Economics 201b
Spring 2010
Solutions to Problem Set 3
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1. **Note in the 2010 version of Professor Anderson's Lecture 4 Notes, the characterization of the firm in a Robinson Crusoe economy is that it maximizes profit over feasible sets. This is incorrect, and the appropriate correction has been made. The correct description should coincide with our understanding of firms and agents from previous exchange economies and more general Arrow-Debreu economies - that is, they do not take into account the size of the social endowment when making their decisions. For this problem set we will make an exception for those students who wrote their solutions for this exercise with the incorrect assumption in mind. However, in the future we will be operating under the correct assumption.**

   Let w and p be the prices of the first and second good.

   The Agent's Utility Maximizing Decision

   $\frac{w}{p} < \frac{1}{\beta} \Rightarrow$ use all wealth for good 1 consumption

   $$(x_1, x_2) = \{(L + \frac{\Pi}{w}, 0)\}$$

   $\frac{w}{p} = \frac{1}{\beta} \Rightarrow$ indifferent between any consumption bundle on budget frontier

   $$(x_1, x_2) = \{(x_1, \frac{w(L - x_1) + \Pi}{p}) | x_1 \in [0, L + \frac{\Pi}{w}]\}$$

   $\frac{w}{p} > \frac{1}{\beta} \Rightarrow$ use all wealth for good 2 consumption

   $$(x_1, x_2) = \{(0, \frac{wL + \Pi}{p})\}$$

   The Firm's Profit Maximizing Decision

   $\frac{w}{p} < \alpha \Rightarrow$ profit can be arbitrarily large, will want arbitrarily large amount good 1 (i.e. demand is empty). Alternatively, under the incorrect assumption, the firm can maximize profit demanding the social endowment of good 1

   $$(z, q) = (L, \alpha L) \quad \text{and} \quad \Pi = p\alpha L - wL$$

   $\frac{w}{p} = \alpha \Rightarrow$ firm makes zero profits, indifferent between any point on production possibility frontier

   $$(z, q) = \{(z, \alpha z) | z \geq 0\} \quad \text{and} \quad \Pi = 0$$

$\frac{w}{p} > \alpha \Rightarrow$ good 1 (the input good) is too expensive, the firm chooses not to produce anything, zero profits

$$(z, q) = (0, 0) \quad \text{and} \quad \Pi = 0$$

Thus this exercise reduces to mixing and matching the quantities $\frac{w}{p}, \frac{1}{\beta}, \alpha$ so that what the agent wants to do fits with what the firm wants to do.

