Economics 202A, Problem Set 4

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1. *Interest rates and consumption.* An individual has the exponential period utility function

$$u(C) = -\gamma e^{-C/\gamma}$$

($\gamma > 0$) and maximizes

$$u(C_t) + \beta u(C_{t+1})$$

($0 < \beta < 1$) subject to the budget constraint

$$C_t + RC_{t+1} = Y_t + RY_{t+1} \equiv W_t$$

[where $R = (1+r)^{-1}$, so that a fall in the real interest rate r means a rise in the market discount factor R].

(a) Solve for C_{t+1} as a function of C_t, R, and β using the consumer's intertemporal Euler equation.

(b) What is the optimal level of C_t, given W_t, R, and β? [In other words, solve for the date t consumption function.]

(c) By differentiating your consumption function (including W_t) with respect to R, show that:

$$\frac{dC_t}{dR} = -\frac{C_t}{1+R} + \frac{Y_{t+1}}{1+R} + \frac{\gamma}{1+R}\left[1 - \log(\beta/R)\right].$$

[Hint: Your consumption function has the form $C = f(W,R)/(1+R)$.Therefore,

$$\frac{dC}{dR} = -\frac{C}{1+R} + \frac{1}{1+R}\left[\frac{\partial f}{\partial W}\frac{dW}{dR} + \frac{\partial f}{\partial R}\right],$$

which gives you half the answer.]

(d) What is the intertemporal substitution elasticity for the exponential utility function? [Calculate this elasticity at an allocation where $C_t = C_{t+1} = \overline{C}$. It is a function $\sigma(\overline{C})$ of $\overline{C}$, not a constant.]

(e) Show that the derivative calculated in part (c) above can be expressed as

$$\frac{\mathrm{d}C_t}{\mathrm{d}R} = \frac{\sigma(C_{t+1})C_{t+1}}{1+R} + \frac{Y_{t+1} - C_{t+1}}{1+R}.$$

(f) Explain intuitively why, for someone with $Y_{t+1} > C_{t+1}$, a rise in R (that is, a fall in the real interest rate r), unambiguously raises consumption on date t.

2. *Optimal consumption with incomplete markets.* A consumer has the quadratic period utility function $u(C) = \alpha C - (\gamma/2)\, C^2$ and maximizes $u(C_t) + \beta u(C_{t+1})$ subject to the constraints

$$A_{t+1} = (1+r)A_t + Y_t - C_t, \quad A_t \text{ given,}$$

$$C_{t+1} = (1+r)A_{t+1} + Y_{t+1}(\mathfrak{s}), \quad \mathfrak{s} \in \{1, 2, ..., S\}.$$

Here, r is the real rate of interest (so that $1/(1+r)$ is the current price of a unit of output delivered next period with probability 1). Let $\pi(\mathfrak{s})$ be the probability of state of nature $\mathfrak{s}$ from the perspective of date t and assume that $\beta = 1/\,(1+r)$.

(a) Ignore for the moment the constraint that date $t+1$ consumption be nonnegative. Compute and interpret the optimal level of C_t.

(b) Suppose the consumer has an infinite horizon and there is uncertainty over output on all future dates. Use your answer to (a) to guess the consumption function and use the "random walk" result to prove that your guess is correct.

(c) Let's return to the 2-period case in part (a) but now take seriously the constraint that

$$C_{t+1}(\mathfrak{s}) \geq 0, \quad \forall \mathfrak{s}.$$

Renumber the states of nature $\mathfrak{s}$ (if necessary) so that

$$Y_{t+1}(1) = \min_{\mathfrak{s}} \{Y_{t+1}(\mathfrak{s})\}.$$

Show the following: If

$$(1+r)A_t + Y_t - \mathrm{E}_t\{Y_{t+1}\} > -(2+r)Y_{t+1}(1)/(1+r),$$

then the result in (a) still holds. (Why, intuitively?) Otherwise, the consumption function is:

$$C_t = (1 + r)A_t + Y_t + Y_{t+1}(1)/(1 + r).$$

(d) Still thinking about the 2=period case, suppose the consumer faces *complete* asset markets such that $p(\mathfrak{s})$, the price in terms of sure date $t + 1$ output of a state-$\mathfrak{s}$ contingent unit of date $t + 1$ output, equals $\pi(\mathfrak{s})$. Compute the optimal value of C_t. Do we gave to worry now about the non-negativity constraint on date $t + 1$ consumption?