s.

A Model of Endogenous Growth: The Basic Idea

The model builds on some of the ideas about differentiated products that also underlie the "new trade theory" developed by Krugman and others in the late 1970s and 1980s. In the model, additional "varieties" of differentiated capital goods will boost productivity, and the process through which new capital goods are invented is endogenized.

In this economy, production of a *final* consumption good is given by

$$Y_t = F(K_{1,t}, ..., K_{A_t,t}, L_{Y,t}) = \left(\sum_{j=1}^{A_t} K_{j,t}^{\alpha} \right) L_{Y,t}^{1-\alpha} = \sum_{j=1}^{A_t} K_{j,t}^{\alpha} L_{Y,t}^{1-\alpha},$$

where $L_{Y,t}$ is the amount of labor employed in the final goods sector at t and $j \in \{1, 2, ..., A_t\}$ indexes the different types of capital that can be used in production as of t. Labor not devoted to final-goods production will, as we shall see, be devoted to research and development into new capital goods.

We assume that the capital depreciation rate is $\delta = 1$, so that the price of a machine is its rental rate.

Note some interesting features of this production setup. At any point in time, there are constant returns to scale with respect to the existing factors of production, no matter how many there are. But while the marginal product of an existing capital good is finite, the marginal product of a *new* capital good is infinite.

A different thought experiment gives a good illustration of why the preceding production function can generate endogenous growth. Imagine combining 1 unit each of N capital goods with 1 unit of Labor; we get $Y = N$. Instead, imagine we combine $N/(N+1)$ units each of $N+1$ capital goods with 1 unit of labor. We get

$$\begin{aligned} Y &= \sum_{j=1}^{N+1}\left(\frac{N}{N+1}\right)^{\alpha} = (N+1)\left(\frac{N}{N+1}\right)^{\alpha} \\ &= N^{a}\left(N+1\right)^{1-\alpha} > N. \end{aligned}$$

So with more capital goods, the output/labor ratio rises holding constant the *amount* of capital input (measured in terms of consumption goods). Thus, the creation of new capital goods has the potential to raise productivity and per-worker output over time.

Notice, finally, that if $K_{j,t} = \tilde{K}_t$ for all varieties j (as is the case in equilibrium when all goods are symmetric), then

$$Y_t = \sum_{j=1}^{A_t} \tilde{K}_t^{\alpha} L_{Y,t}^{1-\alpha} = A_t \tilde{K}_t^{\alpha} L_{Y,t}^{1-\alpha} = \tilde{K}_t^{\alpha}\left(A_t^{\frac{1}{1-\alpha}} L_{Y,t}\right)^{1-\alpha},$$

so the production side looks equivalent to what we assumed for the Solow model. What we will add, as we now show, is a model of how A_t grows endogenously over time.

Production of Capital Goods and Blueprints for New Goods

To produce one unit of capital (of any kind) you need exactly one unit of final output. Capital goods are produced by monopolistic firms. To set up a firm you need to purchase a blueprint for the specific variety j of capital good you will produce. (The cost of the blueprint is sunk.) You can then use a unit of output on date t to yield a unit of your capital good j on date $t+1$, which you sell (rent) at price p_j.

We will assume that more labor devoted to research and development (R&D) results in an expanded set of blueprints allowing the production of more varieties of capital. Specifically, if L_A is labor input to the R&D sector,

$$A_{t+1} - A_t = \theta A_t L_{A,t}. \tag{1}$$

According to eq. (1), labor productivity in R&D is proportional to the existing stock of "knowledge" – so in effect, we have learning by doing. This assumption captures the important idea that, as a public good, new knowledge is *nonrival* (more than one person can use it at the same time) and *nonexcludable* (people cannot be prevented from using knowledge). The learning by doing is external to firms; each firm in R&D behaves competitively.

A blueprint can be put into use the very same period in which it is developed. The total labor force L is constant and fully employed,

$$L = L_Y + L_A.$$

Solving the Model: First Steps

The key is to figure out how the labor force is divided between final-goods production and R&D. The more labor goes into R&D, the faster the growth rate of the economy. The level of output of blueprints, in turn, depends on their price in terms of final goods, p_A.

