Economics 201B–Second Half

Lecture 3, 3/16/10

The Welfare Theorems in the Edgeworth Box

Walrasian Equilibrium with Transfers: An *income transfer* is a vector $T \in \mathbf{R}^2$ such that $T_1 + T_2 = 0$ (budget balance).

$$\begin{aligned} B_i(p,T) &= \{x \in \mathbf{R}^2_+ : p \cdot x \le p \cdot \omega_i + T_i\} \\ D_i(p,T) &= \{x \in B_i(p,T) : x \succeq_i y \text{ for all } y \in B_i(p,T)\} \end{aligned}$$

A *Walrasian Equilibrium with Transfers* is a triple (p, x, T) where

- x is an exact allocation;
- T is an income transfer;
- $x_i \in D_i(p,T)$ $(i = 1,2)$.

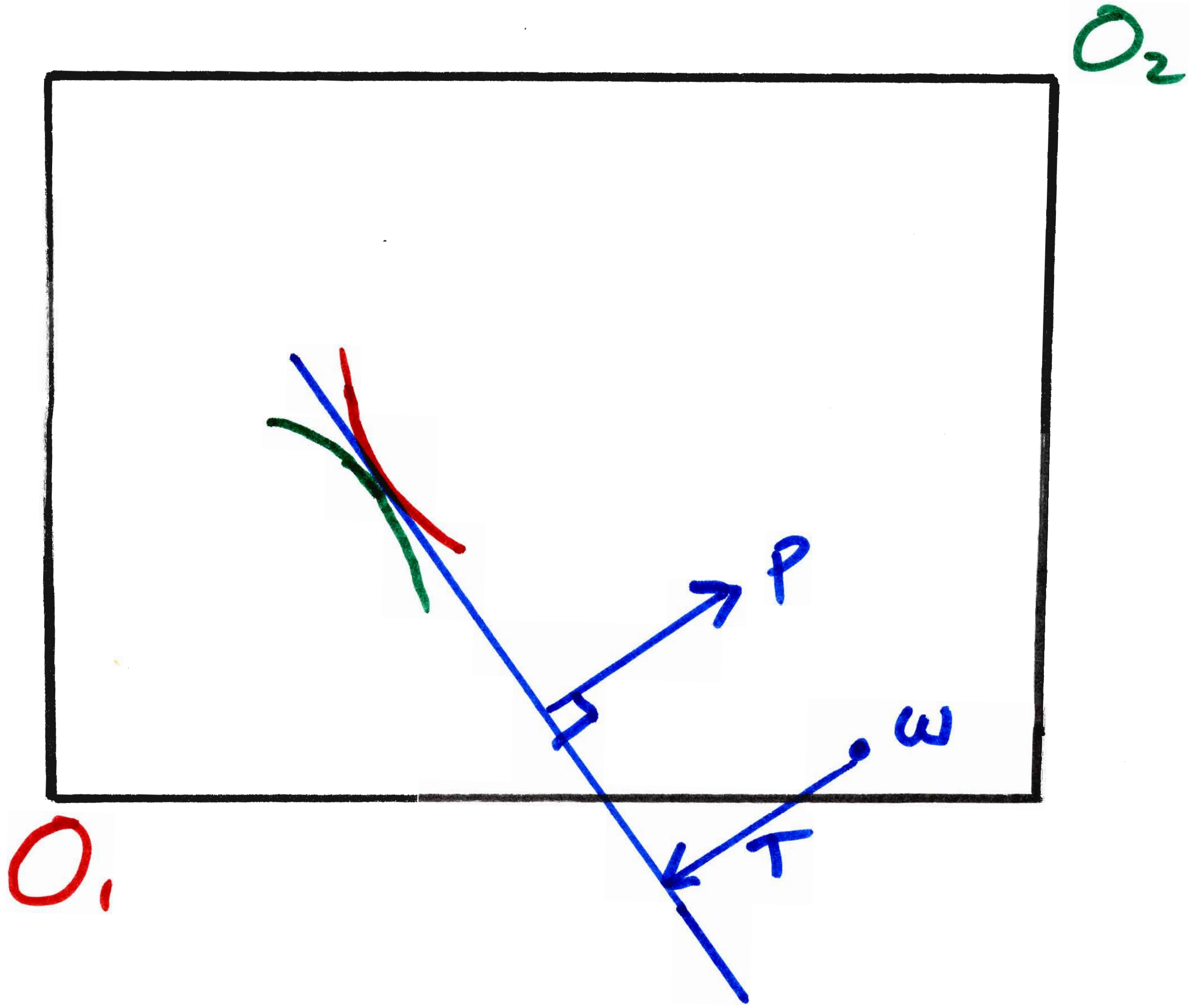
Observe that $(p, x, 0)$ is a Walrasian Equilibrium with Transfers if and only if (p, x) is a Walrasian Equilibrium.

