Economics 201b
Spring 2010
Problem Set 2 Solutions

1. **Competitive Equilibrium(-a) When Preferences Are Kinked.** Recently there have been a surge in decision theory models that are non-differentiable in nature. For example, popular models incorporating loss aversion in prospect theory, or ambiguity aversion as illustrated by Ellsberg Paradox, have kinked indifference curves.

   In this exercise we are going to take a reduced form of these preferences and examine the implications of "kinkiness" on equilibrium prices and allocations in our simplest 2×2 exchange economy, where agents' utility functions are $\forall i \in \{1, 2\}$,

$$U_i(x_{1i}, x_{2i}) = \sqrt{x_{1i}} + \frac{1}{2}\sqrt{x_{2i}} \text{ if } x_{1i} \leq x_{2i}$$
$$\frac{1}{2}\sqrt{x_{1i}} + \sqrt{x_{2i}} \text{ if } x_{1i} > x_{2i}. \tag{1}$$

   Solution. As a normalization, in all of the following questions, we set $p_1 = 1$, and solve for relative price $p = \frac{p_2}{p_1}$.

   Before solving for each specific question, it helps to first derive individual demands for each consumer. Since the utility functions of the two consumers are the same, we calculate for just one. For a consumer i with the utility described above and endowments $(\omega_{1i}, \omega_{2i})$, we want so solve the following optimization problem:

$$\max_{x_{1i}, x_{2i}} U_i(x_{1i}, x_{2i})$$
$$s.t. \quad x_{1i} + p \cdot x_{2i} \leq \omega_{1i} + p \cdot \omega_{2i}$$
$$x_{1i} \geq 0, x_{2i} \geq 0$$

   Notice that $U(x_{1i}, x_{2i})$ is not differentiable and we should be careful when trying to apply Kuhn-Tucker conditions. The mechanical (but safe) way to solve the question is to split the budget set into two subsets by adding constraints $x_{i1} \leq x_{i2}$ and $x_{i1} \geq x_{i2}$, and within each sub-budget set $U(x_{1i}, x_{2i})$ is differentiable. We can solve the two sub-questions, and the solutions to the original question can be obtained by comparing the utility levels of the solutions to the two sub-questions.

   A more intuitive way to solve the problem is to practice with the economic concept of marginal rate of substitution (MRS). Notice that at any interior solution $(0 < x_{1i}, 0 < x_{2i}, \text{ and } x_{1i} \neq x_{2i})$ $MRS_i = \frac{\partial U_i \backslash \partial x_{1i}}{\partial U_i \backslash \partial x_{2i}} = \frac{p_1}{p_2} = \frac{1}{p}$. Also notice that $\frac{\partial U_i}{\partial x_{1i}}|_{x_{1i}=0^+} = \frac{\partial U_i}{\partial x_{2i}}|_{x_{2i}=0^+} = +\infty$, so consumer i will never demand zero

amount of any commodity. Since preferences are strongly monotone, Walras Law applies and the budget constraint is binding at optimal solutions : $x_{1i} + p \cdot x_{2i} = \omega_{1i} + p \cdot \omega_{2i}$.

- When $x_{1i} > x_{2i} > 0$, $U_i = \frac{1}{2}\sqrt{x_{1i}} + \sqrt{x_{2i}}$, we have $MRS_i = \frac{\partial U_i \backslash \partial x_{1i}}{\partial U_i \backslash \partial x_{2i}} = \frac{1}{2}\sqrt{\frac{x_{2i}}{x_{1i}}} < \frac{1}{2}$. Notice if some $x_{1i} > x_{2i} > 0$ is the Walrasian demand, then it is an interior solution, and this happens if and only if $MRS_i = \frac{1}{2}\sqrt{\frac{x_{2i}}{x_{1i}}} = \frac{1}{p}(< \frac{1}{2})$, that is, when $p > 2$. Intuitively, consumer i will want to consumer strictly more amount of commodity 1, only when commodity 2 is more than twice as expensive as commodity 1. Together with $x_{1i} + p \cdot x_{2i} = \omega_{1i} + p \cdot \omega_{2i}$, we can solve for

$$D_i(p) = \left(\frac{p \cdot \omega_{1i} + p^2 \cdot \omega_{2i}}{4 + p}, \frac{4\omega_{1i} + 4p \cdot \omega_{2i}}{4p + p^2} \right)$$

- When $0 < x_{1i} < x_{2i}$, $U_i(x_{1i}, x_{2i}) = \sqrt{x_{1i}} + \frac{1}{2}\sqrt{x_{2i}}$, we have $MRS_i = \frac{\partial U_i \backslash \partial x_{1i}}{\partial U_i \backslash \partial x_{2i}} = 2\sqrt{\frac{x_{2i}}{x_{1i}}} > 2$. Notice if some $0 < x_{1i} < x_{2i}$ is the Walrasian demand, then it is an interior solution, and this happens if and only if $MRS_i = 2\sqrt{\frac{x_{2i}}{x_{1i}}}$, that is, when $p < \frac{1}{2}$. Intuitively, consumer i will want to consumer strictly more amount of commodity 2, only when commodity 1 is more than twice as expensive as commodity 1. Together with $x_{1i} + p \cdot x_{2i} = \omega_{1i} + p \cdot \omega_{2i}$, we can solve for

$$D_i(p) = \left(\frac{4p \cdot \omega_{1i} + 4p^2 \cdot \omega_{2i}}{4p + 1}, \frac{\omega_{1i} + p \cdot \omega_{2i}}{4p^2 + p} \right)$$

