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Financing the Firm: Modigliani-Miller

Firms raise capital by issuing debt as well as equity. Above we priced equity claims to a firm's output. If all claims on a firm are equity claims – claims to dividends – then the value of the firm clearly is the present value derived above.

Consider a firm facing complete markets and a riskless real interest rate of r. There are two periods and $\mathcal{S}$ states of nature in period 2. The firm can issue bonds B and equity shares E to finance its investment in capital K. The payoff to this investment tomorrow is $A(s)F(K)$, for $s \in \mathcal{S}$. Since the firm may be unable fully to repay its bondholders on date 2, its borrowing rate $\tilde{r}$ (the corporate bond rate) will generally exceed r.

For simplicity (and without any real loss of generality) assume the depreciation rate of capital is $\delta = 1$.

Let the firm borrow B on date 1. It will then owe its bondholders $(1+\tilde{r})B$ in every state of nature $s \in \mathcal{S}$,but it declares bankruptcy if it cannot repay them in full. Let $\mathcal{S}_b$ be the set of bankruptcy states and $\mathcal{S}_n$ be the set of nonbankruptcy states, so that $\mathcal{S} = \mathcal{S}_b \cup \mathcal{S}_n$. By definition, in a bankruptcy state, the total output of the firm is insufficient to cover debt payments: $A(s)F(K) < (1+\tilde{r})B$. In this case, bondholders are senior claimants and get whatever there is, while equity holders get nothing.

Thus

$$\text{equity holder payoff} = \begin{cases} A(s)F(K) - (1+\tilde{r})B \text{ for } s \in \mathcal{S}_n \\ 0 \text{ for } s \in \mathcal{S}_b \end{cases},$$

whereas

$$\text{bond holder payoff} = \begin{cases} (1+\tilde{r})B \text{ for } s \in \mathcal{S}_n \\ A(s)F(K) \text{ for } s \in \mathcal{S}_b \end{cases}.$$

At the Arrow-Debreu prices, the value of the firm's equity, sold on date 1, is

$$E = \sum_{s \in \mathcal{S}_n} \frac{p(s)}{1+r}\left[A(s)F(K) - (1+\tilde{r})B\right] = K - B. \tag{1}$$

Lenders, on the other hand, must earn the same return as they would in risk-free lending. Thus, coporate debt must have the same value (in terms of AD prices) as riskless debt. We can express this condition as

$$\sum_{s\in\mathcal{S}_n}\frac{p(s)}{1+r}(1+\tilde{r})B+\sum_{s\in\mathcal{S}_b}\frac{p(s)}{1+r}A(s)F(K)=B$$

and solve for $1+\tilde{r}$ to obtain

$$1+\tilde{r}=\frac{B-\sum_{s\in\mathcal{S}_b}\frac{p(s)}{1+r}A(s)F(K)}{\sum_{s\in\mathcal{S}_n}\frac{p(s)}{1+r}B}.$$

Let's look at the market value of all claims on the firm. It is the sum of equity and debt claims

$$\begin{aligned} V &= E+B \\ &= \underbrace{\sum_{s\in\mathcal{S}_n}\frac{p(s)}{1+r}\left[A(s)F(K)-(1+\tilde{r})B\right]}_{E} + \underbrace{\sum_{s\in\mathcal{S}_n}\frac{p(s)}{1+r}(1+\tilde{r})B+\sum_{s\in\mathcal{S}_b}\frac{p(s)}{1+r}A(s)F(K)}_{B} \\ &= \sum_{s\in\mathcal{S}}\frac{p(s)}{1+r}A(s)F(K)=K. \end{aligned}$$

This is the basic Modigliani–Miller theorem. The firm's market value is simply the value of its ouputs across future states of nature. The division of claims between equity and debt is irrelevant.

