Economics 201b
Spring 2010
Solutions to Problem Set 1
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1(a) The following is a Edgeworth box characterization of the Pareto optimal, and the individually rational Pareto optimal allocations, along with some relevant indifference curves.

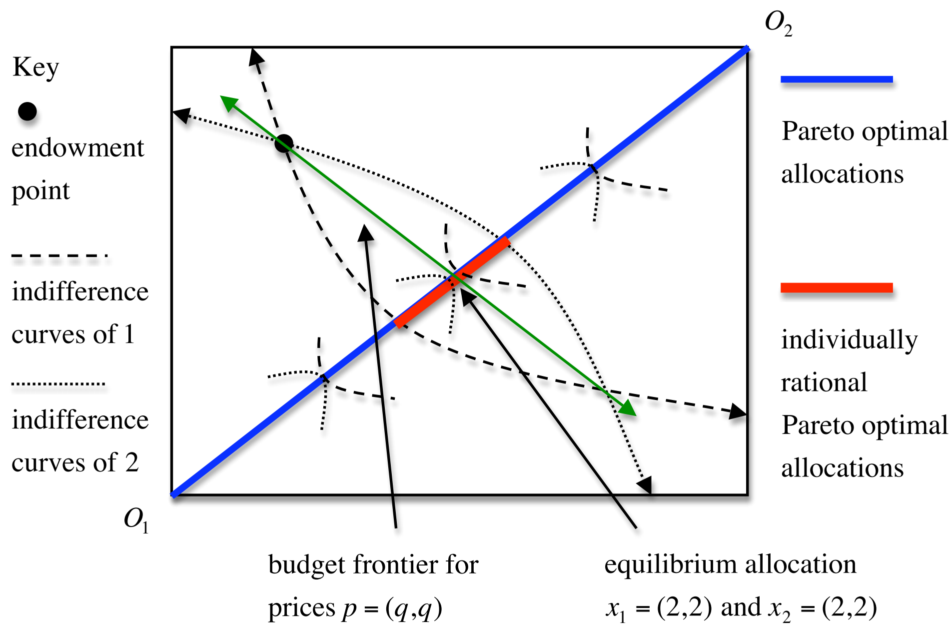

Figure 1: 1(a)-(d)

(b) If either agent has a boundary allocation then his utility is 0, in which case, the pareto optimal allocation is to give everything to the other agent. There are two such pareto optimal (exact) allocations and they correspond to the lower left and upper right corners of the Edgeworth box. Since the common utility function is strictly increasing in both variables in the interior, then any pareto optimal allocation involving interior allocations must also be exact. The marginal rate of substitution of the function $U = x_{1i}x_{2i}$ is $\frac{x_{2i}}{x_{1i}}$. So for any exact allocation $w_1 = (x_{11}, x_{21})$, $w_2 = (4 - x_{11}, 4 - x_{21})$ to be Pareto optimal it must be the case that their marginal rates of substitution (MRS) at this allocation equal:

$$\frac{x_{21}}{x_{11}} = \frac{4 - x_{21}}{4 - x_{11}}$$

The only solutions are when $x_{11} = x_{21}$. Thus the Pareto optimal allocations are

$$\big\{\{x_1 = (s, s),\ x_2 = (4 - s, 4 - s)\}|s \in [0, 4]\big\}$$

The individual rationality condition requires that $s^2 \geq 3 = U_1(\omega_1)$ and $(4 - s)^2 \geq 3 = U_2(\omega_2)$. So the set of individually rational Pareto optimal allocations are

$$\big\{\{x_1 = (s, s),\ x_2 = (4 - s, 4 - s)\}|s \in [\sqrt{3}, 4 - \sqrt{3}]\big\}$$

(c) Define $q = p_1 = p_2$ so that $p = (q, q)$. Agent 1's wealth is $4q$ and his demand $D_1(p)$ is

$$\operatorname*{argmax}_{(x_{11},x_{21})} x_{11}x_{21} \quad \text{s.t.} \quad (x_{11} + x_{21})q \le 4q$$

The constrained maximization problem has a unique solution $(2, 2)$. By symmetry, $D_2(p) = (2, 2)$. So $\{(q, q), x_1 = (2, 2), x_2 = (2, 2)\}$ is an equilibrium.

