Economics 131
Section Notes
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Cost-Benefit Analysis, Externalities, and Prices versus Quantities

1 Cost Benefit Analysis

1.1 Set-Up

Consider a simple cost-benefit problem of determining the level $x \geq 0$ of a certain activity, where the benefits are given the function $B(x)$ and the costs by the function $C(x)$. Mathematically we wish to maximize the net benefit of benefits minus costs

$$\max_{x \geq 0} B(x) - C(x) \tag{CB}$$

Definition of benefits and costs can be somewhat arbitrary. Economically a foregone benefit is a cost, just as a foregone cost is a benefit. However one tries to define things intuitively, and so that $B(0) = C(0) = 0$, $B(x) \geq 0$ and $C(x) \geq 0$ and benefits and costs are increasing over relevant ranges of x. Mathematically $B(x)$ is increasing if $x_2 > x_1$ implies $B(x_2) \geq B(x_1)$, and when differentiable $B'(x) \geq 0$.

Example 1 *A profit-maximizing firm with profits $\pi = px - C(w, x)$. Benefits are given by revenue $B(x) = R(x) = px$, and costs have the same interpretation with $C(x) = C(w, x)$ holding w fixed. In the case of a imperfect competition price may depend on x, i.e. $p = p(x)$ so $B(x) = R(x) = p(x)x$. In either case profit maximization is equivalent to net benefit maximization.*

Example 2 *Imagine a world where consumption x and leisure l are the only goods (see the previous handout) and utility is additively separable so that we can write $U(l, x) = u(l) + v(x)$. To simplify set $M = 0$ and $w = 1$. Then, the time and budget constraints imply $l = T - L = T - px$ and so we can write benefits $B(x) = v(x)$ and costs in terms of foregone leisure $C(x) = u(T) - u(T - px)$. In this case utility maximization is equivalent to (CB) as $B(x) - C(x) = v(x) - u(T) - u(T - px) = U(T - px, x) - u(T)$. $u(T)$, being a constant, does not affect the maximization problem. If utility is not separable between leisure and consumption then benefits and costs of consumption are not separable either and hence are not well defined.*

1.2 Maximization

If $B(x)$ and $C(x)$ are differentiable, then the optimal level of x^* can be found from the first order necessary condition

$$B'(x^*) - C'(x^*) \leq 0 \quad \text{with "=" if } x^* > 0 \tag{FOC}$$

If $x^* > 0$ then $B'(x^*) = C'(x^*)$, i.e. marginal benefit equals marginal cost, i.e. $MB(x^*) = MC(x^*)$. One should check that at this point $B(x^*) - C(x^*) \geq 0 = B(0) - C(0)$, to make sure $x^* \neq 0$.

Oftentimes one should check the second order condition

$$B''(x^*) - C''(x^*) < 0 \tag{SOC}$$

rearranging and using the above definition this means that $MC'(x^*) > MB'(x^*)$ which means that marginal costs are rising faster than marginal benefits. A typical case is when marginal benefits are falling rising and marginal costs rising so $MC'(x) > 0 > MB'(x)$. To solve this problem (i) use the FOC to find all of the candidate x, (ii) eliminate all candidate x's that do not satisfy the SOC, and (iii) find the net benefit $B(x) - C(x)$ of all remaining candidate x's and 0; the one which yields the highest net benefit is x^*.

Example 3 *Assume* $U(l,x) = l + v(x)$ *and* $C(x) = f^{-1}(x)$ *where* $f(L) = f(T-l)$ *is the production function for consumption. In this case utility cost is the same as production costs in terms of leisure, i.e.* $l = T - f^{-1}(x) = T - C(x)$. *We can then write* $B(x) = v(x)$ *and* $C(x)$ *has the obvious definition. The ideal solution finds* $B'(x^*) = v'(x^*) = C'(x^*)$. *Competitive markets produce the same outcome as profit maximization implies* $p = C'\left(x^S\right)$ *and utility maximization implies* $v'\left(x^D\right) = p$ *(check!) and so at the market equilibrium* $v'\left(x^D\right) = C'\left(x^S\right) = p$. *In fact you can see that the demand curve and the marginal benefit curve are the same as* $x^D = (v')^{-1}(p) = MB^{-1}(p)$, *just as the supply curve and marginal cost curve coincide. The net benefit is just the total surplus between the supply and demand curves .*

