Economics 202A, Problem Set 5

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(Due Tuesday, November 18)

1. *Consumer durables.* Our consumption analysis implicitly assumed that consumption is perishable. But if some consumer goods instead were durable (washing machines, autos, etc.), spending in one period would secure an item that yields utility in a number of subsequent periods. This question asks that you analyze consumer behavior when some goods are durable.

Before going further, here is a digression on a solution method, the method of *lag and lead operators*, that should be in your toolkit. For any time series $\{x_t\}$, define the lag operator L by

$$Lx_t = x_{t-1}.$$

Define the lead operator L^{-1} by

$$L^{-1}x_t = x_{t+1}.$$

(Obviously, $LL^{-1}x_t = L^{-1}Lx_t = x_t$, which fact inspires the notation.) You need to know two facts about these operators, both derived from the standard formula for summing geometric series.

FACT #1. Let $y_t = (1 - \theta L)x_t$, where $|\theta| < 1$. Then

$$x_t = (1 - \theta L)^{-1} y_t,$$

where (note the formal similarity to the usual formula)

$$(1 - \theta L)^{-1} = 1 + \theta L + \theta^2 L^2 + \theta^3 L^3 + \ \ldots.$$

Observe that $(1 - \theta L)\left(1 + \theta L + \theta^2 L^2 + \theta^3 L^3 + \ \ldots\right) = 1.$

FACT #2. Let $y_t = (1 - \theta L)x_t$, where $|\theta| > 1$. Then

$$x_t = -\theta^{-1} L^{-1} (1 - \theta^{-1} L^{-1})^{-1} y_t,$$

where

$$(1 - \theta^{-1} L^{-1})^{-1} = 1 + \theta^{-1} L^{-1} + \theta^{-2} L^{-2} + \theta^{-3} L^{-3} + \ \ldots.$$

To establish this last formula, note that

$$(1-\theta^{-1}L^{-1})\left(1+\theta^{-1}L^{-1}+\theta^{-2}L^{-2}+\theta^{-3}L^{-3}+\ ....\right)=1.$$

It is *not* correct to write $x_t = (1-\theta L)^{-1}y_t$ in this case because $|\theta| > 1$, which means that $(1-\theta L)^{-1}$ does not exist. However, if $|\theta| > 1$, then $\left|\theta^{-1}\right| < 1$, and $y_t = (1-\theta L)x_t = \theta L\left(\theta^{-1}L^{-1}-1\right) = -\theta L\left(1-\theta^{-1}L^{-1}\right)x_t$. Because $1-\theta^{-1}L^{-1}$ *is* invertible when $\left|\theta^{-1}\right| < 1$, we can therefore write $x_t = -\theta^{-1}L^{-1}(1-\theta^{-1}L^{-1})^{-1}y_t$.

Now, the model of durables. An individual maximizes

$$\sum_{t=0}^{\infty}\beta^t\left[u(c_t)+v(s_t)\right]$$

where c_t is nondurable consumption and s_t is the stock (measured at the start of period t) of a consumer durable yielding a flow of services proportional to s_t. Let z_t be purchases of durables (which may be resold on a secondary market): if durables do not depreciate, then

$$s_{t+1}=s_t+z_t.$$

Let a_t be the value (in terms of consumption c, at the start of period t) of the individual's financial assets, which have a constant real gross per period yield of $1+r$. If (for simplicity) we assume that the durable good's price in terms of c is constant at 1, then (make sure you see why)

$$a_{t+1}=(1+r)a_t+y_t-c_t-z_t,$$

where y_t is an exogenous flow of income.

(a) Using any method you wish, derive and *interpret* the following first-order conditions for the consumer's problem:

$$u'(c_t)=\beta(1+r)u'(c_{t+1}),$$

$$u'(c_t)=\beta u'(c_{t+1})+\beta v'(s_{t+1}).$$

(b) Write the second f.o.c. above as $-v'(s_t) = (1-\beta^{-1}L)u'(c_t)$ and show that $u'(c_t) = \beta v'(s_{t+1})+\beta^2 v'(s_{t+2})+\ ...$, using the preceding lag-lead formalism. Interpret this condition.

(c) Show that the equation

$$v'(s_t) = \beta(1+r)v'(s_{t+1})$$

holds at the individual optimum. Thus, when $\beta(1+r) = 1$, the consumer will smooth the marginal utility of durable services.

(d) What does this finding imply about the smoothness of *total* spending $c + z$? To think about that question, let $\beta(1+r) = 1$ and assume that $u(c) = \gamma \log(c)$, $v(s) = (1-\gamma)\log(s)$. Then solve explicitly for the paths of c, z, and s.

(e) How would the problem change if we allowed explicitly for a rental market in durables?