(d) Given prices $p = (p_1, p_2)$, agent 1's wealth is $p_1 + 3p_2$ and his demand $D_1(p)$ is

$$\operatorname*{argmax}_{(x_{11},x_{21})} x_{11}x_{21} \quad \text{s.t.} \quad p_1x_{11} + p_2x_{21} \le p_1 + 3p_2 \quad \Rightarrow$$

$$\operatorname*{argmax}_{x_{11}} x_{11}\left[\frac{p_1}{p_2}(1 - x_{11}) + 3\right]$$

Taking the derivative and solving, we get

$$D_1(p) = \left(\frac{p_1 + 3p_2}{2p_1}, \frac{p_1 + 3p_2}{2p_2}\right)$$

and similarly

$$D_2(p) = \left(\frac{3p_1 + p_2}{2p_1}, \frac{3p_1 + p_2}{2p_2}\right)$$

The excess demand function is

$$E(p) = \left(\frac{4p_1 + 4p_2}{2p_1} - 4, \frac{4p_1 + 4p_2}{2p_2} - 4\right)$$

Setting this equal to 0, we get $p_1 = p_2$, in which case we have the equilibrium of part (c). Thus there are no other equilibria.

(e) Given prices $p = (p_1, p_2)$, first assume $p_1, p_2 \neq 0$. Agent 1's wealth is $p_1 + 3p_2$ and his demand $D_1(p)$ is

$$\operatorname*{argmax}_{(x_{11},x_{21})} \min\{x_{11}, x_{21}\} \quad \text{s.t.} \quad p_1x_{11} + p_2x_{21} \le p_1 + 3p_2$$

Since more of one good than the other provides no extra utility but does cost something, a necessary condition for utility maximizing behavior is that $x_{11} = x_{21}$, so the constrained maximization problem can be simplified to

$$\operatorname*{argmax}_{x_{11}} x_{11} \quad \text{s.t.} \quad (p_1 + p_2)x_{11} \le p_1 + 3p_2$$

so

$$D_1(p) = \left(\frac{p_1 + 3p_2}{p_1 + p_2}, \frac{p_1 + 3p_2}{p_1 + p_2}\right)$$

and similarly

$$D_2(p) = \left(\frac{3p_1 + p_2}{p_1 + p_2}, \frac{3p_1 + p_2}{p_1 + p_2}\right)$$

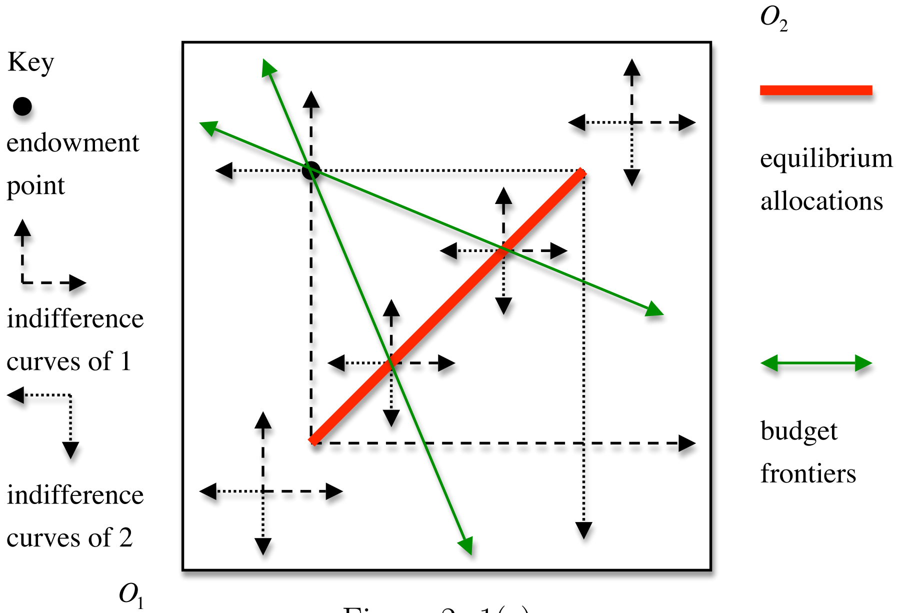

Figure 2: 1(e)

