

Time and Discounting, Expected Utility, Savings and Insurance in a Two State Model

1 Time and Discounting

1.1 Theory of Interest

1.1.1 Stocks

Assume that we can index time periods with $t = 0, 1, 2, \dots, T$. Assume that we let an interesting bearing assets A_t accrue interest at rate r_t and that it obeys the following difference equation $A_{t+1} = (1 + r_t) A_t$. Compounding interest over all of these periods we get that **future value** of assets A_0 at time T is then

$$FV_T(A_0) = A_T = (1 + r_{T-1}) \times (1 + r_{T-2}) \dots \times (1 + r_0) A_0 = A_0 \prod_{t=0}^{T-1} (1 + r_t) \quad (\text{FV Stock})$$

We can also invert the relationship looking at the value of A_T from the perspective of time 0 known as the **present value**

$$PV_0(A_T) = A_0 = A_T \prod_{t=0}^{T-1} \left(\frac{1}{1 + r_t} \right) \quad (\text{PV Stock})$$

Assuming that the interest rate $r_t = r$, $t = 0, \dots, T - 1$, then $FV_T(A_0) = A_0(1 + r)^T$ and $PV_0(A_T) = A_T(1 + r)^{-T}$.

1.1.2 Adding Flows

Assume that at each period there is an addition of s_t , $t = 0, 1, \dots, T - 1$ to the stock of assets so $A_{t+1} = s_t + (1 + r_t) A_t$,

$$A_T = s_{T-1} + (1 + r_{T-1}) s_{T-2} + \dots + s_0 \prod_{t=1}^{T-1} (1 + r_t) = \sum_{k=0}^{T-1} \left[s_k \prod_{t=k+1}^{T-1} (1 + r_t) \right] \quad (\text{FV Flow})$$

Or discounting back using A_0 in (PV Stock)

$$A_0 = \sum_{k=0}^{T-1} \left[s_k \prod_{t=k+1}^{T-1} (1 + r_t) \right] \times \prod_{t=0}^{T-1} \left(\frac{1}{1 + r_t} \right) = \sum_{k=0}^{T-1} \left[s_k \prod_{t=0}^k \left(\frac{1}{1 + r_t} \right) \right] \quad (\text{PV Flow})$$

These expressions are really just accounting tools: while they look difficult, they are pretty straightforward if taken bit by bit.¹

1.2 Geometric Sums

When interest rates are constant then the future value of the flow is gotten with the help of the formula for **geometric sums**

$$\sum_{t=0}^T \delta^t = \frac{1 - \delta^{T+1}}{1 - \delta} \quad (\text{Geom Sum})$$

¹Another thing to take into account is that the formula depends on a few conventions such as whether r_t is the interest gained between periods $t - 1$ and t or between t and $t + 1$, I optef for the latter. Also savings at time t , s_t , do not start to earn interest until time period $t + 1$.

To show this is true multiply both sides of the equation and get

$$(1 - \delta) \sum_{t=0}^T \delta^t = 1 - \delta^{T+1}$$

The left hand side of the equation is then

$$(1 - \delta) \sum_{t=0}^T \delta^t = \sum_{t=0}^T (\delta^t - \delta^{t+1}) = (1 - \delta) + (\delta - \delta^2) + \dots + (\delta^{T-1} - \delta^T) + (\delta^T - \delta^{T+1}) = 1 - \delta^{T+1}$$

If $|\delta| < 1$ then $\lim_{T \rightarrow \infty} \delta^T = 0$ and so the limit of the infinite sum

$$\sum_{t=0}^{\infty} \delta^t = \lim_{T \rightarrow \infty} \left(\sum_{t=0}^T \delta^t \right) = \lim_{T \rightarrow \infty} \left(\frac{1 - \delta^{T+1}}{1 - \delta} \right) = \frac{1}{1 - \delta}$$

Example 1 *Present value of an annuity.* We wish to discount back a constant flow s , at a constant rate r for $T = \infty$. Let $\delta = \frac{1}{1+r}$ to get

$$PV_T(s) = \sum_{k=0}^{\infty} \left[s \prod_{t=0}^k \left(\frac{1}{1+r} \right) \right] = s \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^{t+1} = \frac{s}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t = \frac{s}{1+r} \times \frac{1}{1 - \left(\frac{1}{1+r} \right)} = \frac{s}{r}$$

(PV of Annuity)

so it is just the value of the flow divided by the rate of interest - a very useful rule of thumb. Inversely, a stock A_0 can be turned into a flow of revenue by just taking the interest on it each period $s = rA$ and leaving the principle.