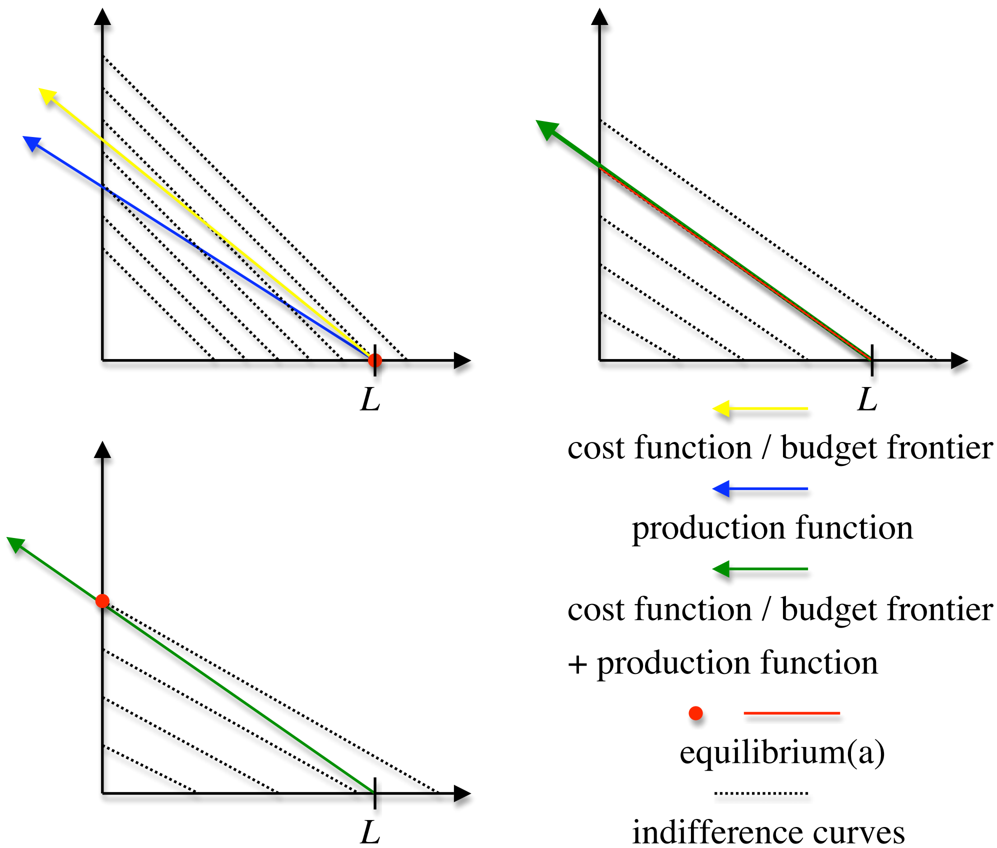


Figure 1: Cases 1 - 3

Case 1: $\frac{1}{\beta} > \alpha$

We can rule out $\frac{w}{p} > \frac{1}{\beta} > \alpha$ and $\frac{1}{\beta} > \alpha > \frac{w}{p}$. However, for any price ratio $\frac{w}{p}$ such that $\frac{1}{\beta} \geq \frac{w}{p} \geq \alpha$ there is a unique equilibrium where the firm produces nothing, and the agent consumes his endowment:

$$\{(x_1(p^*, w^*), x_2(p^*, w^*)) = (L, 0), (z(p^*, w^*), q(p^*, w^*)) = (0, 0), \Pi(p^*, w^*) = 0\}$$

Case 2: $\frac{1}{\beta} = \alpha$

One can show that a necessary condition for equilibrium is $\frac{1}{\beta} = \frac{w}{p} = \alpha$, in which case the set of equilibria is

$$\{(x_1(p^*,w^*), x_2(p^*,w^*)) = (x_1, \frac{w^*(L-x_1)}{p^*}), (z(p^*,w^*), q(p^*,w^*)) = (L-x_1, \alpha(L-x_1)),$$
$$\Pi(p^*,w^*) = 0 \mid x_1 \in [0,L]\}$$

Case 3: $\frac{1}{\beta} < \alpha$

One can show it must be the case $\frac{w}{p} = \alpha$, in which case the unique equilibria is

$$\{(x_1(p^*,w^*), x_2(p^*,w^*)) = (0, \frac{wL}{p}), (z(p^*,w^*), q(p^*,w^*)) = (L, \alpha L), \Pi(p^*,w^*) = 0\}$$

Under the incorrect assumptions, we can let the relative price become lower. So long as $\frac{1}{\beta} \leq \frac{w}{p} \leq \alpha$, we still have equilibrium because the firm will never demand more than L

$$\{(x_1(p^*,w^*), x_2(p^*,w^*)) = (0, \frac{w^*L + p^*\alpha L - w^*L}{p^*}), (z(p^*,w^*), q(p^*,w^*)) = (L, \alpha L),$$
$$\Pi(p^*,w^*) = p^*\alpha L - w^*L\}$$