and the excess demand function is

$$E(p) = \left(\frac{4p_1 + 4p_2}{p_1 + p_2} - 4, \frac{4p_1 + 4p_2}{p_1 + p_2} - 4\right) = 0$$

So we've found some equilibria. Now assume the price is $(p_1, 0)$. Then agent 1's budget frontier is the vertical line through ω_1 and therefore his demand *correspondence* is

$$D_1(p) = \big\{(1, x_{21}) \mid x_{21} \geq 1\big\}$$

and

$$D_2(p) = \big\{(3, x_{22}) \mid x_{22} \geq 3\big\}$$

Since the social endowment is $(4,4)$, one can see the only way for the excess demand to be 0 is if $x_{21} = 1$ and $x_{22} = 3$. We can do a similar analysis for the case when prices are $(0, p_2)$. Collecting all the results together, we can express the set of equilibria as follows

$$\left\{\{(p_1, p_2), x_1 = (\frac{p_1 + 3p_2}{p_1 + p_2}, \frac{p_1 + 3p_2}{p_1 + p_2}), x_2 = (\frac{3p_1 + p_2}{p_1 + p_2}, \frac{3p_1 + p_2}{p_1 + p_2})\}|p = (p_1, p_2) \in \mathbb{R}^2_+\right\}$$

(f) Most of the calculations translate easily over from d. First assume $p_1, p_2 \neq 0$. Then

$$D_1(p) = \left(\frac{p_1 + 3p_2}{p_1 + p_2}, \frac{p_1 + 3p_2}{p_1 + p_2}\right)$$

$$D_2(p) = \left(\frac{4p_1 + p_2}{p_1 + p_2}, \frac{4p_1 + p_2}{p_1 + p_2}\right)$$

and the excess demand function is

$$E(p) = \left(\frac{5p_1 + 4p_2}{p_1 + p_2} - 5, \frac{5p_1 + 4p_2}{p_1 + p_2} - 4\right) = \left(\frac{-p_2}{p_1 + p_2}, \frac{p_1}{p_1 + p_2}\right) \neq 0$$