Now consider the implications for investment. The firm maximizes the value of shareholders' equity E. Suppose the firm borrows an additional dollar of debt to invest in capital. Then B goes up by 1, and the firms' market value V goes up by

$$\sum_{s\in\mathcal{S}}\frac{p(s)}{1+r}A(s)F'(K).$$

If $\sum_{s\in\mathcal{S}}\frac{p(s)}{1+r}A(s)F'(K)>1$, then the firms' market value $V=E+B$ goes up by more than B does, so clearly equity holders gain. The firm should do the investment. We therefore get the same first-order condition for the optimal investment level

$$\sum_{s\in\mathcal{S}}\frac{p(s)}{1+r}A(s)F'(K)=1$$

as in the case without debt (recall we've assumed depreciation $\delta = 1$). Another implication of Modigliani-Miller: the investment rule is unaffected by the mode of finance, debt or equity.

Tobin's q model of investment

Important early work on investment dynamics was done by Dale Jorgenson. He was on the Berkeley faculty, and his very smart research assistant (and coauthor) was Robert Hall (of random walk fame), a Berkeley undergrad. The work basically took the investment first-order condition from a Solow-type growth model to define a long-run target capital stock, given by $AF_K(\bar{K}, L) = r$. The ad hoc dynamics $I = \dot{K} = \gamma(\bar{K} - K)$ were added in to yield an investment equation.

The Tobin's q model is more sophisticated. It assumes that capital is costly to install — and if you want to install it more quickly, you pay more in frictional costs. This setup gives rise to intrinsic dynamics of investment, rather than the imposed Jorgenson-Hall dynamics, as well as to endogenous pricing of installed capital relative to its replacement cost (not including installation expense). The Obstfeld-Rogoff selection in the reader does it in discrete time, but here I will use continuous time and the maximum principle.

The key assumption in the model is that there is a convex installation cost of the form

$$\frac{\chi}{2}(I^2/K)$$

for installing new capital. We can therefore define a firm's present discounted profit stream on date t as

$$\Pi(t) = \int_t^\infty \mathrm{e}^{-r(s-t)}[A(s)F(K(s), L(s)) - w(s)L(s) - I(s) - \frac{\chi}{2}(I(s)^2/K(s))]\mathrm{d}s,$$

which is maximized subject to the constraint

$$\dot{K}(s) = I(s)$$

with K_t given. The interest rate r is assumed to be constant. (In contrast to before, I assume zero depreciation for ismplicity.)

In terms of the language of the Pontryagin Maximum Principle, L and I are the firm's control variables, K the state variable for the maximization problem.

Let us set up the Hamiltonian

$$H = AF(K, L) - wL - I - \frac{\chi}{2}(I^2/K) + qI$$

where q is the costate variable. We differentiate with repect to the two controls, setting the result to zero, to obtain

$$AF_L(K, L) = w,$$

$$\frac{I}{K} = \frac{q-1}{\chi}.$$

The first of these is the standard employment optimality condition, while the second states that investment has the same sign as $q - 1$. As suggested by Tobin (*JMCB* 1969, on the reading list), investment is positive when the value of installed capital exceeds its replacement cost.

Next consider the dynamics of the costate. According to the Maximum Principle, an optimum is characterized by the differential equation

$$\dot{q} - rq = -\frac{\partial H}{\partial K},$$

that is, by

$$\dot{q} - rq = -AF_K(K, L) - \frac{\chi}{2}\left(\frac{I}{K}\right)^2. \tag{2}$$

To derive a phase diagram showing the implied dynamics, let us assume we are looking at the aggregate economy and that the labor market clears with $L = \bar{L}$, the full-employment supply of labor. Then the dynamic equations of the model can be written as:

$$\dot{q} - rq = -AF_K(K, \bar{L}) - \frac{(q-1)^2}{2\chi},$$

$$\dot{K} = \left(\frac{q-1}{\chi}\right) K.$$

The steady state of the model occurs where $\bar{q} = 1$ and $AF_K(K, \bar{L}) = r$. (Alternatively, if we wished to work at the firm level, we could take wages as exogenous to the firm, then solve for L using $AF_L(K, L) = w$, then substitute the solution $L = \phi(K, A, w)$ into (2). Nothing much changes, but the math is slightly more intricate.)