2 Externalities

2.1 Private versus Social Costs and Benefits

Let $PB(x)$ and $PC(x)$ be the *private* benefits and costs, respectively, of the individual who controls x. Left to her own devices, this individual will maximize the private net benefit $PB(x) - PC(x)$ to find the optimal level of x from the private perspective x^P . If *positive externalities* exist then the social benefit $SB(x)$ does not equal the private benefit, but rather $SB(x) = PB(x) + EB(x)$ where $EB(x)$ stands for the *external benefit* to those not controlling x.

In the case of *negative externalities* the social cost $SC(x) = PC(x) + EC(x)$ where $EC(x)$ stands for the *external costs (or damage)* to those not controlling x. The socially optimal level of x, x^S (S now means "social", not "supply") is found by maximizing social net benefit $SB(x) - SC(x)$.

2.2 Private Provision versus Optimal Provision

For now consider the case where there are external benefits to x but no external costs and that all benefits are concave $PB'', EB''(x) < 0$ costs are convex $PC''(x) > 0$. A private individual will choose x^P to satisfy

$$PB'\left(x^P\right) - PC'\left(x^P\right) = 0 \tag{Private FOC}$$

The social optimum on the other hand will satisfy

$$SB\left(x^S\right) - SC\left(x^S\right) = PB'\left(x^S\right) + EB'\left(x^S\right) - PC'\left(x^S\right) = 0 \tag{Social FOC}$$

To see that x is underprovided, i.e. $x^P < x^S$ note that at x^P

$$SB\left(x^P\right) - SC\left(x^P\right) = PB'\left(x^P\right) + EB'\left(x^P\right) - PC'\left(x^P\right) = EB'\left(x^P\right) > 0$$

where the second equality comes from the private FOC. The above expression implies that at x^P that marginal social net benefit is still positive (albeit decreasing) and therefore x should increase. In the case of external costs with no external benefits but with external costs a similar analysis will show that $x^S < x^P$ as the marginal social net benefit at the privately chosen x, $-EC'\left(x^P\right) < 0$, is negative.

2.3 Pigouvian Taxes and Subsidies

Assume now that all costs and benefits are put into money units and that the government steps in and confers a Pigouvian subsidy equal to the marginal external benefit of x at the social optimum, $s = EB'\left(x^S\right)$ so that the private individual will now maximize

$$PB(x) - PC(x) + sx$$

Differentianting yields the FOC

$$PB'\left(x^P\right) - PC'\left(x^P\right) + s = 0 \tag{Pigou FOC}$$

Substituting in the definition of $s = EB'\left(x^S\right)$ this is the same as the social FOC above, so $x^P = x^S$. The individual will choose the social optimum on his own. In the case of a negative externality the government could charge a Pigouvian tax $t = EC\left(x^S\right)$. If there are both benefits and costs the government should pay a net subsidy of $s = EB'\left(x^S\right) - EC\left(x^S\right)$ to insure the optimal allocation. The government may also want to confer a lump-sum tax or transfer, independent of x, to make sure the individual is able to recoup some of the subsidy or to help pay the tax.

This analysis presumes that the government knows the external, as well as private, benefits and costs of x, and thus it can determine x^S and the appropriate tax or subsidy. In this world of perfect certainty it might as well just mandate that x^S be provided, rather than bother with the Pigouvian scheme.