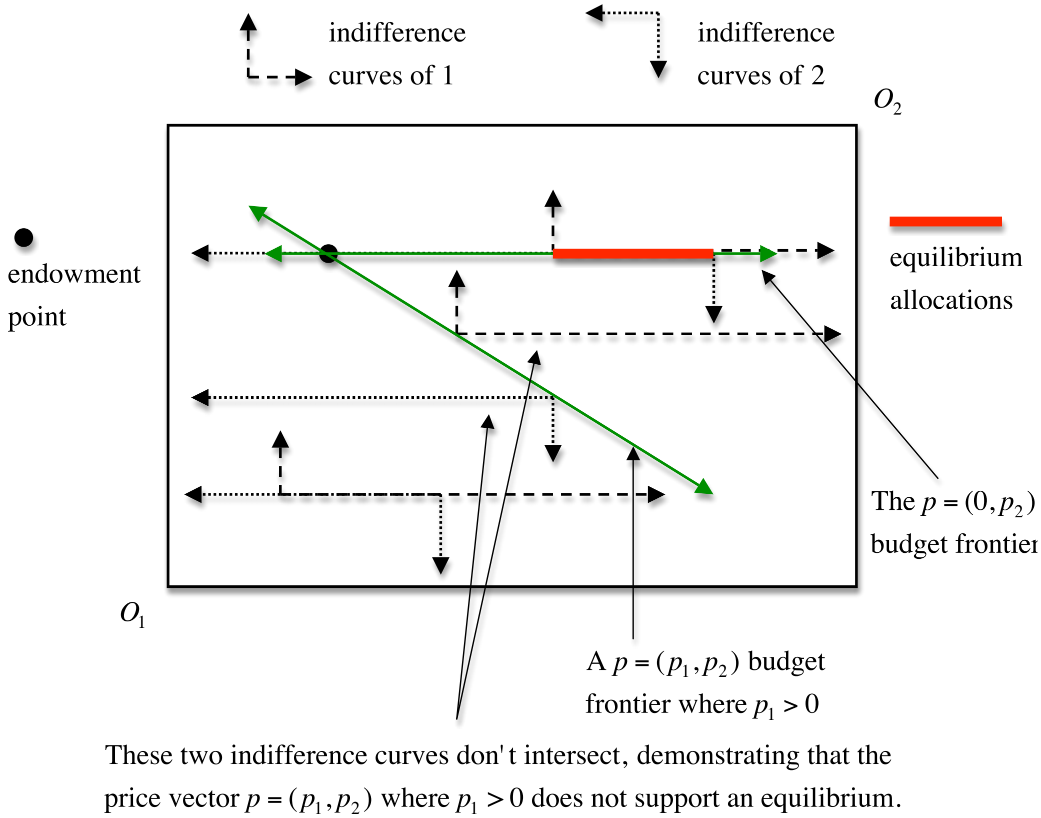

Figure 3: 1(f)

So any equilibrium must have one of the prices as zero. So first suppose that the scarcer good is free (i.e. $p_2 = 0$). Then

$$D_1(p) = \{(1, x_{21}) \mid x_{21} \geq 1\}$$

and

$$D_2(p) = \{(4, x_{22}) \mid x_{22} \geq 4\}$$

Notice there is no way $x_{21} + x_{22} \leq 4$, so markets can't clear for good 2 and this price supports no equilibrium. The last type of price is when the more common good is free (i.e. $p_1 = 0$). In which case

$$D_1(p) = \{(x_{11}, 3) \mid x_{11} \geq 3\}$$

and

$$D_2(p) = \{(x_{12}, 1) \mid x_{12} \geq 1\}$$

It is certainly possible for $x_{11} + x_{12} \leq 5$ and so the set of equilibria is

$$\Big\{\{(0, p_2), x_1 = (3 + r, 3), x_2 = (2 - r, 1)\} | r \in [0, 1]\Big\}$$

Thus, up to normalization there is now a unique equilibrium price $p = (0, 1) \in \Delta$.

2(a) Using an argument similar to that found in exercise 1, we can show all Pareto optimal allocations are exact. Agent 2 achieves her maximal utility, 4, with her endowment ω_2. So a necessary condition for an exact allocation $x_1 = (x_{11}, x_{21})$, $x_2 = (5 - x_{11}, 5 - x_{21})$ to be Pareto optimal and individually rational is that $U_2(x_2) = 4$. This is equivalent to $(5 - x_{11})(5 - x_{21}) \geq 4$. So such an allocation solves:

$$\underset{(x_{11},x_{21})}{\text{argmax}}\ x_{11}x_{21} \ \text{ s.t. } \ (5 - x_{11})(5 - x_{21}) \geq 4 \Rightarrow$$

$$\underset{x_{11}}{\text{argmax}}\ x_{11}(5 - \frac{4}{5 - x_{11}})$$

There is a unique solution $x_{11} = 3$, making the exact allocation $x = \{x_1 = (3, 3), x_2 = (2, 2)\}$ the unique allocation that is Pareto optimal and individually rational for both agents.

