Economics 202A, Problem Set 2
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(Due: Thursday, September 18)

1. (A linearized dynamic model) Consider a case with no population growth and a 100% per period capital depreciation rate, so that the planner maximizes $\sum_{t=0}^{\infty}\beta^t u(c_t)$ subject to $k_{t+1} = f(k_t) - c_t$. Let $u(c) = \log(c)$, $f(k) = k^{\alpha}$, $\alpha < 1$.

(a) Show that the dynamics along the model's optimal path are described by the pair of nonlinear equations:

$$c_{t+1} - c_t = \alpha\beta(k_t^{\alpha} - c_t)^{\alpha-1}c_t - c_t, \quad k_{t+1} - k_t = k_t^{\alpha} - c_t - k_t. \qquad (*)$$

(b) Compute the steady-state values $\bar{c}$ and $\bar{k}$ as functions of α and β. (In an endogenous-growth context you might want to return to this problem and think hard about what happens as $\alpha \to 1$.)

(c) The multivariate version of Taylor's theorem states that if the function $g(x,y)$ is smooth enough, its linear approximation near $(\bar{x},\bar{y})$ is

$$g(x,y) \approx g(\bar{x},\bar{y}) + \frac{\partial g}{\partial x}(\bar{x},\bar{y})(x - \bar{x}) + \frac{\partial g}{\partial y}(\bar{x},\bar{y})(y - \bar{y}).$$

Use this result to show that the system (*) you derived in part (a) has the matrix linear approximation

$$\begin{bmatrix} c_{t+1} - \bar{c} \\ k_{t+1} - \bar{k} \end{bmatrix} = \begin{bmatrix} \alpha + (1-\alpha)/\alpha\beta & -(1-\alpha)(1-\alpha\beta)/\alpha\beta^2 \\ -1 & 1/\beta \end{bmatrix} \begin{bmatrix} c_t - \bar{c} \\ k_t - \bar{k} \end{bmatrix}. \qquad (**)$$

near its steady state.

(d) Show that the characteristic roots of the matrix M in (**) (that is, the solutions γ to the equation $\det(M - \gamma I) = 0$, where I is the identity matrix) are $1/\alpha\beta$ and α. Define the matrix $\Gamma = \begin{bmatrix} 1/\alpha\beta & 0 \\ 0 & \alpha \end{bmatrix}$.

(e) Find any 2×2 matrix X such that $MX = X\Gamma$. [Hint: Columns of X are called <u>eigenvectors</u> belonging to $1/\alpha\beta$ and α; they are rays rather than well-defined vectors but you can tie them down by assuming that $x_{21} = x_{22} = 1$, then solving for x_{11} and x_{12} only.]

(f) Why do we need such an X? Define the transformed variables c', k' by $\begin{bmatrix} c' \\ k' \end{bmatrix} = X^{-1}\begin{bmatrix} c \\ k \end{bmatrix}$. Show that multiplying (**) through by X^{-1} gives the equation

$$X^{-1}\begin{bmatrix} c_{t+1} - \bar{c} \\ k_{t+1} - \bar{k} \end{bmatrix} = \begin{bmatrix} c'_{t+1} - \bar{c}' \\ k'_{t+1} - \bar{k}' \end{bmatrix} = X^{-1}MXX^{-1}\begin{bmatrix} c_t - \bar{c} \\ k_t - \bar{k} \end{bmatrix} = \Gamma\begin{bmatrix} c'_t - \bar{c}' \\ k'_t - \bar{k}' \end{bmatrix}. \qquad (\dagger)$$

Note with satisfaction that Γ (conveniently and not accidentally) is diagonal.

(g) Show that all solutions to system ($\dagger$) take the form:

$$c'_{t+1} - \bar{c}' = (1/\alpha\beta)^{t+1}(c'_0 - \bar{c}'), \qquad k'_{t+1} - \bar{k}' = \alpha^{t+1}(k'_0 - \bar{k}'),$$

where $t = 0$ is an initial date.

(h) Now we want to use these simple solutions to retrieve c and k. Find $c_{t+1} - \bar{c}$ and $k_{t+1} - \bar{k}$ by doing the inverse transformation:

$$X\begin{bmatrix} c' \\ k' \end{bmatrix} = XX^{-1}\begin{bmatrix} c \\ k \end{bmatrix} = \begin{bmatrix} c \\ k \end{bmatrix}.$$

<u>Solve</u> for c_0 by showing that only if $c_0 = \bar{c} + \left(\frac{1}{\beta} - \alpha\right)(k_0 - \bar{k})$ will the system reach its steady state. Solve for $c_t - \bar{c}$ and $k_t - \bar{k}$ along the saddlepath. Show that the planner should set consumption (near the steady state) by the rule

$$c_t = \frac{(1-\alpha\beta)}{\beta} k_t + (\alpha\beta)^{\frac{\alpha}{1-\alpha}}(1-\alpha\beta)(1-\alpha).$$

2. (Zero discounting) In his famous 1928 *Economic Journal* article on optimal saving, Frank P. Ramsey argued that it is morally indefensible to discount the welfare of future generations. He therefore argued that a benevolent economic planner should:

$$\text{maximize } V = \int_0^\infty u[c(t)]dt$$

$$\text{subject to } \dot{k}(t) = f[k(t)] - c(t), \ k(0) \text{ given.}$$

(Ramsey assumed zero population growth.) You can see right away the problem with this formuation: <u>any</u> path that approaches a constant steady-state consumption level will yield an <u>infinite</u> value of V. Thus, it is not clear how to compare such paths and identify one as "optimal."

Ramsey finessed the problem in the following way. He defined $\bar{c}$ to be the "bliss" or maximal steady-state consumption level [the existence of which presupposes that f(k) either eventually becomes decreasing in k or asymptotes to a finite maximum as k goes to ∞]. He then redefined his problem as that of <u>minimizing</u> $\int_0^\infty \{u(\bar{c}) - u[c(t)]\}dt$, society's cumulative distance from "bliss", subject to the above constraints. Note that this integral can be finite if $c(t) \to \bar{c}$ as $t \to \infty$ (and if it isn't, it's not the optimum we seek in any case).

(a) Use the Maximum Principle to derive necessary conditions for a solution to the Ramsey problem. (You can assume a depreciation rate of 0 for capital. The resulting Euler condition is sometimes called the Keynes-Ramsey rule because J. M. Keynes, a friend of Ramsey's and editor of the *Economic Journal*, helped him to interpret it intuitively.) Show the economy indeed should converge to "bliss" (also known as the "golden rule" in growth theory.) Interpret the model's intertemporal Euler condition. You can do so by adressing the following question: Suppose the economy starts with $k(0) < \bar{k}$. Since Ramsey believed in intergenerational equality, why isn't it optimal in his view for each generation simply to consume f[k(0)]?

(b) Let $\{c^*(t)\}_{t=0}^\infty$ denote the Ramsey consumption path starting from an initial capital stock k(0), and let $\{c(t)\}_{t=0}^\infty$ be any other consumption path. Show that the Ramsey path *overtakes* any other feasible consumption path starting from k(0), in the following sense: there exists a finite time $\mathcal{T}$ such that for all $T > \mathcal{T}$, $\int_0^T u[c^*(t)]dt > \int_0^T u[c(t)]dt$.

(c) Ramsey states the Keynes-Ramsey rule as: "*rate of saving multiplied by marginal utility of consumption should always equal bliss minus actual rate of utility enjoyed.*" [By "bliss" he meant $u(\bar{c})$.] Can you derive this rule?