Economics 201B–Second Half
Lecture 9, 4/13/10, Revised 4/13/10

Generic Local Uniqueness of Equilibrium

Comparative Statics: In what direction does the equilibrium move if the underlying parameters of the economy change?

A Foundation for Comparative Statics:

1. *Local Uniqueness*: For every equilibrium price $p^* \in \Delta$, there exists $\delta > 0$ such that there is no equilibrium price $q^* \in \Delta$ such that
$$q^* \neq p^*, \ |q^* - p^*| < \delta$$
2. For a sufficiently small change in the parameters of the economy, the number of equilibria is unchanged and each equilibrium moves
$$\begin{pmatrix} \text{continuously} \\ \text{differentiably} \end{pmatrix}$$
as the parameter changes.

Remark 1 If local uniqueness fails, lattice-theoretic methods may still allow us to establish comparative statics results. We can look at an equilibrium correspondence and we may be able to say that the *set* of equilibria moves in a particular direction in response to a change in the underlying parameters (Milgrom, Shannon, others).

Two-Good Economy: Consider a 2-good economy, normalized prices $p \in \Delta^0$,
$$z(p) = \sum_{i=1}^{I} D_i(p) - \bar{\omega}$$

- Walras' Law with Equality implies that

$$z(p)_1 = 0 \Rightarrow z(p)_2 = 0$$

so we can capture the situation in a diagram in $\mathbf{R}^2$; let

$$\hat{z}(p_1) = z(p_1, 1 - p_1)_1$$

and plot $\hat{z}$ as a function of p_1.

- In Diagram I, a small shift in $\hat{z}$ results in a small shift of the equilibrium price; comparative statics are locally meaningful. Notice that

$$\hat{z}(p_1) = 0 \Rightarrow \hat{z}'(p_1) \neq 0$$

$\hat{z}$ cuts cleanly through 0, so we expect an odd number of equilibria.

- In Diagram II, there are two equilibria p_L^* and p_R^*.
  - A small shift in $\hat{z}$ results in a small shift in p_L^*
  - A small upward shift in $\hat{z}$ causes p_R^* to split in two; one moves left, the other moves right, so no local comparative statics.
  - A small downward shift in $\hat{z}$ cause p_R^* to disappear!
  - Notice that

$$\hat{z}(p_R^*) = 0 \text{ but } \hat{z}'(p_R^*) = 0$$

- Diagram III shows we could even have a whole interval of equilibrium prices. A small change in $\hat{z}$ results in a *discontinuous* shift in equilibrium price.

Multi-Good Case:

- Normalize $p_L = 1$ rather than $p \in \Delta$. Price is represented by

$$\hat{p} = (p_1, \ldots, p_{L-1}) \in \mathbf{R}^{L-1}_{++}$$

- Let
$$\hat{z}(\hat{p}) = (z_1(\hat{p},1),\ldots,z_{L-1}(\hat{p},1))$$
- Walras' Law with Equality and $p \gg 0$ implies that
$$\hat{z}(\hat{p}) = 0 \Leftrightarrow z(\hat{p},1) = 0$$
- Observe that
$$\hat{z} : \mathbf{R}^{L-1}_{++} \to \mathbf{R}^{L-1}$$
so $D\hat{z}$, the Jacobian matrix of $\hat{z}$, is $(L-1) \times (L-1)$
- *Definition*: An equilibrium price p^* is *regular* if
$$\det D\hat{z}\big|_{\hat{p}^*} \neq 0$$
This is equivalent to
$$\text{rank }\ Dz\big|_{p^*} = L-1$$
A *regular economy* is an economy for which every equilibrium price is regular.
- *Maintained Hypotheses for Remainder of 17.D*:
  - z satisfies the hypotheses of the Debreu-Gale-Kuhn-Nikaido Lemma.
  - z is homogeneous of degree zero, i.e.
  $$\forall_{p \in \mathbf{R}^L_{++},\, \lambda > 0} z(\lambda p) = z(p)$$
  Caution: $\hat{z}$ is *not* homogeneous because it is a representation of a normalized price ($p_L = 1$).
  - z is C^1. This appears technical, but it's strong and has economic consequences because it rules out boundary consumptions: demand necessarily has a kink at the price

where demand first hits boundary. This can be weakened to allow boundary consumptions, and the theorems more or less hold.