1.3 Intertemporal Budget Constraints

With almost any decision in economics comes a budget constraint. Imagine an individual who earns nonlabor income M_t as well as earnings wages $w_t L_t$ and earning interest income $r_t A_t$ in each period and spends $p_t x_t$ in consumption. The savings s_t in each period is then just the income minus consumption $s_t = M_t + w_t x_t - p_t x_t$. Plugging this in to the above difference equation we get that each period this individual will face a **flow constraint** of the form

$$A_{t+1} = M_t + w_t x_t - p_t x_t + (1 + r_t) A_t \quad (\text{Flow BC})$$

which states that the amount of A_{t+1} saved in period t is just total income minus total expenditure. Assume an individual dies at time $T + 1$, then there are a total of $T + 1$ of these constraints. Assuming the individual starts with nothing and has nothing left over when she dies, $A_0 = A_{T+1} = 0$, we can combine all of the flow constraints can be combined to yields a **stock constraint**

$$\sum_{t=0}^T \left(\frac{1}{1+r} \right)^t p_t x_t = \sum_{t=0}^T \left(\frac{1}{1+r} \right)^t (w_t L_t + M_t) \quad (\text{Stock BC})$$

This constraint says that the present value of lifetime consumption will equal the present value of lifetime income.

1.4 Discounting Profits and Utility over Time

In making actions firms are generally presumed to take actions which maximize the present-value of profits over time (the present-value of the firm) according to the prevailing interest rate, that is

$$\Pi = \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \pi_t$$

where π_t is per-period profits in period t . For utility economists have developed an analogue theory known as the **Discounting Utility (DU)** model. While its psychological foundations are tenuous, the theory does have some fairly interesting normative properties of how one should discount the future which Koopmans (1960) lays out. The theory implies that utility within each time periods is **additively separable** so we can write utility in period t , from the point of view of period t , as $u(l_t, x_t)$ where (l_t, x_t) are the variables of which affect utility in period t , and u is called the **felicity** function, which is typically concave $u'(x) > 0$ and $u''(x) < 0$. Future utility is discounted at a constant rate ρ , known as **rate of time preference** which is sometimes written as part of a **discount factor** $\beta = \frac{1}{1+\rho}$. Total utility over all time periods for a flow of $\{l_t, x_t\}_{t=0}^T$ is then given by

$$U(l_0, x_0, \dots, x_T, l_T) = \sum_{t=0}^T \left(\frac{1}{1+\rho} \right)^t u(l_t, x_t) \quad (\text{DU})$$

Because of its simplicity and its flexibility the DU model is the most widely chosen by far amongst economist for dealing with decision making over time.

2 Savings

2.1 A Simple Savings Model

We consider a two period model 2 periods $t = 0, 1$ and no leisure-labor choice. An individual needs to simply decide how to allocate his consumption between both periods assuming he starts out with initial assets A . The real interest rate is r so we can ignore prices and set them to $p_0 = p_1 = 1$ in each period. Be warned that if we are trying to model a person's life in 2 periods, r will be much greater than a few percentage points.² Savings in period 0 will then be just $s = A - x_0$ and will earn interest rate r , so that consumption in period 1 is $x_1 = (1+r)(A - x_0)$ which are our stock budget constraint. Utility in both periods is given by $u(x_0) + \frac{1}{1+\rho}u(x_1)$ and so substituting in the budget constraint (as surely x_0 and x_1 are positive in both periods) we solve

$$\max_{x_0} u(x_0) + \frac{1}{1+\rho}u[(1+r)(A - x_0)]$$

using the chain rule appropriately this yields the FOC

$$u'(x_0^*) - \frac{1+r}{1+\rho}u'[(1+r)(A - x_0^*)] = 0$$

rearranging and substituting in $x_1^* = (1+r)(A - x_0^*)$ we get

$$\frac{u'(x_0^*)}{u'(x_1^*)} = \frac{1+r}{1+\rho}$$

Now if the interest rate r is greater than the rate of time preference ρ this implies $\frac{u'(x_0^*)}{u'(x_1^*)} = \frac{1+r}{1+\rho} > 1$ which in turn implies $u'(x_0^*) > u'(x_1^*)$. Because $u''(x) < 0$ (u is concave) then $u'(x)$ is decreasing and so $u'(x_0^*) > u'(x_1^*)$ implies that $x_1^* > x_0^*$, i.e. consumption is higher in the later period. Another way to see this is if that $MRS_{x_0, x_1} = (1+\rho)\frac{u'(x_0)}{u'(x_1)}$ and the $MRT_{x_0, x_1} = 1+r$. Along the 45 degree line where $x_0 = x_1$ has slope $1+\rho$, and so if $r > \rho$ then an indifference curve at $x_0 = x_1$ will cut into the budget constraint at a lower slope, indicating the preference for more x_1 . Of course if $\rho > r$ then the argument can be reversed to show $x_0^* > x_1^*$.