Theorem 1 (Weak First Welfare Theorem, Edgeworth Box) *In the Edgeworth Box, every Walrasian Equilibrium with Transfers is weakly Pareto Optimal.*

Proof: Let (p, x, T) be a Walrasian Equilibrium with Transfers. Suppose x is *not* weakly Pareto Optimal. Then we can find $x' \in (\mathbf{R}^2_+)^2$ such that

$$\begin{aligned} x'_1 &+ x'_2 = \bar{\omega} \\ x'_i &\succ_i x_i \ (i = 1,2) \end{aligned}$$

$x'_i \succ_i x_i$ implies $x_i \not\succeq_i x'_i$; since $x_i \in D_i(p,T)$, $p \cdot x'_i > p \cdot \omega_i + T_i$, so



O2
P
ω
T
O1

$$
\begin{aligned}
p \cdot \bar{\omega} &= p \cdot (x_1' + x_2') \\
&= p \cdot x_1' + p \cdot x_2' \\
&> p \cdot \omega_1 + T_1 + p \cdot \omega_2 + T_2 \\
&= p \cdot \omega_1 + p \cdot \omega_2 \\
&= p \cdot (\omega_1 + \omega_2) \\
&= p \cdot \bar{\omega}
\end{aligned}
$$

so $p \cdot \bar{\omega} > p \cdot \bar{\omega}$, a contradiction which proves that x is weakly Pareto Optimal.■

Note: We used *nothing* except the definition of Walrasian Equilibrium with Transfers.

Theorem 2 (First Welfare Theorem, Edgeworth Box) *In the Edgeworth Box, every Walrasian Equilibrium with Transfers is Pareto Optimal.*

Proof: Let (p, x, T) be a Walrasian Equilibrium with Transfers, so $x_i \in D_i(p, T)$ $(i = 1, 2)$. Suppose x is not Pareto Optimal. Then we may find $x' \in (\mathbf{R}_+^2)^2$ such that

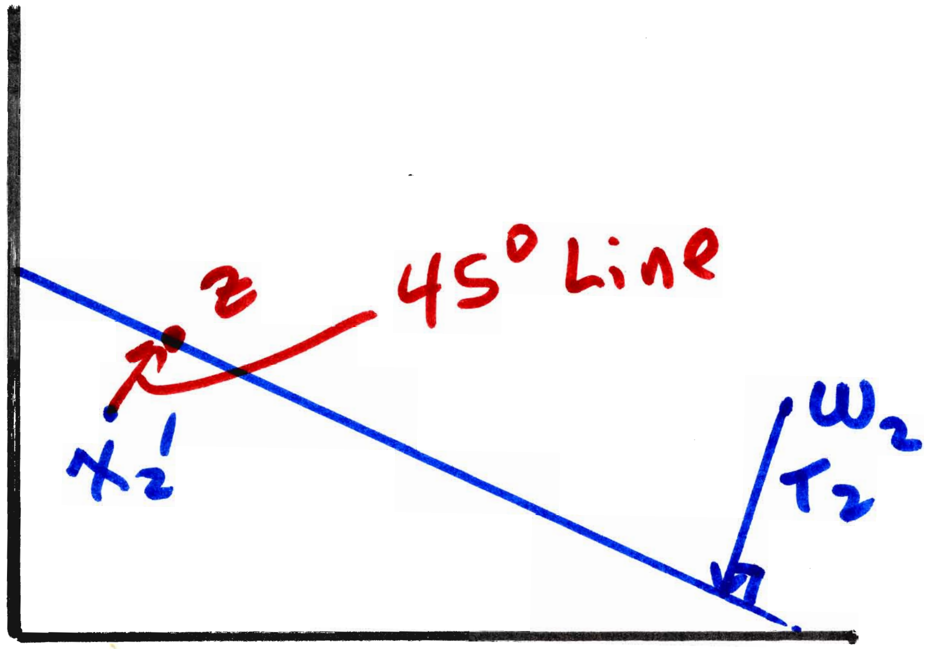
$$
\begin{aligned}
x_1' &+ x_2' = \bar{\omega} \\
x_i' &\succeq_i x_i \ (i = 1, 2) \\
x_i' &\succ_i x_i \text{ (some i, WLOG i=1)}
\end{aligned}
$$

$x_1' \succ_1 x_1$ implies $x_1 \not\succeq_i x_1'$; since $x_1 \in D_1(p, T)$, $p \cdot x_1' > p \cdot \omega_1 + T_1$.