- For $p \in [\frac{1}{2}, 2]$, the only possible demand is at $x_{i1} = x_{i2}$, as we've ruled out the other cases. It's also easy to verify that consumer i does demand at where $x_{1i} = x_{2i}$ hits the budget constraint $x_{1i} + p \cdot x_{2i} = \omega_{1i} + p \cdot \omega_{2i}$. At $x_{1i} = x_{2i}$, when consumer i is considering whether she should increase x_{1i} by a small amount, she calculates MRS using $U_i(x_{1i}, x_{2i}) = \frac{1}{2}\sqrt{x_{1i}} + \sqrt{x_{2i}}$, so the MRS=2. So she is not willing to sacrifice more than $\frac{1}{2}$ unit of consumption of commodity 2 to increase consumption of commodity 1 by 1 unit, and given the exchange price at $p \geq \frac{1}{2}$, she will not want to increase x_{1i}. Similarly she will not want to increase x_{2i}, since at this time she is using $U_i = \sqrt{x_{1i}} + \frac{1}{2}\sqrt{x_{2i}}$, and the $MRS = \frac{1}{2}$. Solving for the interception of 45 degree line and budget constraint, we have

$$D_i(p) = \left(\frac{\omega_{1i} + p\omega_{2i}}{1 + p}, \frac{\omega_{1i} + p\omega_{2i}}{1 + p} \right)$$

.

To sum up, we've verified that a consumer i with this kind of "kinky" prefer-ences, and some general initial endowment $(\omega_{1i}, \omega_{2i})$ will demand as

$$
D_i(p) = \quad \left(\frac{p \cdot \omega_{1i} + p^2 \cdot \omega_{2i}}{4 + p}, \frac{4\omega_{1i} + 4p \cdot \omega_{2i}}{4p + p^2} \right), \text{ for } p > 2;
$$

$$
\left(\frac{\omega_{1i} + p \cdot \omega_{2i}}{1 + p}, \frac{\omega_{1i} + p \cdot \omega_{2i}}{1 + p} \right), \text{ for } p \in \left[\frac{1}{2}, 2 \right]; \qquad (2)
$$

$$
\left(\frac{4p \cdot \omega_{1i} + 4p^2 \cdot \omega_{2i}}{4p + 1}, \frac{\omega_{1i} + p \cdot \omega_{2i}}{4p^2 + p} \right), \text{ for } p < \frac{1}{2}.
$$

And her individual excess demand function is:

$$
E_i(p) = \quad \left(\frac{p^2 \cdot \omega_{2i} - 4\omega_{1i}}{4 + p}, \frac{4\omega_{1i} - p^2 \cdot \omega_{2i}}{4p + p^2} \right), \text{ for } p > 2;
$$

$$
\left(\frac{p \cdot \omega_{2i} - p \cdot \omega_{1i}}{1 + p}, \frac{\omega_{1i} - \omega_{2i}}{1 + p} \right), \text{ for } p \in [\frac{1}{2}, 2]; \qquad (3)
$$

$$
\left(\frac{4p^2 \cdot \omega_{2i} - \omega_{1i}}{4p + 1}, \frac{\omega_{1i} - 4p^2 \cdot \omega_{2i}}{4p^2 + p} \right), \text{ for } p < \frac{1}{2};
$$

(a) Suppose the initial endowments are $\omega_i = (4, 4)$, $\forall i \in \{1, 2\}$. Draw the Edgeworth box for this economy. Find the Pareto optimal allocations. Verify that the initial endowment is an equilibrium allocation. Find the supporting equilibrium price(s).

Solution. When $\omega_i = (\omega_{1i}, \omega_{2i}) = (4, 4), \forall i \in \{1, 2\}$, from demand func-tion (2) we can see for any price in $[\frac{1}{2}, 2]$, both consumers will demand at endowments (4,4), and obviously individual excess demand is 0 for each consumer, the market clears. For a graphic illustration of the equilibrium solution see figure 1(a).

As to the set of all Pareto optimal allocation, we find it by solving the problem (P) for all $\bar{U} \in [U(0), U(\bar{\omega})]$:

$$
\max_{x_1} U(x_1)
$$
$$
s.t. \quad U(x_2) \geq \bar{U},
$$
$$
x_1 \geq 0, x_2 \geq 0,
$$
$$
x_1 + x_2 \leq \bar{\omega}.
$$

where x_i is a consumption vector of agent i. Note again, that with non-differentiable utility function the application of Kuhn-Tacker maximization technique is complicated. However, given the symmetry of the problem, we can solve it quite easily by inspection. For that, fix some utility level of the agent 2, $\bar{U} \in [U(0), U(\bar{\omega})]$. Now, notice that convexity and symmetry of agents' indifference curves implies that the highest utility of the agent 1 is

achieved when both indifference curves meet on the diagonal of Edgeworth box and $x_{1i} = x_{2i}$, $\forall i \in \{1, 2\}$. In other words, the set of all Pareto optimal allocations is

$$\big\{\{w_1 = (s, s),\ w_2 = (8 - s, 8 - s)\}|s \in [0, 8]\big\}.$$

(b) Now suppose the endowments are instead $\omega'_1 = (5, 3)$, $\omega'_2 = (3, 5)$. Find the individual and market excess demand functions (notice that the utility function is not differentiable). Find the competitive equilibrium prices and allocations. Is there a unique equilibrium?