²Say we want a period of 30 years at an interest rate of 5%, then the effective r will be $r = (1.05)^{30} - 1 = 3.3219$.

2.2 Do increases in r increase saving?

Another question is how *changes* in the interest rate will affect savings. Taking the FOC and treating x_0^* as a function of r , $x_0^*(r)$ we can differentiate implicitly to get

$$\left\{ u''(x_0^*) + \frac{(1+r)^2}{1+\rho} u''(x_1^*) \right\} \frac{dx_0^*}{dr} - \frac{1}{1+\rho} u'(x_1^*) - \frac{1}{1+\rho} u''(x_1^*) x_1^* = 0$$

after we substitute back x_1^* (to make the expression shorter). Solving for $\frac{dx_0^*}{dr}$ we get

$$\frac{dx_0^*}{dr} = \frac{u'(x_1^*) + x_1^* u''(x_1^*)}{(1+\rho) u''(x_0^*) + (1+r)^2 u''(x_1^*)}$$

This expression has an uncertain sign. The denominator is negative since $u''(x) < 0$ always, so $\frac{dx_0^*}{dr}$ is negative if and only if the numerator is positive. Since $s^* = A - x_0^*$, $\frac{ds^*}{dr} = -\frac{dx_0^*}{dr}$ then savings increases with r if $\frac{dx_0^*}{dr}$ is negative. This condition can be rearranged to yield

$$\frac{ds^*}{dr} > 0 \Leftrightarrow \frac{dx_0^*}{dr} < 0 \Leftrightarrow u'(x_1^*) + u''(x_1^*) x_1^* > 0 \Leftrightarrow \frac{u''(x_1^*) x_1^*}{u'(x_1^*)} > -1 \Leftrightarrow \gamma(x_1^*) \equiv -\frac{u''(x_1^*) x_1^*}{u'(x_1^*)} < 1$$

Where $\gamma(x)$ is a measure of the concavity of $u(x)$ which is actually the **elasticity of marginal utility with respect to x** .³ It is also known as the or the **coefficient of relative risk aversion** which relates to expected utility theory (see below). Many reasonable utility functions typically yield coefficients of relative risk aversion above and below 1 and so this issue is unresolved. The issues can also be framed in terms of income and substitution effects in terms of x_0 . As r increases x_0 becomes relatively more expensive and so it becomes sensible to save more (this is captured by the $u'(x_1^*) > 0$). However the value of existing savings increases making the individual richer, inducing her to consume more in both periods, which means saving less in period 0 (captured by the $u''(x_1^*) x_1^* < 0$ term).

2.2.1 Social Security

The simplest model of social security is where the government just saves some amount s_G for the individual. This leaves the individual with a private savings of $s = A - x_0 - s_G$. And so the individual consumes $(s_G + s)(1+r) = (s_G + A - x_0 - s_G)(1+r) = (A - x_0)(1+r)$ which is really the same amount. There is no difference. If s_G is set very high this may impose the constraint $x_1 \geq (1+r)s_G$ if the individual cannot borrow against his social security earnings. It might also be that social security savings does not yield the same effective interest as r . For example in a simple overlapping generations model with a pay-as-you-go system the effective interest on s_G is $n + g$ the rate of population growth plus the rate of growth of the economy.