2. *The Lucas "tree" model from* Econometrica*, 1978.* Consider a world with a single representative agent, in which a random and exogenous amount of perishable output y_t falls from a fruit tree each period t. (There is no other output.) Output follows the stochastic process

$$\log y_t = \log y_{t-1} + \varepsilon_t, \qquad \mathrm{E}_{t-1}\varepsilon_t = 0, \tag{1}$$

where the i.i.d. shock ε_t is drawn from a $N(0, \sigma^2)$ normal distribution. There is no way to grow more fruit trees — the supply is fixed.

The agent's lifetime utility function is

$$\mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \mathrm{e}^{-\theta(s-t)} u(c_s) \right\},$$

where $\theta > 0$ is the rate of time preference. Assume that there is a competitive stock market in which people can trade shares in the fruit tree, whose price on date t is p_t. This is the ex dividend price: if you buy a share on date t, you get your first dividend on date $t+1$.

(a) Show that an individual will choose contingent consumption plans such that on each date.

$$p_t u'(c_t) = \mathrm{e}^{-\theta} \mathbb{E}_t \left\{ (y_{t+1} + p_{t+1}) u'(c_{t+1}) \right\}. \tag{2}$$

(You can use the individual finance constraint that $c_t + p_t x_{t+1} \leq (y_t + p_t)\, x_t$, where x_t is the share of the fruit tree the individual holds at the end of period $t-1$.)

(b) Show that in equilibrium, the "fundamentals" price of the tree is

$$p_t^* = \mathbb{E}_t \left\{ \sum_{s=t+1}^{\infty} e^{-\theta(s-t)} \frac{u'(y_s)\, y_s}{u'(y_t)} \right\}.$$

Can you interpret this price in terms of expected dividends and risk factors? What is the sign of these risk factors on the trees?

(c) Let $u(c) = c^{1-\gamma}/(1-\gamma)$ for $\gamma > 0$. Show that the normality assumption in (1) implies (for $s > t$):

$$\mathbb{E}_t \left\{ y_s^{1-\gamma} \right\} = y_t^{1-\gamma} e^{\frac{\sigma^2(1-\gamma)^2}{2}(s-t)}.$$

(Use the *lognormal* distribution: if $\varepsilon \sim N(\mu, \sigma^2)$, then e^{ε} has a lognormal distribution with $\mathbb{E}\{e^{\varepsilon}\} = e^{\mu + \frac{1}{2}\sigma^2}$.)

(d) Deduce from part (c) that if $\theta > \sigma^2(1-\gamma)^2/2$, then $p_t^* = \kappa y_t$, where

$$\kappa = \frac{1}{\{e^{[\theta - \sigma^2(1-\gamma)^2/2]} - 1\}}.$$

(e) Now return to a general strictly concave utility function $u(c)$. Let b_t be the random variable $A y_t^{\rho}/u'(y_t)$, where $\rho = \sqrt{2\theta/\sigma^2}$ and A is an arbitrary constant. Show that $p_t^* + b_t$ will satisfy the individual's Euler equation (2) in equilibrium, so that b_t is a bubble.

(f) Show that $p_t = p_t^* + b_t$ violates the (equilibrium) transversality condition:

$$\lim_{T\to\infty} e^{-\theta(T-t)} \mathbb{E}_t \left\{ u'(y_{t+T}) p_{t+T} \right\} = 0. \tag{3}$$

(g) Together with the equilibrium Euler equation [equation (2) with y substituted for c],

$$p_t u'(y_t) = e^{-\theta} \mathbb{E}_t \left\{ (y_{t+1} + p_{t+1}) u'(y_{t+1}) \right\}, \tag{4}$$

condition (3) is *sufficient* for a stochastic price path $\{p_t\}$ to be an equilibrium of the Lucas model. In this part of the homework we will show that the condition is also *necessary.*

Iterate (4) forward to derive

$$p_t u'(y_t) = \mathbb{E}_t \left\{ \sum_{s=t+1}^{\infty} e^{-\theta(s-t)} u'(y_s)\, y_s \right\} + \lim_{T\to\infty} e^{-\theta(T-t)} \mathbb{E}_t \left\{ u'(y_{t+T}) p_{t+T} \right\}.$$

Argue that free disposal of output ensures that the limit in condition (3) must be nonnegative. Argue that if the limit is strictly positive, we cannot be looking at an equilibrium because individuals can raise expected lifetime utility through the following strategy: sell a tiny amount of the fruit tree today and consume the proceeds now, never repurchasing the portion of the fruit tree just sold (that is, reduce x_t, which equals 1 in the hypothesized equilibrium, permanently). Why is the Euler equation (4) alone not sufficient to rule out the possibility that such a strategy raises lifetime utility?