Proposition 2 (17.D.1) *In a regular economy, (normalized) Walrasian Equilibrium prices are locally unique, and there are only finitely many equilibria.*

Proof: Suppose $\hat{z}(\hat{p}^*) = 0$. Since the economy is regular, $D\hat{z}|_{\hat{p}^*}$ is nonsingular. By the Inverse Function Theorem, there is a neighborhood U of $\hat{p}^*$ and a neighborhood V of 0 and a C^1 function $h : V \to U$, h is one-to-one and onto such that

$$\forall_{v \in V}\ \hat{z}(h(v)) = v,\ \forall_{u \in U}\ h(\hat{z}(u)) = u$$

If $u \in U$ and $\hat{z}(u) = 0$,

$$\begin{aligned} u &= h(\hat{z}(u)) \\ &= h(0) \\ \hat{p}^* &= h(\hat{z}(\hat{p}^*)) \\ &= h(0) \end{aligned}$$

Since h is a function, $u = \hat{p}^*$, so Equilibrium is locally unique.

Now, we show that there are a finite number of equilibria.

Claim: There is a compact set $\hat{K} \subset \mathbf{R}^{L-1}_{++}$ such that if $\hat{z}(\hat{p}^*) = 0$, then $\hat{p}^* \in \hat{K}$.

Suppose the claim is not true. Define $\phi : \mathbf{R}^{L-1}_{++} \to \Delta^0$ by

$$\phi(\hat{p}) = \frac{(\hat{p}, 1)}{\hat{p}_1 + \ldots + \hat{p}_{L-1} + 1}$$

Observe that ϕ is one-to-one and onto, is continuous, and has continuous inverse. Let

$$K_n = \left\{ p \in \Delta^0 : p_\ell \geq \frac{1}{n}\ (1 \leq \ell \leq L) \right\}$$

Then $\phi^{-1}(K_n)$ is the continuous image of a compact set, hence a compact subset of $\mathbf{R}^{L-1}_{++}$, so the set of equilibrium prices cannnot be contained in $\phi^{-1}(K_n)$. Thus, we may find a sequence $\hat{p}^*_n$ of equilibrium prices such that $\phi(\hat{p}^*_n) \notin K_n$. Since Δ is compact, we can find a subsequence

$$\phi(\hat{p}^*_{n_k}) \to p \in \Delta \setminus \Delta^0$$

By the Boundary Condition,

$$|z(\phi(\hat{p}^*_{n_k}))| \to \infty$$

but

$$\begin{aligned} z(\phi(\hat{p}^*_{n_k})) &= z(\hat{p}^*_{n_k}, 1) \text{ (homogeneity of degree zero)} \\ &= 0 \text{ (since } \hat{z}(\hat{p}^*_{n_k}) = 0) \end{aligned}$$

a contradiction which proves the claim.

*** Let $E = \left\{\hat{p} \in \mathbf{R}^{L-1}_{++} : \hat{z}(\hat{p}) = 0\right\}$. Suppose $\hat{p}_n \in E$ and $\hat{p}_n \to \hat{p}$. Since $E \subseteq K$ which is compact, $\hat{p} \in K \subseteq \mathbf{R}^{L-1}_{++}$. $\hat{z}(\hat{p}) = \lim \hat{z}(\hat{p}_n) = 0$, since $\hat{z}$ is continuous, so $\hat{p} \in E$. Thus, E is a closed subset of the compact set K, so E is compact.

For each $\hat{p} \in E$, we may find $\delta_{\hat{p}} > 0$ such that

$$E \cap B\left(\hat{p}, \delta_{\hat{p}}\right) = \{\hat{p}\}$$

The collection

$$\left\{B\left(\hat{p}, \delta_{\hat{p}}\right) : \hat{p} \in E\right\}$$

is an open cover of E, hence has a finite subcover. Since each element of this finite subcover contains exactly one element of E, E is finite. ■

The Index Theorem

- Throughout, we will denote a price in $\Delta = \{p \in \mathbf{R}^L_+ : \Sigma_{\ell=1}^L p_\ell = 1\}$ by p, and the associated price in $\mathbf{R}^{L-1}_{++}$ (with the assumption that the price of good L has been normalized to 1) by $\hat{p}$.

- *Definition*: If p^* is a regular equilibrium price, define

$$\text{index } (p^*) = (-1)^{L-1} \text{sign } \det D\hat{z}|_{\hat{p}^*}$$

- For $L = 2$,

$$\begin{aligned} \text{index } (p^*) &= (-1)^1 \text{sign } \det D\hat{z}|_{\hat{p}^*} \\ &= -\text{sign } \hat{z}'(\hat{p}^*) \end{aligned}$$

  1. index $(p^*) = +1$ means that $\hat{z}'(\hat{p}^*) < 0$; that means demand is downward sloping, so we are in the "normal" case in which an increase in $\hat{z}$, the excess demand for good 1, results in an increase in the equilibrium price of good 1.
  2. index $(p^*) = -1$ means that $\hat{z}'(\hat{p}^*) > 0$; that means demand is upward sloping, so we are in the "abnormal" case in which an increase in $\hat{z}$, the excess demand for good 1, results in an decrease in the equilibrium price of good 1.