2.3 The Full Intertemporal Model (Optional)

Economists typically model an individual as maximizing (DU) subject to the budget constraint (Stock BC) - we ignore any time constraint. We can write the Lagrangean as

$$\mathcal{L}(l_0, x_0, \dots, x_T, l_T, \alpha) = \sum_{t=0}^T \left(\frac{1}{1+\rho} \right)^t u(x_t, l_t) + \alpha \left[\sum_{t=0}^T \left(\frac{1}{1+r} \right)^t (w_t L_t + M_t - p_t x_t) \right]$$

(Intertemporal Lagrangean)

The first order condition for utility for any given x_t is actually not so bad given the separability

$$\frac{\partial \mathcal{L}}{\partial x_t} = \left(\frac{1}{1+\rho} \right)^t \frac{\partial u}{\partial x}(x_t^D, l_t^D) - \alpha \left(\frac{1}{1+r} \right)^t p_t = 0 \quad (\text{Cons FOC})$$

³Recall an elasticity of y with respect to x is the percent change in y due to a 1% change in x , defined mathematically as $\left| \frac{dy}{dx} \frac{x}{y} \right| = \left| \frac{d \ln y}{d \ln x} \right| = \left| \frac{\% \Delta y}{\% \Delta x} \right|$ where "% Δ " means "percent change."

which implies that

$$\frac{\partial u}{\partial x}(x_t^D, l_t^D) = \alpha \left(\frac{1 + \rho}{1 + r} \right)^t p_t$$

Taking the same condition for $t + 1$ we get

$$\frac{\partial u}{\partial x}(x_{t+1}^D, l_{t+1}^D) = \alpha \left(\frac{1 + \rho}{1 + r} \right)^{t+1} p_{t+1}$$

and dividing the first by the second we get that the marginal rate of substitution between periods 1 and 2 is

$$MRS_{x_t, x_{t+1}} = \frac{\frac{\partial u}{\partial x}(x_t^D, l_t^D)}{\frac{\partial u}{\partial x}(x_{t+1}^D, l_{t+1}^D)} = \frac{1 + r}{1 + \rho} \frac{p_t}{p_{t+1}} \quad (\text{ISC})$$

This relationship in economics is known as an **Euler equation**. For small r and ρ and small amounts of inflation this effectively says that at the optimum the $MRS_{x_t, x_{t+1}} = (r - \pi) - \rho$, the real interest rate (the nominal interest rate r minus inflation π) minus the rate of time preference. If the real interest rate and the rate of time preference are equal then the $MRS_{x_t, x_{t+1}} = 1$. Essentially this says that consumers will prefer to consume more later if (i) real rate of return ($r - \pi$) is high, and (ii) they are patient so ρ is low. In general individuals will consume more in periods where p_t is low. The phenomenon by which individuals substitute consumption over time is known as the **intertemporal substitution of consumption**.

The FOC for l_t using the fact that $\frac{\partial L_t}{\partial l_t} = -1$ is

$$\frac{\partial \mathcal{L}}{\partial l_t} = \left(\frac{1}{1 + \rho} \right)^t \frac{\partial u}{\partial l}(x_t^D, l_t^D) - \alpha \left(\frac{1}{1 + r} \right)^t w_t = 0$$

which yields a similar Euler equation for the **intertemporal substitution of leisure**

$$\frac{MRS_{l_t, l_{t+1}}}{1 + \rho} = \frac{\frac{\partial u}{\partial l}(x_t^D, l_t^D)}{\frac{\partial u}{\partial l}(x_{t+1}^D, l_{t+1}^D)} = \frac{1 + r}{1 + \rho} \frac{w_t}{w_{t+1}} \quad (\text{ISL})$$

which says that people will take take leisure (and work less) when the cost of doing so is low, i.e. wages are low. If they are impatient they are more likely to put off work until later in their lives.

Combining the FOC for consumption and leisure we also get the conventional **static condition** between leisure and consumption in period t

$$MRS_{x_t, x_{t+1}} = \frac{\frac{\partial u}{\partial l}(x_t^D, l_t^D)}{\frac{\partial u}{\partial x}(x_t^D, l_t^D)} = \frac{w_t}{p_t} \quad (\text{SC})$$

which should be very familiar if we remove the subscripts t . One word of caution is that while the FOC are relatively easy to interpret, solving these models in all but the simplest situations can be quite burdensome.

3 Expected Utility

3.1 The Basics

Expected Utility (EU) theory is a technique developed by Von Neumann and Morgenstern (1944) to deal with situations of quantifiable risk. It requires preferences to exhibit two additional axioms of *continuity* and *independence*, which are somewhat controversial. Assume that states of nature can be indexed by an $s = 1, \dots, S$, each with a probability of occurring of p_1, \dots, p_S , which as probabilities obey $p_s \geq 0$ and $\sum_{s=1}^S p_s = 1$.⁴ Let x_s be the realization of some random variable, sometimes known as a **prospect** or **lottery**, x in state s , which yields utility $u(x_s)$. The **Expected Utility Theorem** states that if

⁴This approach breaks down if the uncertainty is unquantifiable, i.e. you cannot attach numerical probabilities to each state. Risk is generally defined as quantifiable uncertainty.

consumers have rational preferences that exhibit continuity and independence⁵ then agents will behave *as if* they maximize the expected value of their utility or just **expected utility**

$$E[u(x)] = \sum_{s=1}^S p_s u(x_s) \quad (\text{EU Utility})$$

Similarly, firms can be assumed to maximize expected profits $E[\pi(x)]$ over various states of the world. The nature of the budget constraint will vary considerably upon the situation considered.