2.4 The Coase Theorm

A point made by Coase (1960) is that Pigouvian taxes or subsidies may require too much information and that in many cases individuals may provide the social optimum without government intervention. For example, let person A who decides x have the right to set x to whatever she wants. Say person B benefits from x, $EB(x) > 0$ and $EC(x) = 0$. Person A may offer to pay person B to provide the socially optimal level of x, compensating A for whatever costs or foregone benefits with a minimum payment of

$$\begin{aligned} payment &= \Delta PC - \Delta PB \\ &= PB\left(x^P\right) - PC\left(x^P\right) - \left[PB\left(x^S\right) - PC\left(x^S\right)\right] > 0 \end{aligned}$$

she may experience. Individual B will still having surplus money to spare.

$$\begin{aligned} surplus &= \Delta EB - payment \\ &= EB\left(x^S\right) - EB\left(x^P\right) - \left\{PB\left(x^P\right) - PC\left(x^P\right) - \left[PB\left(x^S\right) - PC\left(x^S\right)\right]\right\} \\ &= EB\left(x^S\right) - EB\left(x^P\right) - PB\left(x^P\right) + PC\left(x^P\right) + PB\left(x^S\right) - PC\left(x^S\right) \\ &= PB\left(x^S\right) + EB\left(x^S\right) - \left[PB\left(x^P\right) + EB\left(x^P\right)\right] - PC\left(x^S\right) + PC\left(x^P\right) \\ &= SB\left(x^S\right) - SB\left(x^P\right) - \left[SC\left(x^S\right) - SC\left(x^P\right)\right] \\ &= \Delta SB - \Delta SC > 0 \end{aligned}$$

How the remaining surplus is split depends on the "bargaining power" of A and B. Only A and B need to know the benefits and costs, and no third party, like the government, needs to get involved. The Coase theorem may breakdown if individuals do not bargain efficiently, i.e. they incur various "transaction costs," which is a whole branch of economics unto itself.

3 Prices versus Quantities

The framework by Weitzman (1974) proposes a non-Coasian way of dealing with uncertainty. It fits best into the above analysis if we think of a case where all costs are internal $SC(x) = PC(x) = C(x)$, for short, and all benefits are external $SB(x) = EB(x) = B(x)$, although it is easily generalized. For example, imagine a firm which can set a level of pollution abatement x, where abating costs are harmful to the profitability of the firm, but pollution reduction benefits society. The government has the option of mandating a level of x or to offer a price (like a Pigouvian subsidy) p for each unit of x provided.

I simplify a bit on Weitzman's notation without sacrificing much of his ideas. Let the benefit curve be given by $B(x) = B'x + B''\frac{x^2}{2}$ where $B' > 0$ and $B'' < 0$ are constants[1] and so $MB(x) = B' + B''x$. Similarly $C(x) = C'x + B''\frac{x^2}{2}$ where $C' > 0$ and $C'' > 0$, and $MC(x) = C' + C''x$. We also assume that $B' > C'$ which will guarantee that the optimal $x^* > 0$.

3.1 With Certainty

The optimal x^* is where $MB(x^*) = MC(x^*)$, i.e. $C' + C''x^* = B' + B''x^*$, which solving[2] gives

$$x^* = \frac{B' - C'}{C'' - B''} \qquad \text{(Optimal x)}$$

[1] This is like assuming that the second derivative of the benefit function is constant and so the first derivative is linear.

[2] $C' + C''x^* = B' + B''x^* \Rightarrow x^*C'' - x^*B'' = x^*(C'' - B'') = B' - C' \Rightarrow x^* = \frac{B'-C'}{C''-B''}$

This is a pretty general method of finding the intersection of two lines with intercepts B' and C' and slopes B'' and C''.

If the government tries to set p optimally this will be simply where $p^* = MC(x^*) = C' + C''x^*$ which substituting in above and doing some algebra[3] means

$$p^* = \frac{C''B' - B''C'}{C'' - B''} \qquad \text{(Optimal p)}$$

Two interesting special cases here are when $C'' = 0$ which means $\bar{p} = C'$ the marginal cost curve, and when $B'' = 0$ then $\bar{p} = B'$ the marginal benefit curve. In general p^* can be seen as a weighted average of B' and C' with weights C'' and $-B''$, respectively.

You can see that in the case of perfect certainty setting quantity or price always produces x^* and so the choice of instrument is irrelevant.