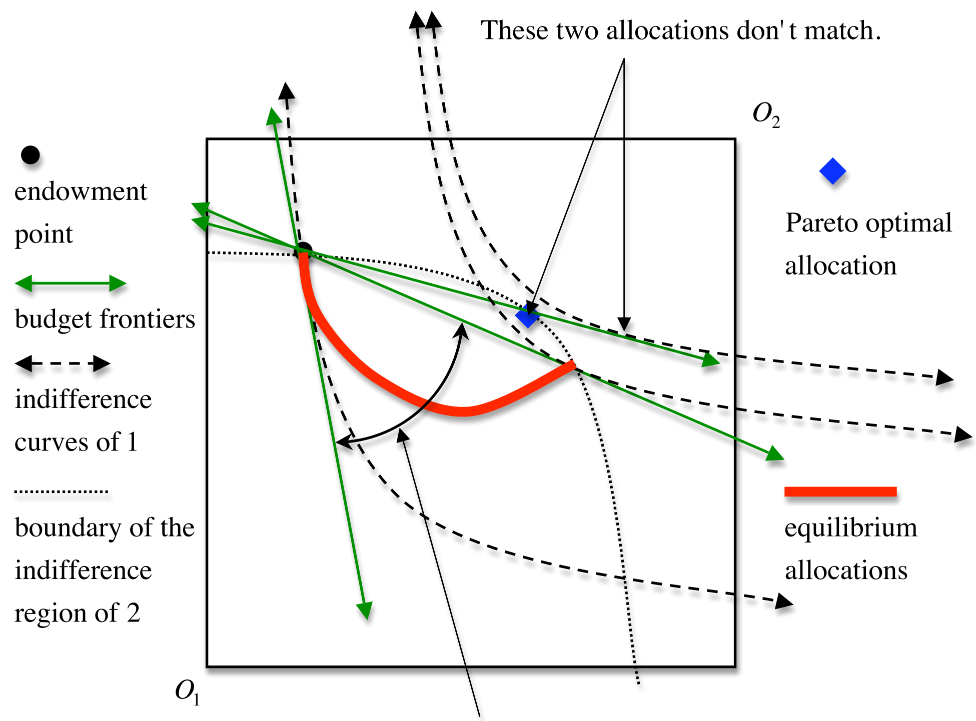

Figure 4: 2(a), (b)

(b) Given prices $p = (p_1, p_2)$, doing the same work as in exercise 1(d) we know that

$$D_1(p) = \left(\frac{p_1 + 4p_2}{2p_1}, \frac{p_1 + 4p_2}{2p_2}\right)$$

In equilibrium, after agent 1 has taken what he can afford, the rest of the social endowment must belong to the demand correspondence of agent 2:

$$(5, 5) - D_1(p) = \left(\frac{9p_1 - 4p_2}{2p_1}, \frac{6p_2 - p_1}{2p_2}\right) \in D_2(p) = \{(x_{12}, x_{22}) | x_{12}x_{22} \geq 4\}$$

Let $r = \frac{p_1}{p_2}$ then since

$$\frac{9p_1 - 4p_2}{2p_1} \cdot \frac{6p_2 - p_1}{2p_2} = \frac{(9r-4)(6-r)}{4r}$$

so the problem reduces to finding all r such that

$$(9r-4)(6-r) \geq 16r \Rightarrow (9r-4)(6-r) - 16r \geq 0 \Rightarrow$$

$$-3r^2 + 14r - 8 \geq 0 \Rightarrow (3r-2)(-r+4) \geq 0 \Rightarrow$$

$$r \in [\frac{2}{3}, 4]$$

So the set of equilibria is

$$\left\{\{(p_1, p_2), x_1 = (\frac{p_1 + 4p_2}{2p_1}, \frac{p_1 + 4p_2}{2p_1}), x_2 = \frac{9p_1 - 4p_2}{2p_1}, \frac{6p_2 - p_1}{2p_2})\}|r = \frac{p_1}{p_2} \in [\frac{2}{3}, 4]\right\}$$

There are a number of ways to show that none of the allocations of this set is the Pareto optimal allocation x. Perhaps the most geometric way to show this is to note that all the budget frontiers of the equilibrium prices are steeper than the budget frontier that goes through x. Thus from agent 1's perspective, x_1 is always out of reach (too expensive) given the equilibrium prices. Indeed the latter budget frontier has slope $-\frac{1}{3}$, whereas the budget frontiers corresponding to the equilibrium prices range from $-\frac{2}{3}$ down to -4 (remember given prices (p_1, p_2) the budget frontier's slope is $-\frac{p_1}{p_2}$). See picture.