3.2 Two State Set Up

The easiest situation to set up is a 2 state set up with $p_1 = p$ and $p_2 = 1 - p$. Individuals maximize

$$E[u(x)] = pu(x_1) + (1 - p)u(x_2)$$

The marginal rate of substitution of x between states is given by

$$MRS_{x_1x_2} = \frac{p}{1 - p} \frac{u'(x_1)}{u'(x_2)}$$

Along the 45 degree line of a graph of x_1 and x_2 , where $x_1 = x_2$, then $u'(x_1) = u'(x_2)$ and so $MRS_{x_1x_2} = p/(1 - p)$ the **odds-ratio** (the proportion of state 1's to state 2's) no matter what the utility function looks like. This is one of the strong restrictions of expected utility theory.

3.3 Risk Aversion

Imagine a person faced with the prospect of a fair 50-50 bet. If the person with money M takes the bet $b > 0$ then $x_1 = M + b$ if he wins and $x_2 = M - b$ if he loses. With no bet $x_1 = x_2 = M$. The bet is fair since $E[b] = \frac{1}{2}b + \frac{1}{2}(-b) = 0$ and so the expected value of the prospect is M . The utility from not taking the bet is just $u(M)$, while the utility taking it is $\frac{1}{2}u(M + b) + \frac{1}{2}u(M - b)$. The person will decline or accept the bet depending on whether

$$u(M) \gtrless \frac{1}{2}u(M + b) + \frac{1}{2}u(M - b)$$

A person who rejects the bet ($>$) is called **risk averse**, takes the bet ($<$) is called **risk loving**, and indifferent about it ($=$) is **risk neutral**.

Generalizing to any prospect x we compare what the utility of its expected value of its expected utility $u[E(x)] \gtrless E[u(x)]$; $>$ implies risk aversion, $<$ risk loving, and $=$ risk neutrality. A mathematical fact known as *Jensen's Inequality* tells us that risk aversion is reflected in a $u(x)$ that is concave, i.e. $u''(x) < 0$ when x is a single variable. Similarly, risk loving implies a convex u , $u''(x) > 0$, and risk neutrality a linear u , $u''(x) = 0$. Of course there are different degrees of risk aversion (or loving), such as that measured by coefficient of relative risk aversion $\gamma(x)$ defined above.

4 Insurance

4.1 Efficient Insurance

Assuming individuals are risk averse and *actuarially fair insurance* exists (i.e. insurance with expected cost to the consumer of zero) then it can be shown that individuals will always choose to *insure fully* (i.e. eliminate all risk). Suppose an agent a utility function which depends only on income, which she has M to start out with. Let $p > 0$ be the chance of an accident which causes a loss d of income. An agent can buy insurance contract (a, b) which has a premium b but pays out a net amount a in case the accident

⁵Intuitively, **continuity** implies that very slight changes in probability will not affect a strict preference of a prospect x over a prospect y . **Independence** implies that if x is preferred to y and we mix x and y each with the same lottery z , so that $x' = x$ with a 50% chance and z with a 50% chance (and y' defined similarly) then x' will be preferred to y' .

occurs. So the expected utility of an agent that buys such a contract is $pu(M - d + a) + (1 - p)u(M - b)$. You can think of state 1 as the accident state and state 2 as the non-accident state with $x_1 = M - d + a$, and $x_2 = M - b$.