3.2 With Uncertainty

Uncertainty manifests itself in parallel up-and-down shifts in the marginal benefit and marginal cost curves. Let $B' = B_0' + \eta$ and $C' = C_0' + \theta$ where η and θ are unknown random variables each with zero mean and uncorrelated so $E(\eta) = E(\theta) = 0$ and $E(\eta\theta) = 0$, where E is the expectations operator. Variances are given by $E(\eta^2) = \sigma_\eta^2$ and $E(\theta^2) = \sigma_\theta^2$. However θ is known by the firm supplying x when it sets x. Then the optimal quantity and price with knowledge of η and θ are the same as above

$$x^* = \frac{B_0' - C_0' + \eta - \theta}{C'' - B''}$$

Of course since η and θ are unknown this cannot be set. The best possible quantity to set is just the average optimal quantity

$$\bar{x} = E(x^*) = \frac{B_0' - C_0'}{C'' - B''} \qquad \text{(Regulated x)}$$

The best price is at the expected marginal cost of $\bar{x}$, so $\bar{p} = E[C' + C''\bar{x}] = C' + C''\bar{x}$ which a bit of math, just like the last footnote, shows

$$\bar{p} = \frac{C''B_0' - B''C_0'}{C'' - B''} \qquad \text{(Regulated p)}$$

The quantity that will arise when $\bar{p}$ is set will also depend on θ at price equal to MC, i.e. $\bar{p} = C_0' + \theta + C''x(\theta)$, which solving gives $x(\theta) = [\bar{p} - C_0' - \theta]/C''$ which substituting in $\bar{p}$ and doing some algebra[4] means

$$x(\theta) = \frac{B_0' - C_0'}{C'' - B''} - \frac{\theta}{C''} = \bar{x} - \frac{\theta}{C''}$$

Therefore quantity regulation and price regulation will produce different x's depending on θ, the marginal cost shock. The differences will be greater, the greater is the size of C''.

3.3 Errors in provision of x

When quantities are set the error is in setting x will be given by the difference $\bar{x} - x^*$ which is quite straightforwardly.

$$error_q = \bar{x} - x^* = \frac{\theta - \eta}{C'' - B''}$$

If prices are set then the error is given by[5]

$$error_p = x(\theta) - x^* = \frac{\frac{B''}{C''}\theta - \eta}{C'' - B''}$$

[3] $\bar{p} = C' + C''\frac{B'-C''}{C''-B''} = \frac{C'(C''-B'')+C''(B'-C')}{C''-B''} = \frac{C'C''-C'B''+C''B'-C''C'}{C''-B''} = \frac{C''B'-C'B''}{C''-B''}$

[4] $x(\theta) = \frac{1}{C''}[\bar{p} - C_0' - \theta] = \frac{1}{C''}\left[\frac{C''B_0'-B''C_0'}{C''-B''} - C_0' - \theta\right] = \frac{1}{C''}\left[\frac{C''B_0'-B''C_0'-C_0'(C''-B'')}{C''-B''} - \theta\right]$
$= \frac{1}{C''}\left[\frac{C''B_0'-B''C_0'-C_0'C''+C_0'B''}{C''-B''} - \theta\right] = \frac{1}{C''}\left[\frac{C''B_0'-C_0'C''}{C''-B''} - \theta\right] = \frac{B_0'-C_0'}{C''-B''} - \frac{\theta}{C''}$

[5] $x(\theta) - x^* = \bar{x} - \frac{\theta}{C''} - x^* = \bar{x} - x^* - \frac{\theta}{C''} = \frac{\theta-\eta}{C''-B''} - \frac{\theta}{C''} = \frac{\theta-\eta-\frac{\theta}{C''}(C''-B'')}{C''-B''} = \frac{\theta-\eta-\theta+\frac{B''}{C''}\theta}{C''-B''} = \frac{\frac{B''}{C''}\theta-\eta}{C''-B''}$