2. Strong monotonicity implies $p \gg 0$ and budget constraints are binding:

$$p \cdot x_i = p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j + T_i \quad \text{for all } i \qquad \Rightarrow$$

$$\sum_{i=1}^{I} p \cdot x_i = \sum_{i=1}^{I} p \cdot \omega_i + \sum_{i=1}^{I}\sum_{j=1}^{J} \theta_{ij} p \cdot y_j + \sum_{i=1}^{I} T_i \qquad \Rightarrow$$

$$\sum_{i=1}^{I}\sum_{l=1}^{L} p_l x_{li} = \sum_{i=1}^{I}\sum_{l=1}^{L} p_l \omega_{li} + \sum_{i=1}^{I}\sum_{j=1}^{J} \big(\theta_{ij} \sum_{l=1}^{L} p_l y_{lj}\big) \qquad \Rightarrow$$

$$\sum_{l=1}^{L} \big(p_l \sum_{i=1}^{I} x_{li}\big) = \sum_{l=1}^{L} \big(p_l \sum_{i=1}^{I} \omega_{li}\big) + \sum_{l=1}^{L} \big(p_l \sum_{j=1}^{J}\sum_{i=1}^{I} \theta_{ij} y_{lj}\big) \qquad \Rightarrow$$

$$\sum_{l=1}^{L} \big(p_l \sum_{i=1}^{I} x_{li}\big) = \sum_{l=1}^{L} p_l \bar{\omega}_l + \sum_{l=1}^{L} \big(p_l \sum_{j=1}^{J} y_{lj}\big) \tag{1}$$

For $l \leq L-1$, the market clears, so

$$\sum_{i=1}^{I} x_{li} = \bar{\omega}_l + \sum_{j=1}^{J} y_{lj} \quad \text{for } l = 1, 2, \ldots L-1$$

or equivalently

$$p_l \sum_{i=1}^{I} x_{li} = p_l \bar{\omega}_l + p_l \sum_{j=1}^{J} y_{lj} \quad \text{for } l = 1, 2, \dots L-1$$

Summing over $l = 1, \dots, L-1$

$$\sum_{l=1}^{L-1} \left(p_l \sum_{i=1}^{I} x_{li} \right) = \sum_{l=1}^{L-1} p_l \bar{\omega}_l + \sum_{l=1}^{L-1} \left(p_l \sum_{j=1}^{J} y_{lj} \right) \tag{2}$$

subtracting (2) from (1) we get

$$p_L \sum_{i=1}^{I} x_{Li} = p_L \bar{\omega}_L + p_L \sum_{j=1}^{J} y_{Lj}$$

Since $p_L > 0$, we may divide by p_L and deduce the market clearing condition for good L:

$$\sum_{i=1}^{I} x_{Li} = \bar{\omega}_L + \sum_{j=1}^{J} y_{Lj}$$

3(a). Fix an arbitrary price vector $p = (p_1, p_2)$. The firm's profit maximization problem is

$$\underset{y_1}{\text{argmax}} \;\; p_2 e \log(1 - y_1) + p_1 y_1$$

Taking the derivative, we have the first order condition

$$\frac{-p_2 e}{1 - y_1} + p_1 = 0 \Rightarrow$$

$$y_1 = \min\{1 - e\frac{p_2}{p_1}, 0\}$$

We need the min because when $\frac{p_2}{p_1} \le \frac{1}{e}$, we have a boundary solution of $y = 0$. However we will see that this is a non-binding condition because if $\frac{p_2}{p_1} \le \frac{1}{e}$, then the markets will not clear. Now the first agent's utility maximization problem is

$$\underset{x_{11}, x_{21}}{\text{argmax}} \;\; \frac{\log(x_{11})}{e} + x_{21} - \frac{1}{e} \quad \text{s.t.} \;\; p_1 x_{11} + p_2 x_{21} \le p_1 e + \theta_1 \Pi \quad \Rightarrow$$

$$\underset{x_{11}}{\text{argmax}} \;\; \frac{\log(x_{11})}{e} + \frac{p_1 e + \theta_1 \Pi - p_1 x_{11}}{p_2}$$

Taking the derivative, we have the first order condition

$$\frac{1}{e x_{11}} = \frac{p_1}{p_2} \qquad \Rightarrow \qquad x_{11} = \frac{1}{e}\frac{p_2}{p_1}$$

Similarly, we can find agent 2's x_{12}:

$$x_{12} = \frac{p_2}{p_1}$$

Thus market clearing condition for good 1 dictates

$$x_{11} + x_{12} = \frac{p_2}{p_1}(1 + \frac{1}{e}) = e^2 + e + 1 - e\frac{p_2}{p_1} = \bar{\omega}_1 + y_1 \quad \Rightarrow$$

$$\frac{1}{e}\frac{p_2}{p_1}(e^2 + e + 1) = e^2 + e + 1 \quad \Rightarrow \quad \frac{p_2}{p_1} = e$$