The expected profit of a firm which offers this contract is $\pi = p(-a) + (1 - p)b$ (we leave out the expectations operator E) The presence of actuarially fair insurance can be justified by assuming insurance markets are *competitive* and firms are *risk neutral*: competition among firms will drive expected profits to zero $\pi = 0 \Rightarrow ap = b(1 - p) \Rightarrow b = ap/(1 - p)$. The effective "budget constraint" has a $MRT_{x_1x_2} = -\frac{dx_2}{dx_1} = \frac{db}{da} = \frac{p}{1-p}$. The choice of optimal insurance is found by finding the optimal a

$$\max_a pu(M - d + a) + (1 - p)u\left(M - a\frac{p}{1-p}\right)$$

Taking the FOC with respect to a

$$pu'(M - d + a^*) + (1 - p)u'\left(M - a^*\frac{p}{1-p}\right)\left(-\frac{p}{1-p}\right) = 0$$

which rearranging and cancelling out $(1 - p)$ and p implies

$$u'(M - d + a^*) = u'\left(M - a^*\frac{p}{1-p}\right)$$

If agents are risk averse then concavity implies u' is decreasing and so $u'(x_1) = u'(x_2)$ implies $x_1 = x_2$ and so agents will have the same effective income in either state, they are fully insured. We can solve for the premium as

$$M - d + a^* = M - a^*\frac{p}{1-p} \Rightarrow a^* = (1 - p)d$$

and so $b^* = a^*p/(1 - p) = pd$ and the optimal contract will be $(a^*, b^*) = ((1 - p)d, pd)$. In each state income is $x_1^* = x_2^* = M - pd$. This is shown in a diagram since the zero profit condition implies that the marginal rate of transformation $MRT_{x_1x_2} = \frac{p}{1-p}$ equals the marginal rate of substitution $MRS_{x_1x_2}$ along the 45 degree line (see above) where the two allocations are equal.

4.2 Adverse Selection and Insurance

The problem of adverse selection in insurance markets was laid about by Rothschild and Stiglitz (1976). Take the same setup as above except assume there are two types of individuals - low risk types L and high risk types H with accident probabilities p_L and p_H respectively, where $p_L < p_H$, but where M, u , and d are the same for both types. Competition among firms will have some interesting implications as it implies firms will always offer contracts which can make at least zero profits. Say proportion λ are low risk types and so $(1 - \lambda)$ are high risk types. If insurance companies can tell the two apart then they can just offer each type the efficient contract $(a_L^*, b_L^*) = ((1 - p_L)d, p_Ld)$ and $(a_H^*, b_H^*) = ((1 - p_H)d, p_Hd)$.

4.2.1 Failure of Efficient Insurance

If the types are not observable by the insurance firms then the efficient contracts no longer work as firms cannot prevent one type from taking the other type's efficient contract. The high types H all want to pretend to be low types L as the accident benefit is higher as $a_L^* = (1 - p_L)d > (1 - p_H)d = a_H^*$ and premium costs are lower $b_L^* = p_Ld < p_Hd = b_H^*$. Since firms compete to get the low-risk types $\pi_L = -p_La_L^* + (1 - p_L)b_L^* = 0$ and so overall profits when high types take the efficient low risk contract will turn negative:

$$\begin{aligned} \lambda\pi_L + (1 - \lambda)\pi_H &= 0 + (1 - \lambda)[(1 - p_H)b_L^* - p_Ha_L^*] \\ &= (1 - \lambda)[(1 - p_H)p_Ld - p_H(1 - p_L)d] \\ &= d(1 - \lambda)[p_L - p_H] < 0 \end{aligned}$$

where the inequality comes from the fact that $p_L < p_H$. Therefore the efficient contracts cannot be an equilibrium, and some other suboptimal equilibrium must be found. There are two main possibilities to consider: (i) where firms offer a one-size-fits-all or "pooling" contract (a^P, b^P) which both types will take and (ii) where firms offer "separating" contracts, one for low risk types (a_L^S, b_L^S) and one for high risk types (a_H^S, b_H^S) which are designed so that each type voluntarily self-selects into buying the contract made for it.

4.2.2 Pooling Contracts

Say a firm tries to institute a pooling contract (a^P, b^P) so that everyone buys it. The question is can this contract can work as an equilibrium. (The answer is no) If a firm can offer a profitable contract which at least one type will take, then the equilibrium will fall apart. In fact, for any pooling equilibrium there is always a contract (a', b') that is better for the low risk types and is profitable. Therefore any pooling equilibrium will fall apart.