3(a) In order to find the offer curves, we need to find each agent's demand. So fix a price $p = (p_1, p_2)$.

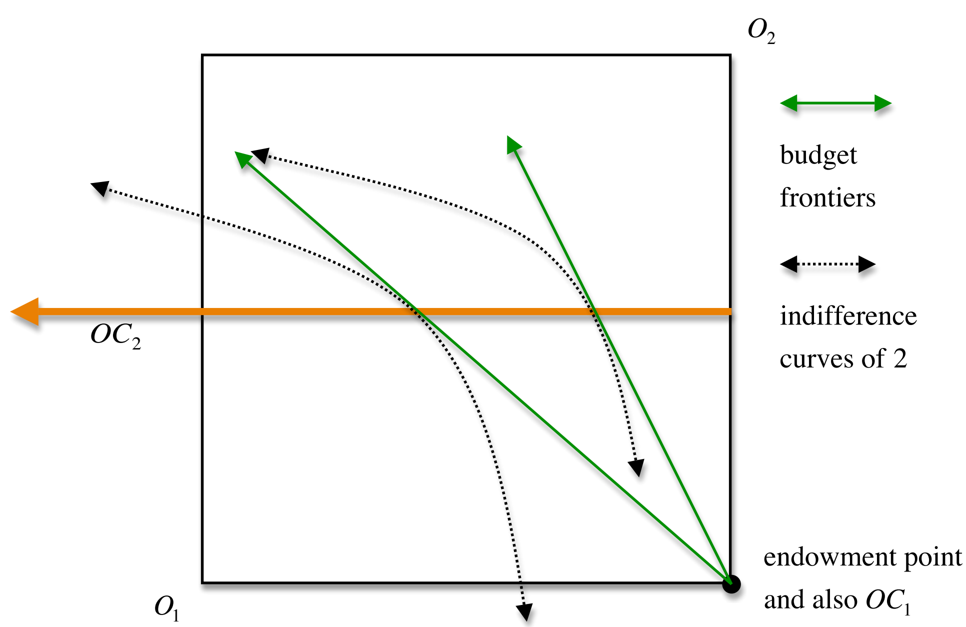

Figure 5: 3(a), (b)

The offer curve for agent 1 is just a single point - his endowment. This follows from a combination of two facts: one, he only cares about the first good, and two, all he has is the first good. Thus no matter the prices, he can't afford to get more of the first good - since it's all he has to begin with! Agent 2's wealth is $2p_2$ and her demand $D_2(p)$ is

$$\operatorname*{argmax}_{(x_{12},x_{22})} x_{22} + \log x_{12} \text{ s.t. } \ p_1 x_{12} + p_2 x_{22} = 2p_2 \Rightarrow$$

$$\operatorname*{argmax}_{x_{12}} \frac{2p_2 - p_1 x_{12}}{p_2} + \log x_{12} \Rightarrow$$

$$x_{12} = \frac{p_2}{p_1} \Rightarrow x_{22} = 1$$

So $D_2(p) = (\frac{p_2}{p_1}, 1)$ and OC_2 is the horizontal line with height 1 in agent 2's consumption set $\mathbb{R}^2_{++}$.

(b) Recall that the two offer curves must intersect in the Edgeworth box for there to be an equilibrium. Since $OC_1 \cap OC_2 = \emptyset$ there is no equilibrium.

4(a) Checking for Pareto optimality is straightforward - since the allocation is in the interior, and utilities are smooth, it suffices to check that the MRS equal. In the first exercise we calculated that the MRS at the allocation (x_{1i}, x_{2i}) is $\frac{x_{2i}}{x_{1i}}$. Thus the MRS of each agent under the allocation is 1 and so we have Pareto optimality.

(b) The agents 1 and 2 can get together and split their goods down the middle so that each agent receives the allocation $(3.5, 3.5)$ which is strictly preferred to x_1 and x_2. In other words, these two agents would be mutually strictly better off splitting from the proposed allocation, and trading amongst themselves, leaving out agent 3.