Indeed, if $\frac{p_2}{p_1} \leq \frac{1}{e}$, we would have $x_{11} + x_{12} \leq \frac{1}{e^2} + \frac{1}{e} < e^2 + e = \bar{\omega}_1 + y_1$. From here, we can then find y_2, Π, x_{21} and x_{22}:

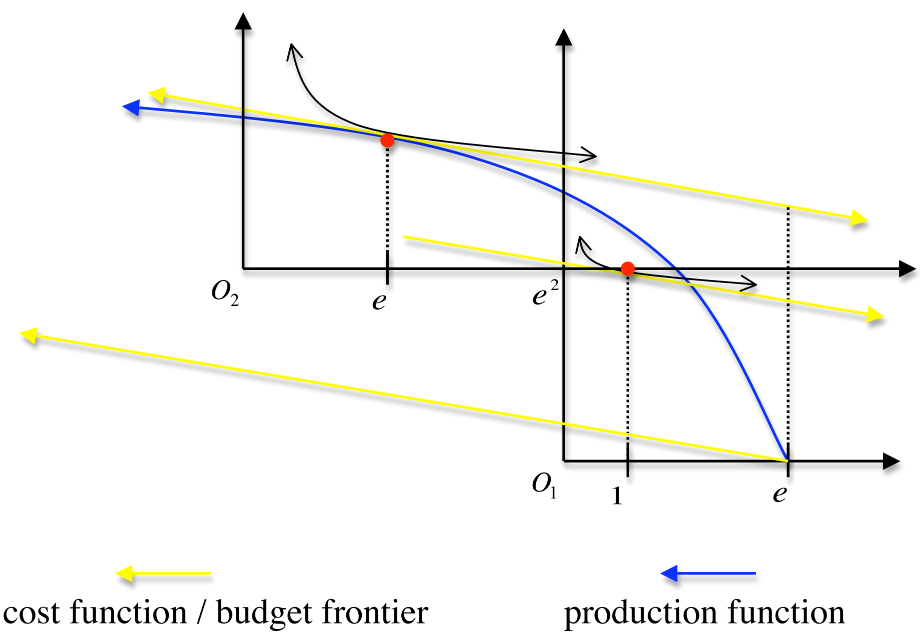
$$y_1 = 1 - e^2 \quad \Rightarrow \quad y_2 = 2e \quad \text{and} \quad \Pi = p_2 2e + p_1(1 - e^2) = p_2(2e - \frac{e^2 - 1}{e}) = p_2(e + \frac{1}{e})$$

$$x_{21} = \frac{p_1 e + \theta_1 \Pi - p_1 x_{11}}{p_2} = 1 - \frac{1}{e} + \theta_1(e + \frac{1}{e})$$

$$x_{22} = \frac{p_1 e^2 + \theta_2 \Pi - p_1 x_{12}}{p_2} = e - 1 + \theta_2(e + \frac{1}{e})$$

Now because of exercise 2 we know that the second good must automatically clear as well, which means we have found the equilibrium:

$$\left(\frac{p_2^*}{p_1^*} = e,\ x^* = \left\{\left(1, 1 - \frac{1}{e} + \theta_1(e + \frac{1}{e})\right),\ \left(e, e - 1 + \theta_2(e + \frac{1}{e})\right)\right\},\ y^* = (1 - e^2, 2e)\right)$$



(b). The phrase - "no access to firm's technology" - literally means that they cannot use the technology of the firm, which means there is no second good, and the agents then simply consume their endowment. In particular "no access to firm's technology" does not have anything to do with whether the agents get a share of the firm's profits. Some of you asked for clarification at office hours. In general this is a good idea if you are ever unsure. However, if you misunderstood the meaning to be zero shares but did the analysis correctly otherwise, I will give you full credit.