The overall probability of accident $p = \lambda p_L + (1 - \lambda) p_H$ and so $p_L < p < p_H$. The zero profit condition is that $\pi^P = -p a^P + (1 - p) b^P = 0$ and thus $b^P = a^P p / (1 - p)$. Therefore the $MRT_{x_1 x_2} = \frac{p}{1-p}$. Now at any point the indifference curves of high types and low types will cross, as high types will have a higher $MRS_{x_1 x_2}^H$.

$$MRS_{x_1 x_2}^H > MRS_{x_1 x_2}^L \Leftrightarrow \frac{p^H}{1-p^H} \frac{u'(x_1^P)}{u'(x_2^P)} > \frac{p^L}{1-p^L} \frac{u'(x_1^P)}{u'(x_2^P)} \Leftrightarrow \frac{p^H}{1-p^H} > \frac{p^L}{1-p^L} \Leftrightarrow p^H > p^L$$

which makes sense since high types will value consumption in the accident state 1 more than in the non-accident state 2. However the pooling contract that satisfies $MRS_{x_1 x_2}^H > \frac{p}{1-p} > MRS_{x_1 x_2}^L$, will not be attractive to low risk types when offered another contract.

Let $K = (MRS_{x_1 x_2}^L + MRS_{x_1 x_2}^H) / 2$ be the average of the $MRS_{x_1 x_2}$ for both types. Now for some small $\varepsilon > 0$, consider the following contract that offers just slightly less coverage ε , but requires a lower premium $K\varepsilon$, $(a', b') = (a^P - \varepsilon, b^P - K\varepsilon)$, implying $x_1' = x_1^P - \varepsilon$, and $x_2' + K\varepsilon$. The low types will take this contract since it will yield a higher utility than (a^P, b^P) , which can be shown using a differential argument from calculus⁶

$$\begin{aligned} U_L(x_1^P, x_2^P) - U_L(a', b') &= p_L [u(x_1^P - \varepsilon) - u(x_1^P)] + (1 - p_L) [u(x_2^P + K\varepsilon) - u(x_2^P)] \\ &\simeq p_L u'(x_1^P) (-\varepsilon) + (1 - p_L) u'(x_2^P) (K\varepsilon) \end{aligned}$$

This quantity is positive since

$$(1 - p_L) u'(x_2^P) (K\varepsilon) - p_L u'(x_1^P) (\varepsilon) > 0 \Leftrightarrow K > \frac{p^L}{1-p^L} \frac{u'(x_1^P)}{u'(x_2^P)} = MRS_{x_1 x_2}^L$$

For the same reason high risk types will not like this contract since $K < MRS_{x_1 x_2}^H$ because the reduction in coverage is not made up enough for them by the reduction in the premium. Firms will want to offer such a contract as they can make a positive profit from it

$$\begin{aligned} \pi' &= -p_L (a^P - \varepsilon) + (1 - p_L) (b^P - K\varepsilon) \\ &= [-p_L a^P + (1 - p_L) b^P] + \varepsilon [p_L - (1 - p_L) K] \\ &= \left[-p_L a^P + (1 - p_L) \frac{p}{1-p} a^P \right] + \varepsilon [p_L - (1 - p_L) K] \\ &= a^P \left[\frac{p - p_L}{1-p} \right] + \varepsilon [p_L - (1 - p_L) K] \end{aligned}$$

⁶Remember the definition of a derivative is

$$u'(x) = \lim_{\varepsilon \rightarrow 0} \frac{u(x+\varepsilon) - u(x)}{\varepsilon}$$

and so for a small $\varepsilon > 0$, $u'(x) = \frac{u(x+\varepsilon) - u(x)}{\varepsilon}$ which rearranging implies $u(x + \varepsilon) - u(x) = \varepsilon u'(x)$.

The first term is always positive while the second term is negative. The firm will just pick an $\varepsilon > 0$ small enough the it can make the second term negligibly small and assure itself positive profits. Intutively, the pooling contract makes positive profits from the low risk types which are offset by losses from the high risk types. This provides an incentive for another firm to offer a contract which is slightly more attractive to low risk types and slightly less attractive to high risk types. Low risk types will switch to the new contract, and so the new firm makes all of the positive profits while the old firm is saddled with the high risk types and loses money - its contract doesn't work.

4.2.3 Separating Contracts

We know that no pooling contract will ever work as it will lose out to a separating contract. However that separating contract is not an equilibrium since the former pooling contract which serves the high types no longer works. An equilibrium pair of separating contracts (a_L^S, b_L^S) and (a_H^S, b_H^S) must be stable for both types. As we saw earlier it is impossible for both types to get their respective efficient contracts as high risk types prefer the low risk efficient contract to their own optimal contract. However, it is possible for high risk types to get their efficient contract will low types get an *inefficient* contract, since low types do not want the high risk efficient contract. In, fact competition amongst firms for the high types business will assure that the high risk types will get their efficient contract $(a_H^S, b_H^S) = (a_H^*, b_H^*)$ in a separating equilibrium.