Let us *conjecture* that in equilibrium we will observe a constant real rate of interest r, constant relative prices, a constant demand for each type of capital, and a constant allocation of labor to sectors of the economy. (Later we show that these guesses are all correct.) Let us start by considering the demand of final-goods firms for capital goods, given by the solution to

$$\max_{\{K_j\}} \sum_{j=1}^{A_t} K_j^{\alpha} L_Y^{1-\alpha} - \sum_{j=1}^{A_t} p_j K_j - w L_Y,$$

where p_j (once again) is the output price of capital of type j and w is the wage in terms of final output. The first-order condition for a maximum for K_j is

$$p_j = \alpha K_j^{\alpha-1} L_Y^{1-\alpha}. \tag{2}$$

Thus, the demand for a capital good is[1]

$$K_j = \left(\frac{\alpha}{p_j}\right)^{\frac{1}{1-\alpha}} L_Y.$$

[1] We also see that $(1-\alpha)\sum_{j=1}^{A_t} K_j^{\alpha} L_Y^{-\alpha} = w$.

What does this imply for producers of the intermediate capital goods? The (intertemporal) profits of intermediate producer j are

$$\Pi_j = \frac{p_j K_j}{1+r} - K_j = \frac{\alpha K_j^{\alpha} L_Y^{1-\alpha}}{1+r} - K_j.$$

Maximizing Π_j with respect to K_j yields:

$$\frac{\alpha^2 K_j^{\alpha-1} L_Y^{1-\alpha}}{1+r} - 1 = 0$$

or

$$\bar{K} = \left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y$$

(where the j subscript has been dropped, as all capital goods are symmetric). Substituting this equation into eq. (2) yields the (constant) relative price of a (generic) intermediate capital good:

$$\begin{aligned}
\bar{p} &= \alpha \bar{K}^{\alpha-1} \bar{L}_Y^{1-\alpha} \\
&= \alpha \left[\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y\right]^{\alpha-1} \bar{L}_Y^{1-\alpha} \\
&= \frac{1+r}{\alpha}.
\end{aligned}$$

For a constant elasticity demand function, a standard result is that a monopolist's price is a constant markup over cost.[2] Here we see that

$$\frac{\text{Price}}{\text{Cost}} = \frac{\bar{p}}{1+r} = \frac{1}{\alpha} = \frac{\frac{1}{1-\alpha}}{\frac{1}{1-\alpha} - 1}.$$

The cost of production is 1 on date $t-1$, and the price obtained (also from the perspective of date $t-1$) is $\bar{p}/(1+r)$.

Given all this, what is the profit that a capital-good producer earns? We need to know this because the requirement that the stream of profits covers sunk cost is what ties the model down. Substitution yields:

$$\begin{aligned}
\bar{\Pi} &= \frac{\bar{p}\bar{K}}{1+r} - \bar{K} = \left(\frac{\bar{p}}{1+r} - 1\right)\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y \\
&= \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y. \qquad (3)
\end{aligned}$$

[2] If the price elasticity of demand is η, the markup is $\eta/(\eta-1)$, which goes to 1 as $\eta \to \infty$. In the present model, $\eta = 1/(1-\alpha)$.

There is free entry into producing intermediate goods, so the price of a blueprint must equal the present discounted value of $\bar{\Pi}$ above, or

$$\begin{aligned} \bar{p}_A &= \sum_{t=0}^{\infty} \frac{\bar{\Pi}}{(1+r)^t} = \frac{1+r}{r}\bar{\Pi} \\ &= \frac{1+r}{r}\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y \end{aligned} \tag{4}$$

An important point: if we did not have monopoly in the capital-producing sector, there would be no stream of monopoly profits to cover the sunk cost of blueprints, and so blueprints would never be purchased. In the market setting we have assumed, monopoly – and some degree of monopoly inefficiency – is necessary to sustain positive growth.

Equilibrium Growth Rate

Equilibrium growth in the number of capital goods is given by

$$g = \frac{A_{t+1} - A_t}{A_t} = \theta \bar{L}_A.$$

Production of each specific capital good will remain constant at $\bar{K}$.