Solution. Substitute $\omega'_1 = (5, 3)$, $\omega'_2 = (3, 5)$ into (3) and set the excess demand to zero $E_{11}(p) + E_{12}(p) = 0$ (notice that by Walras' Law, we only need to check clearing of one market). We have:

- $\forall p > 2$, $E_{11}(p) + E_{12}(p) = \frac{8p^2 - 32}{4 + p} > 0$, markets will not clear,
- $\forall p \in [\frac{1}{2}, 2]$, $E(p) = (0, 0)$, markets clear.
- $\forall p < \frac{1}{2}$, $E_{11}(p) + E_{12}(p) = \frac{32p^2 - 8}{4p + 1} < 0$, markets will not clear.

So in this case, every price in $[\frac{1}{2}, 2]$ is an equilibrium price. At every equilibrium price $p^* \in [\frac{1}{2}, 2]$, the equilibrium allocation is $x_1^* = (\frac{5 + 3p^*}{1 + p^*}, \frac{5 + 3p^*}{1 + p^*})$, $x_b^* = (\frac{3 + 5p^*}{1 + p^*}, \frac{3 + 5p^*}{1 + p^*})$. For a graphic illustration of the equilibrium solution see figure 1(b). The equilibrium is not unique.

Remark: in both (a) and (b), we have a continuum of (thus uncountably many) equilibrium prices. This is called "price indeterminacy" in the literature. Notice that this result is mainly driven by the "kink" in the indifference curve, the nondifferentiability at the kink makes the MRS when approximating from the left and the right differ, thus there exist a range of price at which we can observe demand "bundling" at the kink. For a long time the literature associates the kinky preference with price indeterminacy, but actually it is not always true. The next exercise illustrates this point.

(c) Now suppose the endowments are instead $\omega''_1 = (8, 2)$, $\omega''_2 = (3, 5)$. Repeat the calculations that you have done in (b). (Note that the Edgeworth box is different in this case). Comment on how the kinkiness of preferences affect the size of competitive equilibria.

Solution.   Again we substitute into (3) and sum up $E_{11}(p) + E_{12}(p)$ to calculate market excess demand:

- $\forall p > 2$, $E_{11}(p) + E_{12}(p) = \frac{7p^2 - 44}{4 + p}$. Market clearing $E_{11}(p) + E_{12}(p) = 0$ occurs when $p^* = \sqrt{\frac{44}{7}}$, and the equilibrium allocation is

$$(x_{11}^*, x_{21}^*) = \left(\frac{8p^* + 2(p^*)^2}{4 + p^*}, \frac{8p^* + 32}{4p^* + (p^*)^2}\right),$$

$$(x_{12}^*, x_{22}^*) = \left(\frac{3p^* + 5(p^*)^2}{4 + p^*}, \frac{20p^* + 12}{4p^* + (p^*)^2}\right)$$

- $\forall p \in [\frac{1}{2}, 2]$, $E_{11}(p) + E_{12}(p) = \frac{-3p}{1+p}$, markets will not clear.
- $\forall p < \frac{1}{2}$, $E_{11}(p) + E_{12}(p) = \frac{28p^2 - 11}{4p+1} < 0$, markets will not clear.

For a graphic illustration of the equilibrium solution see figure 1(c). In this case, we are back to the case we only have a unique equilibrium. So even with "kinky" preference, price indeterminacy is the equilibrium price could be uniquely determined.

(d) Does the First Welfare Theorem holds in this economy? What about the Second Welfare Theorem?

Solution.   Yes, both theorems hold as their hypotheses are satisfied.

2. **More Fun with Offer Curves!** Consider simple two-person, two-good economy in which agents' utility functions are given by

$$U_1(x_{11}, x_{21}) = \min\{x_{11}, x_{21}\}, \text{ and } U_2(x_{12}, x_{22}) = \min\{4x_{12}, x_{22}\}. \qquad (4)$$

and endowments are $\omega_1' = (30, 0)$, $\omega_2' = (0, 20)$.

(a) If neither agents can have negative consumption of either good, what is Walrasian equilibrium?

Solution.   With Leontiev preferences, the indifference curves of both agents are right-angles. For agent 1 their vertices lie on the line $x_{11} = x_{21}$, whereas, for agent 2, they lie on the line $x_{22} = 4x_{12}$. These lines are respective offer curves of both agents when both prices are strictly positive. For the case when one of the prices is zero, we have following offer curves:

$$OC_1(p_1, p_2) = \begin{cases} (x_{11}, \omega_{21}) : x_{11} \geq \omega_{21} & \text{if } (p_1, p_2) \in \{0\} \times \mathbb{R}_{++} \\ (\omega_{11}, x_{21}) : x_{21} \geq \omega_{11} & \text{if } (p_1, p_2) \in \mathbb{R}_{++} \times \{0\} \end{cases}$$

$$OC_2(p_1, p_2) = \begin{cases} (x_{11}, \omega_{21}) : x_{11} \geq \frac{1}{4}\omega_{21} & \text{if } (p_1, p_2) \in \{0\} \times \mathbb{R}_{++} \\ (\omega_{11}, x_{21}) : x_{21} \geq 4\omega_{11} & \text{if } (p_1, p_2) \in \mathbb{R}_{++} \times \{0\} \end{cases}$$

For $\omega_1' = (30, 0)$, $\omega_2' = (0, 20)$ it is easy to see that the line $x_{11} = x_{21}$ starting at agent 1's origin, OC_1, does not intersect the line $x_{22} = 4x_{12}$ with respect to agent 2's origin, OC_2. Thus, there cannot be an equilibrium with both goods having non-zero price (equivalently, with both agents consuming non-zero amounts of both goods).