The differences in errors of quantity regulation relative to price regulation is given by

$$\Delta error = error_q - error_p = \frac{\theta}{C'' - B''}\left(1 - \frac{B''}{C''}\right) \qquad \text{(Error diff)}$$

as the η terms cancel out (check), not surprising since only marginal costs shocks produce different outcomes. Therefore shocks which affect the marginal benefit shock η have the same effects with either choice, and the difference depends only on marginal cost shocks θ. Quantities produce a greater error than prices if $|\Delta error| > 0$, as the absolute value measures the distance. So quantities are worse if $\left|\frac{\theta}{C''-B''}\left(1 - \frac{B''}{C''}\right)\right| > 0$, which is true if $|1 - B''/C''| > 0$ or $1 > |B''| / |C''|$ or finally, $|C''| > |B''|$, which means the marginal cost curve has a greater slope than the marginal benefit curve. If on the other hand $|B''| > |C''|$ price regulation produces a greater error.

3.4 The Deadweight Loss of Errors

The error itself is not what matters so much as the *welfare loss* caused by the error. The welfare loss of the error is equal to the size of the deadweight loss triangle it causes. The length of the triangle is just the $error$ term while the height of the triangle is given by $(C'' - B'') \times error$. Therefore the area of the triangle is $\frac{1}{2}(C'' - B'') \times error^2$.[6] Already you can see that the size of the deadweight loss is proportional to the size of the error squared, and so a greater error will always produce a greater welfare loss.

Substituting in the above equations into the triangle formula gives the deadweight loss in each case[7]:

$$DWL_q(\eta, \theta) = \frac{1}{2}\frac{(\theta - \eta)^2}{(C'' - B'')} \qquad DWL_p(\eta, \theta) = \frac{1}{2}\frac{\left(\frac{B''}{C''}\theta + \eta\right)^2}{(C'' - B'')}$$

The expected deadweight losses over all values (η, θ) are then[8]

$$E[DWL_q] = \frac{1}{2}\frac{\sigma_\theta^2 + \sigma_\eta^2}{(C'' - B'')} \qquad E[DWL_p] = \frac{1}{2}\frac{\left(\frac{B''}{C''}\right)^2 \sigma_\theta^2 + \sigma_\eta^2}{(C'' - B'')}$$

which makes use of the fact that η and θ are uncorrelated. The difference in deadweight loss between quantity and price regulation is

$$\Delta E[DWL] = E[DWL_q] - E[DWL_p] = \frac{\sigma_\theta^2}{2(C'' - B'')}\left[1 - \left(\frac{B''}{C''}\right)^2\right] \qquad \text{(DWL diff)}$$

Again you can see the term with η, σ_h^2 dropping out. The similarity of $\Delta E[DWL]$ and $\Delta error$ is pretty striking. Quantity regulation creates larger losses when $\Delta E[DWL] > 0$ which means $1 - (B''/C'')^2 > 0$ which is equivalent to $(C'')^2 > (B'')^2$ or just $|C''| > |B''|$ which is consistent with what we saw for errors in instituting the optimal x. Therefore in such cases it would be optimal to institute price regulation. In cases where $|B''| > |C''|$ quantity regulation would be best.

[6] Recall from geometry that a triangle with a length L and height H has area $\frac{1}{2}LH$.

[7] $DWL_q(\eta, \theta) = \frac{1}{2}\left[\frac{(\theta-\eta)}{(C''-B'')}\right]^2 (C'' - B'') = \frac{1}{2}\frac{(\theta-\eta)^2}{(C''-B'')^2}(C'' - B'') = \frac{1}{2}\frac{(\theta-\eta)^2}{(C''-B'')}$ The case of DWL_p is quite similar.

[8] $E[DWL_q] = E\left[\frac{(\theta-\eta)^2}{2(C''-B'')}\right] = \frac{\left((\theta-\eta)^2\right)}{2(C''-B'')}E = \frac{E\left(\theta^2-2\eta\theta+\eta^2\right)}{2(C''-B'')} = \frac{E(\theta^2)-2E(\eta\theta)+E\left(\eta^2\right)}{2(C''-B'')} = \frac{\sigma_\theta^2-0+\sigma_\eta^2}{2(C''-B'')} = \frac{\sigma_\theta^2+\sigma_\eta^2}{2(C''-B'')}$