Economics 202A Lecture Outline, October 28-30, 2008 (version 1.3)

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Stock Prices as Present Values

The most basic theory of the stock market is that a stock's price is the present value of expected future dividends.

Suppose the real interest rate is r, and is constant. Suppose the stocks real dividend in period t is d_t and the stock's *ex dividend* real price (i.e., in terms of output, or more generally, in terms of the CPI basket), is q_t.

Then in a risk-neutral world, we would have the arbitrage condition

$$1+r=\mathrm{E}_t\left\{\frac{d_{t+1}+q_{t+1}}{q_t}\right\}, \tag{1}$$

which equates the gross return on bonds to that on stocks (dividends + capital gains). This works for a time-dependent interest rate r_t as well — do that case as an exercise.

To see how the preceding return relationship translates into a theory of stock pricing, write

$$\begin{aligned}
q_t &= \mathrm{E}_t\left\{\frac{d_{t+1}+q_{t+1}}{1+r}\right\} \\
&= \mathrm{E}_t\left\{\frac{d_{t+1}}{1+r}\right\}+\mathrm{E}_t\left\{\frac{q_{t+1}}{1+r}\right\} \\
&= \mathrm{E}_t\left\{\frac{d_{t+1}}{1+r}\right\}+\mathrm{E}_t\left\{\frac{1}{1+r}\mathrm{E}_{t+1}\left\{\frac{d_{t+2}+q_{t+2}}{1+r}\right\}\right\} \\
&= \mathrm{E}_t\left\{\frac{d_{t+1}}{1+r}\right\}+\mathrm{E}_t\left\{\frac{d_{t+2}}{(1+r)^2}\right\}+\mathrm{E}_t\left\{\frac{q_{t+2}}{(1+r)^2}\right\}.
\end{aligned}$$

Here, I have used the law of iterated conditional expectations, $\mathrm{E}_t\{\mathrm{E}_{t+1}x_{t+2}\}=\mathrm{E}_t\{x_{t+2}\}$.

One can continue the iterative substitution procedure above indefinitely, successively substituting the versions of eq. (1) for dates $t+2$, $t+3$, etc.

The result is

$$q_t = \sum_{i=1}^{\infty} \mathrm{E}_t \left\{ \frac{d_{t+i}}{(1+r)^i} \right\} + \lim_{i \to \infty} \mathrm{E}_t \left\{ \frac{q_{t+i}}{(1+r)^i} \right\}.$$

What to make of the term $\lim_{i\to\infty} \mathrm{E}_t \left\{ \frac{q_{t+i}}{(1+r)^i} \right\}$? This term represents a potential *speculative bubble* (of one particular "rational" kind) in the stock price: it captures the idea of a self-fulfilling frenzy in the asset price. More on this later; for now let's assume there is no bubble. In that case

$$q_t = \mathrm{E}_t \left\{ \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1+r)^i} \right\}, \tag{2}$$

and the stock's price is the expected present value of future dividends.

An important implication of this formula is that changes in stock prices reflect *news*.

Suppose that, within a particular trading instant, people change their expected dividend stream to be $\mathrm{E}_t' \{d_{t+1}\}$. Then the stock price will jump by the amount

$$q_t' - q_t = \mathrm{E}_t' \left\{ \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1+r)^i} \right\} - \mathrm{E}_t \left\{ \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1+r)^i} \right\}, .$$

where this change is uncorrelated with any information available before the revision in market expectations. This is the basic idea of the "random walk" theory of stock prices, or, more broadly, the "efficient markets" view.

As another application, consider the behavior of the stock price from period to period. We have

$$q_{t+1} - q_t = \mathrm{E}_{t+1} \left\{ \sum_{i=1}^{\infty} \frac{d_{t+1+i}}{(1+r)^i} \right\} - \mathrm{E}_t \left\{ \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1+r)^i} \right\}.$$

Let dividends follow the AR(1) process

$$d_{t+1} = \rho d_t + \varepsilon_{t+1},$$

where $\mathrm{E}_t \varepsilon_{t+1} = 0$. Then

$$q_t = \frac{\rho}{1+r-\rho} d_t$$

and

$$\begin{aligned} q_{t+1} - q_t &= \frac{\rho}{1+r-\rho}\left(d_{t+1} - d_t\right) \\ &= \frac{\rho}{1+r-\rho}\left[(\rho-1)d_t + \varepsilon_{t+1}\right]. \end{aligned}$$

Changes in stock prices are proportional to changes in dividends (as in Shiller's excess volatility tests). Also, for ρ near 1, or for a very small time interval, the change in the stock price is essentially proportional to the "news" ε_{t+1} — the innovation in dividends.