So, consider first the case $p_1 = 0$, $p_2 > 0$. The offer curves are: $OC_1(0, p_2) = \{(x_{11}, 0) : x_{11} \geq 0\}$ and $OC_2(0, p_2) = \{(x_{12}, 5) : x_{12} \geq 20\}$. Their points of intersection are given by the set

$$\left\{ \left((x_{11}^*, x_{21}^*), (x_{12}^*, x_{22}^*)\right) = \left((30 - s, 0), (s, 20)\right) : s \in [5, 30] \right\}.$$

Any such point is a competitive equilibrium.

For the case $p_1 > 0, p_2 = 0$, the offer curves are: $OC_1(p_1, 0) = \{(30, x_{21}) : x_{21} \geq 30\}$ and $OC_2(p_1, 0) = \{(0, x_{22}) : x_{22} \geq 0\}$. Since agent 1's demand for good 2 cannot be satisfied by the social endowment, we cannot have an equilibrium for $p_1 > 0, p_2 = 0$.

(b) Now suppose the first agent starts only with 10 units of good 1 instead of 30 and none of the second. What is Walrasian equilibrium in this case? Explain briefly your results. *Hint: be sure to find all Walrasian equilibria.*

Solution.   In this case, line $x_{11} = x_{21}$ starting at agent 1's origin, OC_1, intersect the line $x_{22} = 4x_{12}$ with respect to agent 2's origin, OC_2, at E. This corresponds to the allocation

$$(x_{11}^*, x_{21}^*) = \left(\frac{20}{3}, \frac{20}{3}\right), \text{ and } (x_{12}^*, x_{22}^*) = \left(\frac{10}{3}, \frac{40}{3}\right).$$

It can be supported as a competitive equilibrium for any strictly positive price vector, $(p_1, p_2) \in \mathbb{R}^2_{++}$.

When $p_1 = 0$, $p_2 > 0$, the offer curves are similar to those described above. They intersect at any point in the set

$$\left\{ \left((x_{11}^*, x_{21}^*), (x_{12}^*, x_{22}^*)\right) = \left((10 - s, 0), (s, 20)\right) : s \in [5, 10] \right\}.$$

Again, any such point is a competitive equilibrium.

For the case $p_1 > 0, p_2 = 0$, the offer curves are: $OC_1(p_1, 0) = \{(10, x_{21}) : x_21 \geq 10\}$ and $OC_2(p_1, 0) = \{(0, x_{22}) : x_{22} \geq 0\}$. They intersect at any point in the set

$$\left\{ \left((x_{11}^*, x_{21}^*), (x_{12}^*, x_{22}^*)\right) = \left((10, s), (0, 20 - s)\right) : s \in [10, 20] \right\}.$$

Any such point is also a competitive equilibrium.

Notice that when the social endowment of good 1 decreases from 30 to 10, we obtain equilibria where the price of good 1 is no longer zero. Intuitively, good 1 can now be scarce enough, relative to good 2, to have a positive price and, therefore, to allow agent 1 positive consumption of both goods. The price of good 2 can now even be driven down to zero, in which case agent 1 essentially consumes the entire social endowment of both goods, the situation that was previously not possible.

(c) Suppose that an agent decides to throw away part of her endowment to change the equilibrium prices in the economy. Can agent be better off in the new equilibrium than in the equilibrium with the original endowment? Provide an example and explain. (The example does not have to be analytic, however, it must be described *clearly* and *coherently*).

Solution.   Note that is exactly what we have just observed in (b). However, this observation applies not only when one of the prices is zero, but also much more generally, as income effects can lead to quite "surprising"

changes in the welfare properties of competitive equilibria. Finally, it is important to mention that agent 1 should throw away part of her endowment *before* Walrasian "auctioneer" starts quoting up prices, otherwise argent 1 won't be acting as a price-taker.

3. **Equilibrium with "Bads."** Consider an exchange economy that contains two consumers with utility function of the form $\forall i \in \{1,2\}$:

$$U_i(x_{1i}, x_{2i}) = x_{1i}(4 - x_{2i}) \tag{5}$$

defined over consumption set $[0,5] \times [0,3] \subset \mathbb{R}_+^2$. Notice that the second commodity is "bad." Endowments are given by $\omega_1 = (1,3)$ and $\omega_2 = (3,1)$.

(a) Show that feasible allocation x is Pareto optimal if and only if $x_{11} + x_{21} = 4$.

Solution.   To find set of all feasible allocations we solve the program (P), which is both a sufficient and necessary condition for an allocation to be Pareto optimal

$$\max_{x_{11}, x_{21}} x_{11}(4 - x_{21})$$
$$s.t. \quad (4 - x_{11})\, x_{21} \geq \bar{U},$$
$$x_1 \geq 0, x_2 \geq 0.$$

where x_1, x_2 are the consumption vectors of agents. The first order conditions yield $(4 - x_{21})(4 - x_{11}) = x_{21}\, x_{11}$, which directly implies the result we need to demonstrate.