The low types can only get the most efficient contract (a_L^S, b_L^S) that high risk types will *not* want to take. We model this by making *high types indifferent about both contracts*, assuming they take the one for the high types. Let (x_{1H}^*, x_{2H}^*) be the amounts in each state from the efficient high type separating contract and (x_{1L}^S, x_{2L}^S) be that for the low types. Since utility for the high types is $U_H^* = p_H u(x_{1H}^*) + (1 - p_H) u(x_{2H}^*)$ then

$$U_H^* = p_H u(x_{1L}^S) + (1 - p_H) u(x_{2L}^S) = p_H u(M - d + a_L^S) + (1 - p_H) u(M - b_L^S) \quad (\text{Cond 1})$$

is the implicit restriction on (x_{1L}^S, x_{2L}^S) : be careful to note that it is the high risk probabilities for the low risk contract. The contract (a_L^S, b_L^S) that will result in a separating equilibrium will satisfy the above condition and satisfy the *zero profit condition* for the low risk types

$$b_L^S = a_L^S p_L / (1 - p_L) \quad (\text{Cond 2})$$

Substituting in (Cond 2) into (Cond 1) we have

$$U_H^* = p_H u(M - d + a_L^S) + (1 - p_H) u(M - a_L^S p_L / (1 - p_L))$$

This one equation implicitly defines the one variable a_L^S . Differentiating with respect to a_L^S we get

$$\begin{aligned} 0 &= p_H u'(M - d + a_L^S) + (1 - p_H) u'(M - a_L^S p_L / (1 - p_L)) \left(-\frac{p_L}{1 - p_L} \right) \\ \Rightarrow u'(M - d + a_L^S) &= \frac{1 - p_H}{p_H} \frac{p_L}{1 - p_L} u' [M - a_L^S p_L / (1 - p_L)] \end{aligned}$$

and so the marginal rate of substitution is

$$MRS_{x_1 x_2}^L = \frac{u'(x_{1L}^S)}{u'(x_{2L}^S)} = \frac{1 - p_H}{p_H} \frac{p_L}{1 - p_L} \neq \frac{p_L}{1 - p_L}$$

and so this results in low risk types being underinsured relative to what would be efficient.

So far we have found a set of equilibrium contracts that Because the contract for low types is inefficient there are lots of possible contracts that could preferred by low risk types that would increase profits. However a more efficient contract for the low risk types will also attract the high risk types. The question is whether there are pooling contracts which could also sustain high risk types *and* still make a profit. If λ is big and there are relatively few high profit types then such a pooling contract exists. Then the separating equilibrium we just solved is not a true equilibrium as some firm can offer the pooling contract which will

cause both types to abandon the separating contracts. In this case there will be *no market equilibrium whatsoever*. Any potential pooling equilibrium will be abandoned for separating contracts and any potential separating equilibrium will be abandoned for a pooling one!

This contract (\tilde{a}, \tilde{b}) has to make at least zero profits so

$$p\tilde{b} = (1 - p)\tilde{a}$$

and it has to be better for low risk types (which automatically makes it better for high risk types) which substituting in the above means

$$p_L u(M - d + \tilde{a}) + (1 - p_L) u(M - \tilde{a}p / (1 - p)) > p_L u(M - d + a_L^S) + (1 - p_L) u(M - a_L^S p_L / (1 - p_L))$$

So rearranging, if there there exists an \tilde{a} that satisfies this above condition then there will be no separating equilibrium either. As the proportion of low risk types goes to one, $\lambda \rightarrow 1$, then $p \rightarrow p_L$ and this condition becomes whether or not there exists an \tilde{a} such that

$$p_L u(M - d + \tilde{a}) + (1 - p_L) u(M - \tilde{a}p_L / (1 - p_L)) > p_L u(M - d + a_L^S) + (1 - p_L) u(M - a_L^S p_L / (1 - p_L))$$

which holds by the inefficiency of a_L^S showed earlier. Graphically this can be checked by seeing if the zero profit constraint for a pooled contract lies beneath the indifference curve for low types at the separating equilibrium. If it is beneath then the above condition is not satisfied and the separating contracts are an equilibrium. If it crosses this indifference curve then there is no equilibrium. An interesting conclusion of this paper is that it takes only a few high risk types (λ close to one) for the entire insurance market to fall apart.