We get at the essence of the "efficient markets" hypothesis by examining the ex post *excess return*

$$e_{t+1} = \frac{d_{t+1} + q_{t+1}}{q_t} - 1 - r.$$

Our arbitrage condition guarantees that this is uncorrelated with date t information. In our particular AR(1) example,

$$\begin{aligned} \frac{d_{t+1} + q_{t+1}}{q_t} - 1 - r &= \frac{d_{t+1} + \frac{\rho}{1+r-\rho}d_{t+1}}{\frac{\rho}{1+r-\rho}d_t} - 1 - r \\ &= \frac{(1+r)d_{t+1}}{\rho d_t} - 1 - r \\ &= \frac{(1+r)\left(\rho d_t + \varepsilon_{t+1}\right)}{\rho d_t} - 1 - r \\ &= \frac{\varepsilon_{t+1}}{\rho d_t}. \end{aligned}$$

For any random variable x_t realized as of date t, $\mathrm{E}_t\left\{\frac{\varepsilon_{t+1}}{\rho d_t}x_t\right\} = \frac{x_t}{\rho d_t}\mathrm{E}_t\varepsilon_{t+1} = 0$. The excess return is *unpredictable.*

Note: Even if there is a "rational bubble" in the stock price the preceding implication of unpredictable excess returns will hold. That is because the result follows entirely from eq. (1), rather than from eq. (2).

Summers's Critique on the Interpretation of Efficiency Tests

Some financial economists argued that if one fails to find lagged variables helping to predict excess returns e_t, one can infer that the PDV formula (2) for a stock's price is valid: stocks are priced according to their fundamentals. Larry Summers offers a persuasive critique of this inference in his paper "Does the Stock Market Rationally Reflect Fundamental Values?" on the reading list.

Let q_t^* (temporarily, for this section) denote the PDV price given in equation (2) and imagine that, perhaps do to "fads" in investment preferences or the like, the *actual* stock price q_t is given by

$$q_t = q_t^* \mathrm{e}^{u_t},$$

where the log discrepancy u_t follows an autoregressive process

$$u_t = \alpha u_{t-1} + v_t, \qquad |\alpha| \leq 1,$$

where the innovation v_t is uncorrelated with all economic variables at all leads and lags. (It is a pure "sunspot.") In this alternative model, stock prices can differ from fundamental values due to a slow moving pricing error that can be expected to diminish over time if $|\alpha| < 1$. The question Summers asks is: will standard tests of excess return predictability disclose the presence of this — possibly large — pricing error? His answer is no.

Let's see why. Define the efficient excess return as

$$e_{t+1} = \frac{d_{t+1} + q_{t+1}^*}{q_t^*} - 1 - r$$

and, following Summers, define the actual excess return as

$$z_{t+1} = \frac{d_{t+1} + q_{t+1}}{q_t} - 1 - r$$

Let's adopt the approximations $\frac{q_{t+1}-q_t}{q_t} \approx \log q_{t+1} - \log q_t$ and $\mathrm{e}^{-u_t} \approx 1 - u_t$. (The latter is not going to be a great approximation unless u_t is relatively small, but I am getting closer to the right answer than Summers does. He assumes that $\frac{d_{t+1}}{q_t^*} \approx \frac{d_{t+1}}{q_t}$, which amounts to the very bad approximation $\mathrm{e}^{-u_t} \approx 1$. I worry about Summers's approximation because it is only good when $q_t \approx q_t^*$, whereas the whole point of this exercise is to argue that the two q's can diverge widely.) Then we may write

$$\begin{aligned}
z_{t+1} &\approx \log q_{t+1} - \log q_t + \frac{d_{t+1}}{q_t^*}\mathrm{e}^{-u_t} - r \\
&\approx \log q_{t+1}^* - \log q_t^* + u_{t+1} - u_t + \frac{d_{t+1}}{q_t^*} - \frac{d_{t+1}}{q_t^*}u_t - r \\
&\approx e_{t+1} + u_{t+1} - u_t - \frac{d_{t+1}}{q_t^*}u_t.
\end{aligned}$$

To finish up, imagine that dividends follow the AR(1) process $d_{t+1} = \rho d_t + \varepsilon_{t+1}$. To make life easier, let us take $\rho = 1$. As per our earlier result, we have $q_t^* = d_t\,/r$ and so

$$\frac{d_{t+1}}{q_t^*}u_t = \frac{\rho d_t}{q_t^*}u_t + \frac{\varepsilon_{t+1}}{q_t^*}u_t \approx r u_t,$$

assuming that $\frac{\varepsilon_{t+1}}{q_t^*}u_t$ is small. So

$$\begin{aligned}
z_{t+1} &\approx e_{t+1} + u_{t+1} - u_t - r u_t \\
&= e_{t+1} + v_{t+1} + (\alpha - r - 1)\,u_t.
\end{aligned}$$