(b) Compute excess demand functions and find the Walrasian equilibrium. Illustrate it with Edgeworth box diagram.

Solution.   As a normalization, in all of the following questions, we set $p_1 = 1$, and solve for relative price $p = \frac{p_2}{p_1}$. Solving consumer's utility maximization problem

$$\max_{x_{1i}, x_{2i}} x_{1i}(4 - x_{21})$$
$$s.t. \quad x_{1i} + p \cdot x_{2i} \leq \omega_{1i} + p \cdot \omega_{2i}$$
$$x_{1i} \geq 0, x_{2i} \geq 0$$

we obtain following excess demand functions

$$ED_i = \left(\frac{\omega_{1i} + p \cdot \omega_{2i} - 4p}{2} - \omega_{1i},\ \frac{\omega_{1i} + p \cdot \omega_{2i} + 4p}{2p} - \omega_{2i} \right)$$

Setting market excess demand to zero, $E_{11} + E_{12} = 0$, we obtain $p^* = -1$ and equilibrium allocation that is economy's endowment $(x_1^*, x_2^*) = \big((1,3), (3,1)\big)$.

(c) What happens to the Walrasian equilibrium if the first consumer has the right to dump all of her endowment of the second commodity onto the second consumer?

Solution.   Consumer 1 deciding to dump all "bad" onto the second consumer creates new endowment $(\omega_1, \omega_2)' = \big((1,0),(3,4)\big)$. Again, setting market excess demand to zero, $E_{11} + E_{12} = 0$, we obtain $-2p = 2$. Thus, the equilibrium price has not changed, $p^* = -1$, but we obtained new equilibrium allocation with $(x^*_{11}, x^*_{21}) = \big(\frac{5}{2}, \frac{3}{2}\big)$ and $(x^*_{12}, x^*_{22}) = \big(\frac{3}{2}, \frac{5}{2}\big)$.

(d) What happens to the Walrasian equilibrium if there is an *ad valorem* tax t on any sale of good 2 paid by the seller, which is then shared equally between two agents?

Solution.   Notice that the imposition of the tax on *net trade* in second commodity creates kinks in the budget set of the consumers at the endowment points.

$$B_i(p, R_i) = \Big\{ x_i \in [0,5] \times [0,3] \subset \mathbb{R}^2_+ :$$
$$x_{1i} - \omega_{1i} + p \cdot (x_{2i} - \omega_{2i}) + t \cdot p \cdot \max\{x_{2i} - \omega_{2i}, 0\} \le R_i \Big\}$$

where R_i are the rebated tax receipts. For instance, in our 2×2 exchange economy, budget line will generally have two kinks. Consequently, relative prices can change as a result of *ad valorem* tax. Solving for a new equilibrium would be entirely non-trivial because we would have to consider carefully different cases.

However, our results in (a) and (b) suggest that the only competitive equilibrium in the economy is the original endowment point, i.e. we do not observe any net trades taking place. Thus, the *ad valorem* tax on the "bad" is irrelevant here, and we retain old equilibrium with $(p^*, (x^*_1, x^*_2))$.

4. **Importance of Assumptions.** Give examples of the following, and illustrate them using an Edgeworth box. Please be *clear* and *precise*.

   (a) A Pareto optimal allocation that can't be sustained as a Walrasian equilibrium with transfers.

   Solution.   Consider two-person, two-good economy with agent's utility function $U_i(x_{1i}, x_{2i}) = x^2_{1i} + x^2_{2i}$ and endowment vector $\omega_i = (1,1)$ $\forall i \in \{1,2\}$. It is easy to show that non-convexity of preferences generates a situation where none of the Pareto optimal allocations, except for $(2,0)$ and $(0,2)$, can be supported as a Walrasian equilibrium with transfers (see figure 4(a)).

   (b) A Walrasian equilibrium that is not Pareto optimal (Please do not use externalities).

   Solution.   See question 2(b) on the problem set 1, where the existence of a satiation point for second consumer generates a Walrasian equilibrium that is not Pareto optimal.

5. **Computing the Transfers.** Consider again the exchange economy from the question 1 of the problem set 1: a two-person, two-good exchange economy

where the agents' utility functions are $U_1(x_{11}, x_{21}) = x_{11}x_{21}$ and $U_2(x_{12}, x_{22}) = x_{12}x_{22}$, and the initial endowments are $\omega_1 = (1, 3)$ and $\omega_2 = (3, 1)$. Show directly that every interior Pareto optimal allocation in this economy is a price equilibrium with transfers by finding the associated prices and transfers.

Solution.   Recall from the solution of the previous problem set that given two prices $p_1 > 0$, $p_2 > 0$ we had following demand schedules:

$$D_i(p_1, p_2, \omega_i) = \left(\frac{p \cdot \omega_i}{2p_1}, \frac{p \cdot \omega_i}{2p_2} \right)$$

thus, with transfers $T_1 + T_2 = 0$ and endowments $\omega_1 = (1, 3)$ and $\omega_2 = (3, 1)$ we have:

$$D_1(p_1, p_2, \omega_1) = \left(\frac{p_1 + 3p_2 + T}{2p_1}, \frac{p_1 + 3p_2 + T}{2p_2} \right) \text{ and}$$
$$D_2(p_1, p_2, \omega_2) = \left(\frac{3p_1 + p_2 - T}{2p_1}, \frac{3p_1 + p_2 - T}{2p_2} \right).$$

where $T = T_1$ is the only transfer in the economy from agent 2 to agent 1. Now, notice that the price will always be equal for both goods and in equilibrium we will have the following allocations:

$$(x_{11}^*, x_{21}^*) = \left(2 + \frac{T}{2p^*}, 2 + \frac{T}{2p^*} \right) \text{ and } (x_{12}^*, x_{22}^*) = \left(2 - \frac{T}{2p^*}, 2 - \frac{T}{2p^*} \right).$$