Claim: $p \cdot x_2' \geq p \cdot \omega_2 + T_2$

If not, $p \cdot x_2' < p \cdot \omega_2 + T_2$. Let

$$
\begin{aligned}
\delta &= \frac{p \cdot \omega_2 + T_2 - p \cdot x_2'}{|p_1| + |p_2|} > 0 \\
z &= x_2' + (\delta, \delta) > x_2'
\end{aligned}
$$



45° Line
z
x_2'
w_2
T_2

$$
\begin{aligned}
p \cdot z &= p \cdot x_2' + p_1\delta + p_2\delta \\
&\leq p \cdot x_2' + (|p_1| + |p_2|)\delta \\
&= p \cdot x_2' + p \cdot \omega_2 + T_2 - p \cdot x_2' \\
&= p \cdot \omega_2 + T_2
\end{aligned}
$$

so $z \in B_i(p,T)$, $z \succ_2 x_2' \succeq_2 x_2$. By Transitivity, $z \succ_2 x_2$, so $x \notin D_2(p,T)$, a contradiction that proves the claim that $p \cdot x_2' \geq p \cdot \omega_2 + T_2$.

$$
\begin{aligned}
p \cdot \bar{\omega} &= p \cdot (x_1' + x_2') \\
&= p \cdot x_1' + p \cdot x_2' \\
&> p \cdot \omega_1 + T_1 + p \cdot \omega_2 + T_2 \\
&= p \cdot \omega_1 + p \cdot \omega_2 \\
&= p \cdot (\omega_1 + \omega_2) \\
&= p \cdot \bar{\omega}
\end{aligned}
$$

so $p \cdot \bar{\omega} > p \cdot \bar{\omega}$, a contradiction which proves that x is Pareto Optimal. ■

Theorem 3 (Second Welfare Theorem, Edgeworth Box) *In the Edgeworth Box Economy, every Pareto Optimum is a Walrasian Equilibrium with Transfers.*

Interpretation:

- Unbiasedness of Walrasian Equilibrium: You can give all consumption to agent 1, or all consumption to agent 2, or anything in between. Walrasian Equilibrium doesn't bias distribution, as long as we allow for income transfers.

- Income transfers are lump sum taxes. Tax depends on your innate characteristics, not on your choices. This is not like an income tax, which is based on your trade (you sell your leisure to earn income to buy goods). In an income tax, you are taxed on the income you earned, not the income you could have earned. Real taxes induce distortions which thwart Pareto Optimality. However, in some cases (such as externalities) where Walrasian equilibrium is not Pareto Optimal, taxes can be used to improve efficiency.

- Informational efficiency: A social planner need not know people's preferences and assign a specific allocation. Instead, she imposes lump sum taxes and lets the market operate. If she doesn't like the resulting distribution, she can fine tune the transfers until she is satisfied.

"**Proof:**" Suppose x is Pareto Optimal, so x is an exact allocation. Let p be the "support price" at x, i.e.

$$y \succ_i x_i \Rightarrow p \cdot y > p \cdot x_i \ (i = 1, 2)$$

(As before, we defer proving existence of the support price until we do the general case.) Let

$$\begin{aligned}
T_i &= p \cdot x_i - p \cdot \omega_i \\
T_1 + T_2 &= p \cdot x_1 - p \cdot \omega_1 + p \cdot x_2 - p \cdot \omega_2 \\
&= p \cdot (x_1 + x_2) - p \cdot (\omega_1 + \omega_2) \\
&= p \cdot \bar{\omega} - p \cdot \bar{\omega} \\
&= 0 \text{ (that's Budget Balance!)} \\
B_i(p, T) &= \{z \in \mathbf{R}^2_+ : p \cdot z \leq p \cdot \omega_i + T_i\} \\
&= \{z \in \mathbf{R}^2_+ : p \cdot z \leq p \cdot \omega_i + p \cdot x_i - p \cdot \omega_i\} \\
&= \{z \in \mathbf{R}^2_+ : p \cdot z \leq p \cdot x_i\} \text{ (so } x_i \in B_i(p, T)\text{!)} \\
x_i \not\succeq_i y &\Rightarrow y \succ_i x_i \text{ (completeness)} \\
&\Rightarrow p \cdot y > p \cdot x_i \text{ (support price)}
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow \quad p \cdot y > p \cdot \omega_i + p \cdot x_i - p \cdot \omega_i \\
&\Rightarrow \quad p \cdot y > p \cdot \omega_i + T_i \\
&\Rightarrow \quad y \notin B_i(p, T) \\
y \in B_i(p, T) &\Rightarrow \quad x_i \succeq_i y \text{ (so } x_i \in D_i(p, T)\text{!)}
\end{aligned}
$$

so (p, x, T) is a Walrasian Equilibrium with Transfers. ■