Using this approximation, and the fact that $\sigma_u^2 = \sigma_v^2/(1-\alpha^2)$, we find that the variance of z is

$$\begin{aligned}
\sigma_z^2 &= \sigma_e^2 + \sigma_u^2(1-\alpha^2) + (\alpha - r - 1)^2\,\sigma_u^2 \\
&= \sigma_e^2 + \left[2(1+r)(1-\alpha) + r^2\right]\sigma_u^2.
\end{aligned}$$

Since the interest rate r is the rate from month to month, it is small in magnitude and this formula is close to Summers's. Let ρ_1 be the first lagged autocorrelation of z, $\rho_1 \equiv \mathrm{Corr}(z_{t+1}, z_t)$. It is proportional to the covariance

$$\begin{aligned}
&\mathrm{E}\left[e_{t+1} + v_{t+1} + (\alpha - r - 1)\,u_t\right]\left[e_t + v_t + (\alpha - r - 1)\,u_{t-1}\right] \\
=\ &\mathrm{E}\left[e_{t+1} + (\alpha - r - 1)\,\alpha u_{t-1} + (\alpha - r - 1)\,v_t\right]\left[e_t + v_t + (\alpha - r - 1)\,u_{t-1}\right] \\
=\ &\left[(\alpha - r - 1)\,(1-\alpha^2) + \alpha\,(\alpha - r - 1)^2\right]\sigma_u^2 = [\alpha - (1+r)][1 - \alpha(1+r)]\sigma_u^2.
\end{aligned}$$

Thus

$$\rho_1 = -\frac{[1 + r - \alpha][1 - \alpha(1+r)]\sigma_u^2}{\sigma_e^2 + \left[2(1+r)(1-\alpha) + r^2\right]\sigma_u^2},$$

which is less than 0 unless α is very close to 1. When $r = 0$, this is the same as in the Summers paper.

How big is the autocorrelation likely to be? Summers suggests taking $\alpha = 0.98$ for monthly data. In this case the fraction of a u innovation that has not decayed after three years is $0.98^{36} = 0.483$. That is, the half-life of a "fad" is about 3 years. Summers also suggests that we take $\sigma_e^2 = 0.001$ (making the monthly standard deviation of returns about 3.2 percent). Finally, he looks at the case $\sigma_u^2 = .08$, meaning that roughly 30 percent of the unconditional variance of q (a large fraction) is due to non-fundamental noise. Finally, I add the assumption that $r = 0.00325$ or 0.325 per cent per month, giving an annual real interest rate of about 4 percent. Then we find that

$$\rho_1 = -0.00743,$$

which is slightly smaller than Summers estimate of -0.008. Of course, higher-order autocorrelations are even smaller. Summers's point is that it would take thousands of years of monthly data to reliably detect such a small autocorrelation in excess returns — even though q deviates persistently from q^* by large amounts. The intuition for this example is that the deviation $\log q^* - \log q$ is so persistent that it can barely be detected by looking at the autocorrelation in returns. The overall deviation u has a high variance, but its innovation v need not, so the example is not terribly far off from adding a constant to the stock price. In contrast, a more variable higher-frequency noise would be easier to detect.

Risk and Equity Pricing

When people are risk averse, the relevant arbitrage condition between bonds and equities is more complicated. The model of Robert E. Lucas, Jr. ("Asset Prices in an Exchange Economy," *Econometrica*, November 1978) deals with this case. The starting point is the Euler equation for the stock, which can be written (for a representative agent) as

$$q_t = \mathrm{E}_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} (d_{t+1} + q_{t+1}) \right\}. \tag{3}$$

Notice that the "risk neutral" formula we used before is different. Because the bond Euler equation states that

$$\frac{1}{1+r} = \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\},$$

the preceding risk-neutral hypothesis that $q_t = \mathrm{E}_t\left\{\frac{d_{t+1}+q_{t+1}}{1+r}\right\}$ would, in the Lucas model, imply the invalid relationship

$$q_t = \mathrm{E}_t\left\{\mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\}(d_{t+1}+q_{t+1})\right\},$$

which is generally the same as eq. (3) if the marginal utility of consumption is constant (no risk aversion) but not otherwise. By using eq. (3), we are also allowing for non-constant real interest rates.