Thus, for the Pareto optimal allocation

$$\{(s, s) \text{ for agent 1, and } (4 - s, 4 - s) \text{ for agent 2}\}$$

the transfer is $T = 2p^*(s - 2)$ from agent 2 to agent 1.
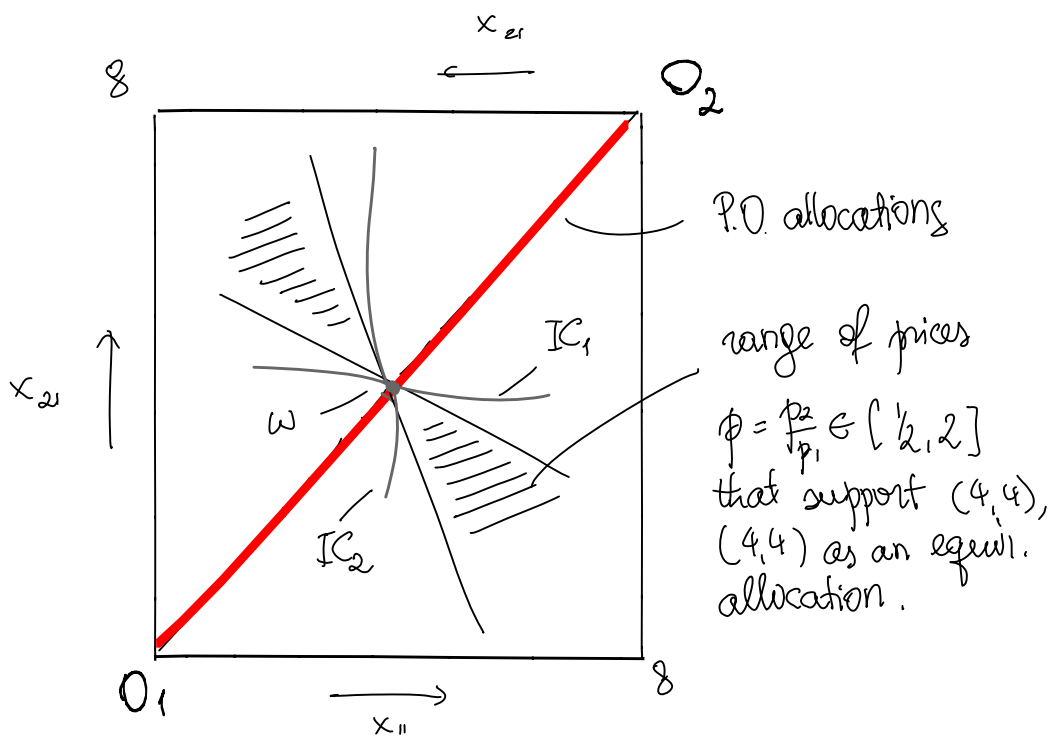
Figure 1(a)

x_{2r}

8

O_2

P.O. allocations

IC_1

x_{2l}

ω

range of prices

$p = \frac{p_2}{p_1} \in [\frac{1}{2}, 2]$
that support $(4,4)$,
$(4,4)$ as an equi.
allocation.

IC_2

O_1

8

x_{1l}

Figure 1 (b)

x_{21}

8

O_2

x_{21}

equilibrium
allocations

range of prices

IC_1'

IC_1''