One can easily show that the reservations utilities $\underline{U}_i$ are both 0. So it reduces to maximizing the following expression with respect to θ_1:

$$\underset{\theta_1}{\text{argmax}} \quad U_1\Big(1, 1 - \frac{1}{e} + \theta_1(e + \frac{1}{e})\Big)U_2\Big(e, e - 1 + (1-\theta_1)(e + \frac{1}{e})\Big) \Rightarrow$$

$$\underset{\theta_1}{\text{argmax}} \quad \Big(1 - \frac{2}{e} + \theta_1(e + \frac{1}{e})\Big)\Big(1 + e - 1 + (1-\theta_1)(e + \frac{1}{e}) - 2\Big)$$

Taking the derivative, we get the first order condition for θ_1

$$(e + \frac{1}{e})\Big(e - 2 + (1-\theta_1)(e + \frac{1}{e})\Big) - (e + \frac{1}{e})\Big(1 - \frac{2}{e} + \theta_1(e + \frac{1}{e})\Big) = 0 \quad \Rightarrow$$

$$e - 2 + e + \frac{1}{e} - 1 + \frac{2}{e} = \theta_1 2(e + \frac{1}{e}) \Rightarrow$$

$$2e^2 - 3e + 3 = \theta_1 2(e^2 + 1) \quad \Rightarrow \quad \theta_1 = \frac{2e^2 - 3e + 3}{2(e^2+1)} \quad \text{and} \quad \theta_2 = \frac{3e - 1}{2(e^2+1)}$$

If you interpreted the problem to mean zero shares, then a similar analysis can be performed and the answer is $(\theta_1, \theta_2) = (\frac{1}{2}, \frac{1}{2})$.

4(a). Regardless of γ, the exact allocations $\{(1,0),(0,\gamma)\}$ and $\{(0,\gamma),(1,0)\}$ are always Pareto Optimal - whoever gets $(1,0)$ achieves maximal utility given the social endowment, so if there is a Pareto improvement, it must improve the utility of the agent with $(0,\gamma)$. But this can only be done if either he receives more of good 2, which is impossible, or receives strictly more than γ of good 1, which is either also impossible or would leave the other agent strictly worse off.

Case 1: $\gamma < \frac{1}{2}$
Consider an arbitrary exact allocation $\{(a,b),(c,d)\}$.

$a \le \gamma$: then consider the following sequence of changes:

$$\{(a,b),(c,d)\} \to \{(a,b+d),(c,0)\} \to \{(0,b+d),(c+a,0)\} = \{(0,\gamma),(1,0)\}$$

* since $c \ge 1-\gamma > \gamma \ge d$ the first change is weakly Pareto improving
* since $b + d = \gamma \ge a$ the second change is weakly Pareto improving
* if $d > 0$ then the first change is strictly Pareto improving
* if $a > 0$ then the second change is strictly Pareto improving
* Thus the only exact Pareto Optimal allocations when $a \le \gamma$ is $\{(0,\gamma),(1,0)\}$.