- For $L = 3$,

$$\begin{aligned} \text{index } (p^*) &= (-1)^2 \text{sign } \det D\hat{z}|_{\hat{p}^*} \\ &= \text{sign } \det D\hat{z}|_{\hat{p}^*} \end{aligned}$$

  The sign of the determinant is +1 if orientation is preserved, -1 if orientation is reversed:

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$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is obtained from } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by a counterclockwise rotation.

$$D\hat{z}|_{\hat{p}^*}\begin{pmatrix}0\\1\end{pmatrix} \text{ is obtained from } D\hat{z}|_{\hat{p}^*}\begin{pmatrix}1\\0\end{pmatrix}$$

by a rotation; counterclockwise (orientation preserved) if

$$\det D\hat{z}|_{\hat{p}^*} > 0, \text{ index } (\hat{p}^*) = +1$$

and clockwise (orientation reversed) if

$$\det D\hat{z}|_{\hat{p}^*} < 0, \text{ index } (\hat{p}^*) = -1$$

- For $L = 4$,

$$\begin{aligned}\text{index } (p^*) &= (-1)^3 \text{sign } \det D\hat{z}|_{\hat{p}^*} \\ &= -\text{sign } \det D\hat{z}|_{\hat{p}^*}\end{aligned}$$

The sign of the determinant is +1 if orientation is preserved, -1 if orientation is reversed:

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$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

is a right-handed system.

$$\det D\hat{z}|_{\hat{p}^*}\begin{pmatrix}1\\0\\0\end{pmatrix}, \det D\hat{z}|_{\hat{p}^*}\begin{pmatrix}0\\1\\0\end{pmatrix}, \det D\hat{z}|_{\hat{p}^*}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

is right-handed (orientation preserved) if

$$\det D\hat{z}|_{\hat{p}^*} > 0, \text{ index } (\hat{p}^*) = -1$$

and left-handed (orientation reversed) if

$$\det D\hat{z}|_{\hat{p}^*} < 0, \text{ index } (\hat{p}^*) = +1$$

- *Connection to Tatonnement Stability*: Consider the Tatonnement Price Dynamics

$$\frac{d\hat{p}}{dt} = \hat{z}(\hat{p}) \tag{1}$$

  - This is a nonlinear differential equation, but we can approximate its behavior near an equilibrium price $\hat{p}^*$ by considering the linear differential equation

$$\frac{d\hat{p}}{dt} = D\hat{z}\big|_{\hat{p}^*} (\hat{p} - \hat{p}^*) \tag{2}$$

  - Let $\lambda_1, \ldots, \lambda_{L-1}$ be the eigenvalues of $D\hat{z}\big|_{\hat{p}^*}$.
  - *Fact*:

$$\det D\hat{z}\big|_{\hat{p}^*} = \prod_{\ell=1}^{L-1} \lambda_\ell$$

    This is obvious if the matrix is diagonalizable, but is true in general.

    * Some of the eigenvalues are real; the others come in conjugate pairs.
    * If $a + bi$ and $a - bi$ are a conjugate pair of eigenvalues

$$(a + bi)(a - bi) = a^2 + b^2 > 0$$

    * Thus,

$$\text{sign } \det D\hat{z}\big|_{\hat{p}^*} = \prod_{\lambda_\ell \in \mathbf{R}} \text{sign } (\lambda_\ell)$$

      is the product of the signs of the real eigenvalues.
    * Each complex eigenvalue represents a rotation, which does not change orientation.
    * Each real, negative eigenvalue represents a change of orientation. Orientation is unchanged if there are an even number of real, negative eigenvalues.

– Equation (1) is locally asymptotically stable near $\hat{p}^*$ if all solutions to Equation (2) converge to $\hat{p}^*$, which is true if and only if

$$\Re(\lambda_1) < 0, \ldots, \Re(\lambda_{L-1}) < 0$$

If $\Re(\lambda_\ell) > 0$ for any ℓ, then Equation (1) is not locally asymptotically stable.

Suppose $L-1$ is odd. Since there are an even number of complex eigenvalues, there are an odd number of real eigenvalues, so if all of them are negative, the determinant is negative and

$$\text{index } (\hat{p}^*) = (-1)^{L-1} \text{sign } \det D\hat{z}\big|_{\hat{p}^*} = +1$$

On the other hand, suppose $L-1$ is even. Since there are an even number of complex eigenvalues, there are an even number of real eigenvalues, so if all of them are negative, the determinant is positive and

$$\text{index } (\hat{p}^*) = (-1)^{L-1} \text{sign } \det D\hat{z}\big|_{\hat{p}^*} = +1$$

Thus, we have

$$\text{Tatonnement Stability near } \hat{p}^* \Rightarrow \text{index } (\hat{p}^*) = +1$$

but the converse is false. Thus, the Index Theorem lets us quickly determine that some equilibria are unstable, and allow us to concentrate a search for stable equilibria on those with index +1, which *might* be stable.