$\phi = \frac{p_2}{p_1} \in [\frac{1}{2}, 2]$
that support $(5,3)$
$(3,5)$ as an eqwl.
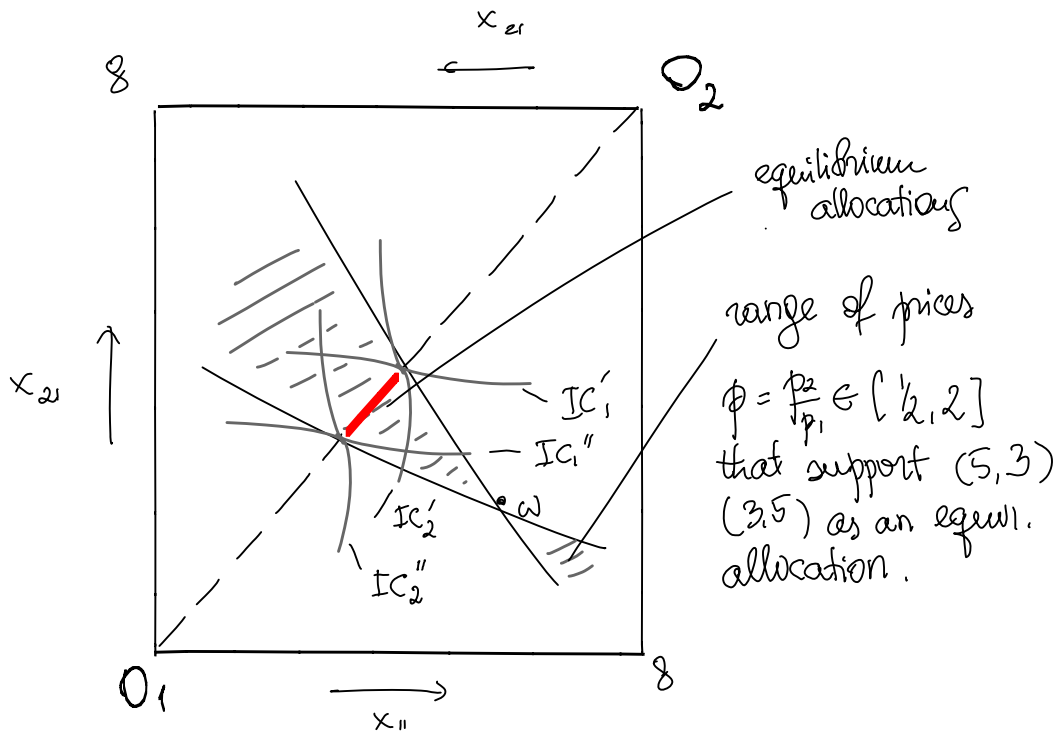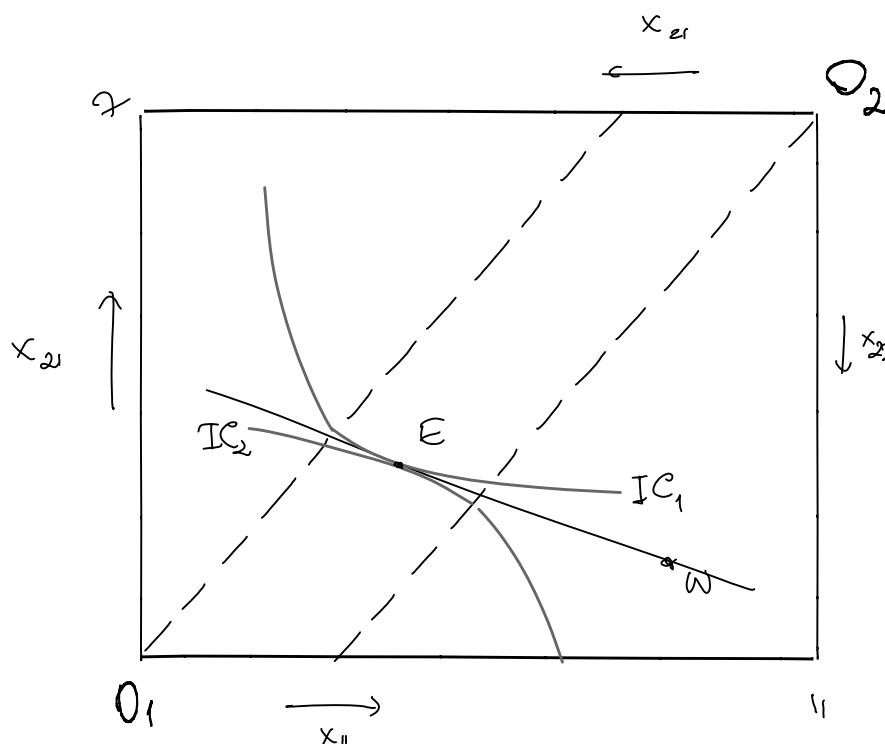allocation.

ω

IC_2'

IC_2''

O_1

8

x_{11}

Figure 1(c)

Unique equilibrium with $p^* = \sqrt{44}/7$
for $\omega_1 = (8,2)$, $\omega_2 = (3,5)$

Figure 2(a)

Figure 2(b)