What ties down the equilibrium allocation of labor, and hence g, is the preceding eq. (4) for $\bar{p}_A$. Suppose there are too many workers in final goods production (relative to the equilibrium) because workers are paid more in final goods than in R&D. Then the demand for capital (to equip those workers) will be high, raising the profits of intermediate producers and causing them to bid up the price of blueprints $\bar{p}_A$. But that development, in turn will raise the wages paid in the R&D sector, drawing workers out of final goods. The process will continue until wages in the two sectors are equal.

We formalize the requirement that workers have the same marginal value product in both sectors by requiring that

$$\text{MVPL in R\&D} = \bar{p}_A \theta A = (1-\alpha) L_Y^{-\alpha} \sum_{j=1}^{A} \bar{K}^{\alpha} = (1-\alpha) A \bar{K}^{\alpha} L_Y^{-\alpha} = w.$$

The solution is

$$
\begin{aligned}
\bar{L}_Y &= \left[\frac{(1-\alpha)}{\bar{p}_A\theta}\right]^{\frac{1}{\alpha}} \bar{K} \\
&= \left[\frac{r(1-\alpha)}{(1+r)\bar{\Pi}\theta}\right]^{\frac{1}{\alpha}} \bar{K} \\
&= \left[\frac{r(1-\alpha)}{(1+r)\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y\theta}\right]^{\frac{1}{\alpha}} \left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y \Rightarrow \\
1 &= \left[\frac{\alpha r}{(1+r)\bar{L}_Y\theta}\right]^{\frac{1-\alpha}{\alpha}} \left(\frac{\alpha^2}{1+r}\right)^{-\frac{1-\alpha}{\alpha}} \Rightarrow \\
1 &= \left[\frac{\alpha r}{(1+r)\bar{L}_Y\theta}\right] \left(\frac{\alpha^2}{1+r}\right)^{-1} \Rightarrow \\
\bar{L}_Y &= \frac{r}{\alpha\theta}.
\end{aligned}
$$

This is consistent, by the way, with the assumption we made that $\bar{L}_Y$ is constant. We can now also find the long-run rate of growth, which is

$$\bar{g} = \theta\bar{L}_A = \theta(L - \bar{L}_Y) = \theta L - \frac{r}{\alpha}. \tag{5}$$

Notice that there is a "scale effect" here: a bigger work force implies more innovation and hence faster growth. Higher interest rates retard growth – though we have yet to solve for the equilibrium rate of interest r.

Let's do so next. If the lifetime utility function of the representative consumer is

$$U_0 = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}},$$

then the intertemporal Euler equation is

$$C_t^{-\sigma} = \beta(1+r)C_{t+1}^{-\sigma}.$$

In balanced-growth equilibrium, consumption, like productivity, grows at the (gross) rate $1+g$, so

$$1 + g = \frac{C_{t+1}}{C_t} = (1+r)^{\sigma}\beta^{\sigma},$$

which tells us the interest rate is

$$r = \frac{(1+g)^{\frac{1}{\sigma}}}{\beta} - 1.$$

Now combine this solution with eq. (5),

$$\bar{g} = \theta L - \frac{1}{\alpha}\left[\frac{(1+\bar{g})^{\frac{1}{\sigma}}}{\beta} - 1\right],$$

to infer the *equilibrium* rate of growth as the solution to

$$\alpha\beta\bar{g} + (1+\bar{g})^{\frac{1}{\sigma}} = \beta\left(1+\alpha\theta L\right).$$

For example, when $\sigma = 1$, we find that

$$\bar{g} = \frac{\alpha\beta\theta L - (1-\beta)}{1+\alpha\beta}.$$

Growth is higher for higher L, α, β, σ, and θ. (Why?)

Government policy can certainly affect the economic growth rate in this model. For example, suppose the government imposes a fixed fee τ that new firms have to pay for a license to enter the capital-goods industry. This will increase the sunk cost of entry into the production of new capital goods. The break-even condition, based on eqs. (3) and (4), now becomes

$$\bar{p}_A + \tau = \frac{1+r}{r}\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}}\bar{L}_Y.$$

Intuitively, as τ rises from 0, $\bar{p}_A$ will fall and $\bar{L}_Y$, will rise. But with $\bar{L}_A = L - \bar{L}_Y$ therefore lower, the pace of productivity growth will be lower as well.