Theorem 3 (Index Theorem) *For any regular economy,*

$$\sum_{\hat{p}^* \in \mathbf{R}^{L-1}_{++}, \hat{z}(\hat{p}^*)=0} \text{index } (\hat{p}^*) = +1$$

Corollary 4 *For any regular economy, there are an odd number of equilibria. Since* 0 *is even, every regular economy has an equilibrium.*

Intuition behind Index Theorem: index $(\hat{p}^*)$ indicates the direction in which $\hat{z}$ passes through zero near $\hat{p}^*$. The Boundary Condition implies that $\hat{z}$ starts on one side of zero and ends up on the other side of zero, so every equilibrium price with index -1 must be paired with an equilibrium price with index +1, and exactly one equilibrium price with index +1 must be left unpaired.

Genericity: Almost All Economies are Regular

Review notion of Lebesgue measure zero from 204: This is a natural formulation of the notion that A is a small set:

"If you choose $x \in \mathbf{R}^n$ at random,
the probability that $x \in A$ is zero."

Regular and Critical Points and Values:

Suppose $X \subseteq \mathbf{R}^n$ is open. Suppose $f : X \to \mathbf{R}^m$ is differentiable at $x \in X$. Then $df_x \in L(\mathbf{R}^n, \mathbf{R}^m)$, so

$$\text{rank } (df_x) \leq \min\{m, n\}$$

- x is a *regular point* of f if rank $(df_x) = \min\{m, n\}$.
- x is a *critical point* of f if rank $(df_x) < \min\{m, n\}$.
- y is a *critical value* of f if there exists $x \in X$, $f(x) = y$, x is a critical point of y.
- y is a *regular value* of f if y is not a critical value of f (notice this has the counterintuitive implication that if $y \notin f(X)$, then y is automatically a regular value of f).

A function may have many critical points; for example, if a function is constant on an interval, then every element of the interval is a critical point. But it can't have many critical *values*.

Theorem 5 (2.4, Sard's Theorem) *Let $X \subseteq \mathbf{R}^n$ be open, $f : X \to \mathbf{R}^m$, f is C^r with $r \geq 1 + \max\{0, n - m\}$. Then the set of all critical values of f has Lebesgue measure zero.*

Recall that our definition of critical point differed from de la Fuente's in the case $m > n$. If $m > n$, then every $x \in X$ is critical using de la Fuente's definition, because

$$\text{rank } Df(x) \leq n < m$$

Consequently, every $y \in f(X)$ is a critical value, using de la Fuente's definition. This does not contradict Sard's Theorem, since one can show that $f(X)$ is a set of Lebesgue measure zero when $m > n$ and $f \in C^1$.

The Transversality Theorem is a particularly convenient formulation of Sard's Theorem for our purposes:

Theorem 6 (2.5', Transversality Theorem) *Let*

$$\begin{aligned} X \subseteq \mathbf{R}^n \textit{ and } \Omega &\subseteq \mathbf{R}^p \textit{ be open} \\ F : X \times \Omega &\to \mathbf{R}^m \in C^r \\ &\qquad \textit{with } r \geq 1 + \max\{0, n - m\} \end{aligned}$$

Suppose that

$$F(x, \omega) = 0 \Rightarrow DF(x, \omega) \textit{ has rank } m.$$

Then there is a set $\Omega_0 \subseteq \Omega$ such that $\Omega \setminus \Omega_0$ has Lebesgue measure zero such that

$$\omega \in \Omega_0,\ F(x, \omega) = 0 \Rightarrow D_x F(x, \omega) \textit{has rank } m$$

In particular, if $m = n$, and $\omega_0 \in \Omega_0$,

- *there is a local implicit function*

$$x^*(\omega)$$

characterized by

$$F(x^*(\omega), \omega) = 0$$

where x^* is a C^r function of ω

- *the equilibrium correspondence*

$$\omega \to \{x : F(x, \omega) = 0\}$$

is lower hemicontinuous at ω_0.

Remark: If $n < m$, rank $D_x F(x, \omega) \leq \min\{m, n\} = n < m$. Therefore,

$$\begin{aligned} & (F(x, \omega) = 0 \Rightarrow DF(x, \omega) \text{ has rank m}) \\ \Rightarrow \; & \text{for all } \omega \text{ except for a set of Lebesgue measure zero} \\ & F(x, \omega) = 0 \text{ has no solution} \end{aligned}$$