Imagine a phase diagram with q on the vertical axis and K on the horizontal axis. The locus along which $\dot{K} = 0$ is horizontal at $\bar{q} = 1$, with K

rising above it and falling below. On the other hand, consider the slope of the schedule along which $\dot{q}=0$. It is given by

$$\left.\frac{\mathrm{d}q}{\mathrm{d}K}\right|_{\dot{q}=0}=\frac{\chi A F_{KK}}{\chi r-(q-1)}.$$

At $q=1$ the slope is negative, but as q rises the slope becomes vertical and then positive. To the right of this schedule q is rising and to the left q is falling. We get a saddlepath adjustment to the steady state, with q falling as K rises.

This picture can be used to analyze even anticipated future shocks (e.g., to productivity A) by requiring that, at the time the shock occurs, the variable q not take a discrete (anticipated) jump.

What does q represent? The general solution to a differential equation such as (2) is a forward-looking integral expression (as you can verify by differentiating).[1] (All of this is still true if we allow the real interest rate r to vary over time.) The general solution (for a constant r) is

$$q(t)=\int_t^{\infty}\mathrm{e}^{-r(s-t)}\left\{AF_K\left[K(s),L(s)\right]+\frac{\chi}{2}\left[\frac{I(s)}{K(s)}\right]^2\right\}\mathrm{d}s+b\mathrm{e}^{rt}.$$

Above, I have made the substitution

$$\frac{(q-1)^2}{2\chi}=\frac{\chi}{2}\left(\frac{I}{K}\right)^2$$

and b is an arbitrary constant. The economically relevant solution imposes the transversality condition $\lim_{t\to\infty}\mathrm{e}^{-rt}q(t)K(t)=0$,which obliges us to set $b=0$ in the general solution for $q(t)$ above. In that case,

$$q(t)=\int_t^{\infty}\mathrm{e}^{-r(s-t)}\left\{AF_K\left[K(s),L(s)\right]+\frac{\chi}{2}\left[\frac{I(s)}{K(s)}\right]^2\right\}\mathrm{d}s,$$

[1] Use the following fact from calculus. Let

$$f(t)=\int_{a(t)}^{b(t)}g(s,t)\mathrm{d}s.$$

Then

$$f'(t)=g\left[b(t),t\right]b'(t)-g\left[a(t),t\right]a'(t)+\int_{a(t)}^{b(t)}\frac{\partial g(s,t)}{\partial t}\mathrm{d}s.$$

which means that the shadow value of a unit of installed capital equals the discounted future marginal products, plus the future contributions to lowering the installation costs of optimal investments (it is cheaper at the margin to add capital to a larger pre-existing capital stock).

The q variable defined above is *marginal* q, the shadow value of an *extra* unit of capital, given K. Empirical work on investment, however, does not have access to this variable: researchers must use as a proxy stock-market value divided by total capital-in-place, Π/K, which amounts to *average* q. What is the relationship between average and marginal q? This was clarified in a famous 1982 article in *Econometrica* by Fumio Hayashi.

Notice that

$$\begin{aligned}
\frac{\mathrm{d}\,(qK)}{\mathrm{d}t} &= q\dot{K} + \dot{q}K \\
&= rqK - \left(AF_K K + \frac{\chi I^2}{2K}\right) + qI \\
&= rqK - \left[AF(K,L) - wL + \chi\frac{I^2}{2K}\right] + I\left(1 + \chi\frac{I}{K}\right) \\
&= r\,(qK) - \left[AF(K,L) - wL - I - \chi\frac{I^2}{2K}\right].
\end{aligned}$$

Imposing the transversality condition, we can integrate forward (solving for the composite variable qK) to conclude that

$$q(t)K(t) = \int_t^\infty \mathrm{e}^{-r(s-t)}[A(s)F(K(s),L(s)) - w(s)L(s) - I(s) - \frac{\chi}{2}(I(s)^2/K(s))]\mathrm{d}s = \Pi(t).$$

We see that marginal q and average q are equal:

$$q = \frac{\Pi}{K}.$$

The key facts used to derive this prediction are that markets are competitive and that the function $\psi(K,L,I) = F(K,L) - \frac{\chi I^2}{2K}$ displays constant returns to scale, i.e., for any nonnegative number λ, $\psi(\lambda K, \lambda L, \lambda I) = \lambda\psi(K,L,I)$.