To see how this case differs from the risk-neutral, let us again write the excess return on the equity as

$$e_{t+1} = \frac{d_{t+1}+q_{t+1}}{q_t} - (1+r)$$

and express (3) as

$$1 = \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}(e_{t+1}+1+r)\right\},$$

or as

$$0 = \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}e_{t+1}\right\},$$

where we have recalled that

$$\mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\} = \frac{1}{1+r}.$$

An equivalent expression is

$$\begin{aligned}
0 &= \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\}\mathrm{E}_t\left\{e_{t+1}\right\} + \mathrm{Cov}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}, e_{t+1}\right\} \Longleftrightarrow \\
0 &= \frac{\mathrm{E}_t\left\{e_{t+1}\right\}}{1+r} + \mathrm{Cov}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}, e_{t+1}\right\}.
\end{aligned}$$

Thus we find that

$$\frac{\mathrm{E}_t\left\{e_{t+1}\right\}}{1+r} = -\mathrm{Cov}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}, e_{t+1}\right\}. \tag{4}$$

The expected excess return (in terms of today's consumption) equals minus the covariance between the excess return and the (ex post) marginal rate of substitution of present for future consumption. This formula also implies that excess returns can be predictable – in principle, by any information in the information set underlying the conditional covariance in (4).

How should one interpret the fundamental relationship (4)? Imagine that the covariance in the equation is positive. Since $u''(c) < 0$, this means that the excess return tends to be high when consumption is low (i.e., when the marginal utility of consumption is high). In this case the stock provides good *consumption insurance* because it tends to do well when other sources of income are underperforming. So the expected excess return will be negative — the expected return is less than the risk-free rate, because the asset reduces the risk of the overall portfolio. Conversely, if the covariance is negative, we have an asset whose payoff is high when consumption is high. This asset does not help insure against consumption risk, so its expected return must offer a (positive) *risk premium* over the risk-free rate. Note that the relevant concept of risk is *not* the variance of the return; it is the covariance with consumption. That is why this model is often called the consumption-based capital asset pricing model (CCAPM).

A useful approximation can be derived as follows. Assume CRRA preferences and take the second-order Taylor approximation around the point $e_t = 0$, $c_{t+1}/c_t = 1$:

$$\left(\frac{c_{t+1}}{c_t}\right)^{-R} e_{t+1} \approx e_{t+1} - R\left(\frac{c_{t+1}}{c_t} - 1\right) e_{t+1}.$$

Then we may estimate

$$\begin{aligned} 0 &= \mathrm{E}_t\left\{\frac{\beta c_{t+1}^{-R}}{c_t^{-R}} e_{t+1}\right\} \\ &\approx \mathrm{E}_t\left\{\beta e_{t+1} - \beta R\left(\frac{c_{t+1}}{c_t} - 1\right) e_{t+1}\right\}. \end{aligned}$$

This implies that

$$\mathrm{E}_t\left\{e_{t+1}\right\} = R\mathrm{Cov}_t\left\{e_{t+1}, \frac{c_{t+1}}{c_t} - 1\right\}$$

(assuming that the product of the expected excess return and the expected growth rate of per capita consumption is small).

This way of expressing the equity risk premium shows that it depends on two factors:

1. Relative risk aversion.

2. The covariance of the excess return with the growth rate of per capita consumption.

We may now get a handle on the famous "equity premium puzzle" of Raj Mehra and Ed Prescott (in the *Journal of Monetary Economics*, March 1985). They use 1870-1979 U.S. data, in which the standard deviation of annual per capita consumption growth is 0.036 (surely an overestimate, based on Christina Romer's famous study of prewar U.S. macro data); that of the excess equity return 0.167 (including the Great Depression); the correlation coefficient between equity excess returns and consumption growth is 0.4; and the realized long-run average equity return premium is 0.062 per annum. What degree of risk aversion is needed to rationalize this? Solve for R using

$$0.062 = R \times 0.4 \times 0.036 \times 0.167.$$

Because consumption growth is so smooth for the United States, the answer of $R = 25.8$ is much larger than most economists would regard as reasonable.

Several potential solutions have been suggested. One now in vogue is the possibility of some catastrophic negative shock, whose likelihood is understated by reliance on historical probability frequencies (such as a recent paper by Robert J. Barro, "Rare Disasters and Asset Markets in the Twentieth Century," *Quarterly Journal of Economics,* August 2006).