x_{12}

20

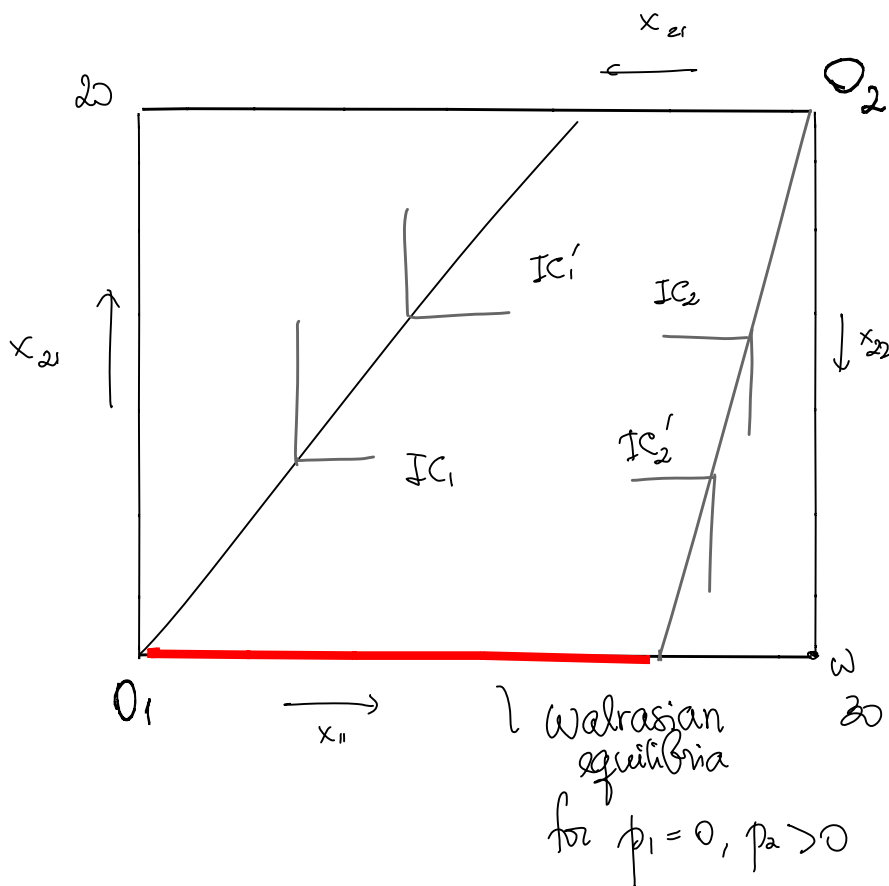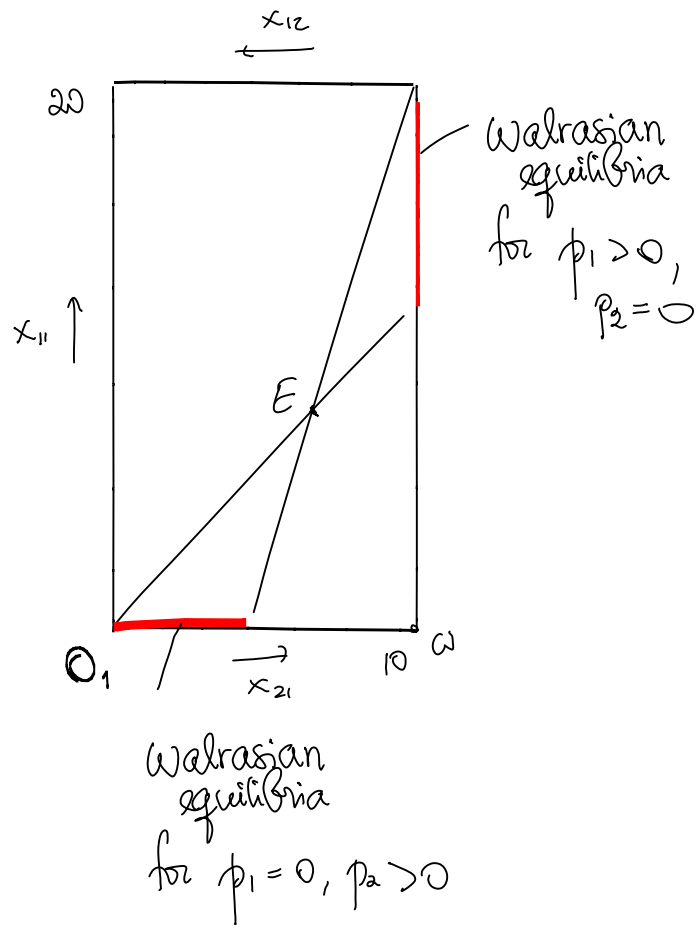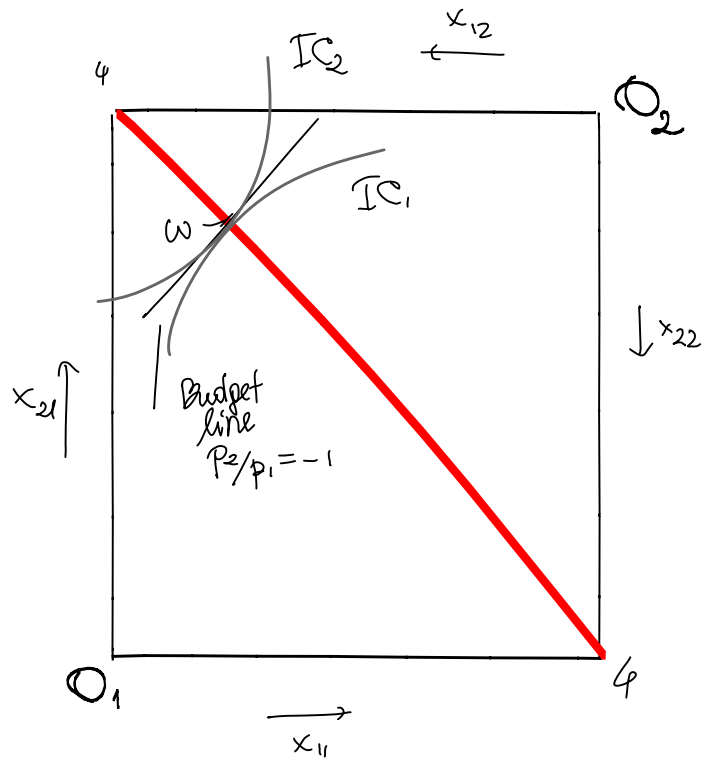x_{11}

E

Walrasian
equilibria

for $p_1 > 0$,
              $p_2 = 0$

O_1

x_{21}

10   ω

Walrasian
equilibria

for $p_1 = 0$, $p_2 > 0$

Figure 3(a)

Figure 4(a)

agent 1
demands
(given prices $\vec{p}$
and transf. T)

agent 2
demands

P.O. allocation that
can't be supported
as walrasian
equil. w/ transf.

O_2

x_{12}

IC_2^-

$\vec{p}$

x_{21}

ω

transfer T

O_1

x_{11}

2

2

2