Now let's look at multiperiod equity pricing. Let us recall the Euler equation,

$$q_t = \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\left(d_{t+1} + q_{t+1}\right)\right\},$$

and substitute recursively to get

$$q_t = \mathrm{E}_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} d_{t+1} + \frac{\beta^2 u'(c_{t+2})}{u'(c_t)} d_{t+2} + \frac{\beta^2 u'(c_{t+2})}{u'(c_t)} q_{t+2} \right\}.$$

Going to the limit, we find that

$$q_t = \mathrm{E}_t \left\{ \sum_{i=1}^{\infty} \frac{\beta^i u'(c_{t+i})}{u'(c_t)} d_{t+i} \right\} + \lim_{i \to \infty} \mathrm{E}_t \frac{\beta^i u'(c_{t+i})}{u'(c_t)} q_{t+i},$$

and if we assume the transversality condition that $\lim_{i\to\infty} \mathrm{E}_t \beta^i q_{t+i} u'(c_{t+i})/u'(c_t) = 0$,

$$q_t = \sum_{i=1}^{\infty} \mathrm{E}_t \left\{ \frac{\beta^i u'(c_{t+i})}{u'(c_t)} d_{t+i} \right\}. \tag{5}$$

This is the analog of equation (2) for the model with risk aversion. (This PDV relation is true for any individual's consumption.)

There is another interpretation of this condition that makes the comparison with equation (2) clearer. Define $R_{t,t+i}$ to be the price, in terms of date t's output, of a unit of output delivered with certainty on date $t+i$. If the real interest rate is constant at r, then $R_{t,t+i} = 1/(1+r)^i$. In general, $R_{t,t+i}$ is the inverse of the *long-term interest rate* between dates t and $t+i$. The usual logic of Euler equations tells us that in equilibrium,

$$R_{t,t+i} = \mathrm{E}_t \left\{ \frac{\beta^i u'(c_{t+i})}{u'(c_t)} \right\}.$$

Now use the decomposition we invoked earlier to rewrite (5) as

$$q_t = \sum_{i=1}^{\infty} R_{t,t+i} \mathrm{E}_t \{d_{t+i}\} + \sum_{i=1}^{\infty} \mathrm{Cov}_t \left\{ \frac{\beta^i u'(c_{t+i})}{u'(c_t)}, d_{t+i} \right\}.$$

The stock price can be expressed as the PDV (at market interest rates) of expected future dividends – as in the risk-neutral pricing model – plus a risk correction. If consumption tends to be positively conditionally correlated with dividends, the stock price is depressed relative to the PDV model, and in the opposite case, it is raised. Of course, a lower stock price, all else equal, implies a higher expected rate of return.

More on Rational Bubbles

For this section let's again denote the price in (5) by q_t^*. This price obviously satisfies the Euler equation for each date,

$$q_t^* = \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\left(d_{t+1}+q_{t+1}^*\right)\right\}.$$

Are there other solutions? Let $\tilde{q}_t = q_t^* + b_t$. For this to be a solution, the variables $\{b_t\}$ must satisfy

$$b_t = \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}b_{t+1}\right\}.$$

Mathematically, there can be many types of bubble. The simplest might be to specify

$$b_t = \frac{k}{\beta^t u'(c_t)}$$

for any constant k. Then

$$\begin{aligned}\mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}b_{t+1}\right\} &= \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\frac{k}{\beta^{t+1}u'(c_{t+1})}\right\} \\ &= b_t.\end{aligned}$$

Clearly, because $\beta < 1$, this bubble will tend to explode over time.

To take a more subtle example proposed by Olivier Blanchard in *Economics Letters* (1979), imagine that our bubble has the form

$$b_t = \begin{cases} \frac{k}{\pi^t\beta^t u'(c_t)} & \text{(with probability } \pi) \\ 0 & \text{(with probability } 1-\pi) \end{cases}$$

conditional on $b_{t-1} > 0$; but if $b_{t-1} = 0, b_t = 0$ with probabilty 1. The transition probabilities are independent of the rest of the economy. Then we once again have a bubble because if $b_t > 0$,

$$\begin{aligned}\mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}b_{t+1}\right\} &= \pi\mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\frac{k}{(\pi\beta)^{t+1}u'(c_{t+1})}\right\} + (1-\pi)\cdot 0 \\ &= \frac{k}{(\pi\beta)^t u'(c_t)} = b_t.\end{aligned}$$

This is a bubble that "crashes" permanently to 0 with probability $1-\pi$, and so it grows faster prior to the crash. An interesting (and realistic) feature of this crashing bubble is that it must crash in finite time with probability 1.

The problem set contains an even weirder example, and discusses arguments for excluding rational bubbles of this kind on theoretical grounds (at least in nonmonetary models). Note that the uniqueness-of-equilibrium proof given in the Lucas (1978) paper does not cover the possibility of rational stock-price bubbles. (Lucas proves uniqueness only within a class of pricing functions that does not admit potentially unbounded bubbles.)