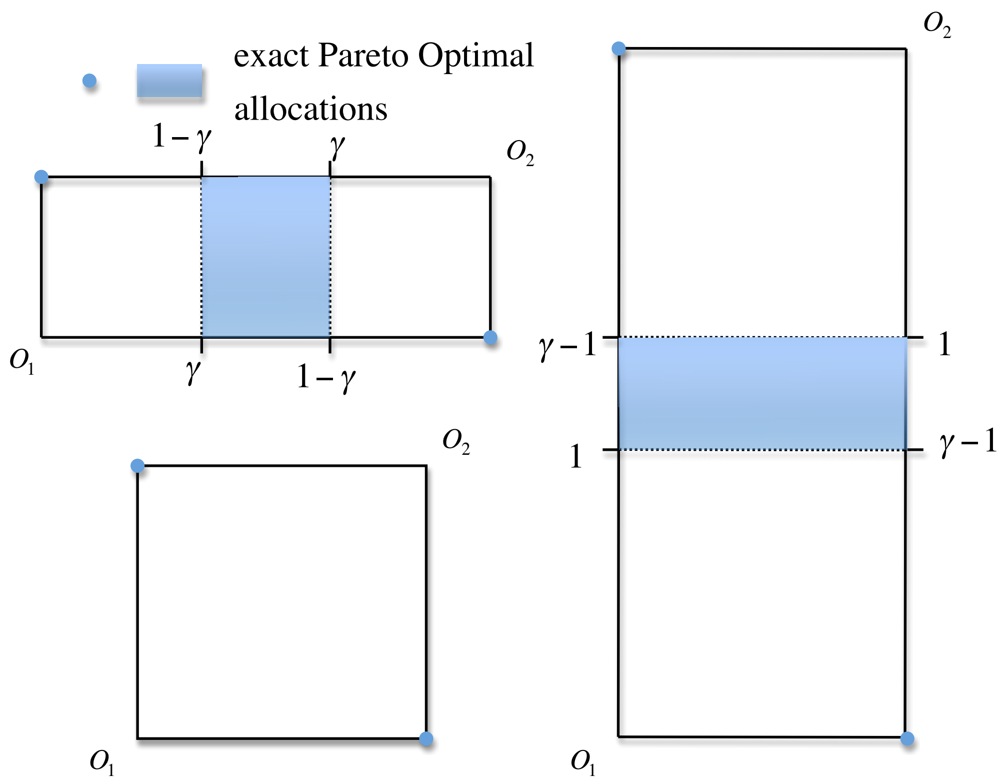
$c \le \gamma$ (equivalently, $a \ge 1-\gamma$): by symmetry the only exact Pareto Optimal allocation when $c \le \gamma$ is $\{(1,0),(0,\gamma)\}$.

$a \in (\gamma, 1-\gamma)$ (equivalently, $c \in (\gamma, 1-\gamma)$): Any exact allocation is Pareto Optimal because both agents derive their utility from their allocation of the first good. Even if a particular agent had all of good 2 transferred to himself, it is less than what he's consuming in good 1, so it does not constitute a strict Pareto improvement.

– So the set of all exact Pareto Optimal allocations is

$$\{(1,0),(0,\gamma)\} \cup \{(0,\gamma),(1,0)\} \cup \{(x_{11},x_{21}),(1-x_{11},\gamma-x_{21}) \mid x_{11} \in (\gamma,1-\gamma)\ \ x_{21} \in [0,\gamma]\}$$

Case 2: $\frac{1}{2} \leq \gamma \leq 1$ There are only two possibilities: either $a \leq \gamma$ or $c \leq \gamma$. We can use the exact same argument to get to the only two exact Pareto Optimal allocations $\{(1,0),(0,\gamma)\}$ and $\{(0,\gamma),(1,0)\}$.



Now I claim that by symmetry we have already basically solved the cases where $\gamma \geq 1$ as well. First convince yourself that the analysis of cases where the social endowments only differ by a constant factor are essentially the same. Furthermore, because of the symmetry of the utility function, convince yourself that the analysis of cases where the social endowment for each good is reversed is also essentially the same.

Thus for $\gamma \geq 1$, the $(1,\gamma)$ case is basically the same as the $(\frac{1}{\gamma},1)$ case which is basically the same as the $(1,\frac{1}{\gamma})$ case, which we have already considered since $\frac{1}{\gamma} \leq 1$.

So for $1 \leq \gamma \leq 2$, the only exact Pareto Optimal allocations are $\{(1,0),(0,\gamma)\}$ and $\{(0,\gamma),(1,0)\}$ just like Case 2. And for $\gamma > 2$, the set of exact Pareto Optimal allocations is

$$\{(1,0),(0,\gamma)\} \ \cup \ \{(0,\gamma),(1,0)\} \ \cup \ \{(x_{11},x_{21}),(1-x_{11},\gamma-x_{21}) \mid x_{11} \in [0,1] \ \ x_{21} \in [1,\gamma-1]\}$$

which is similar to Case 1.

4(b). The Pareto Optimal allocations $\{(1,0),(0,\gamma)\}$ and $\{(0,\gamma),(1,0)\}$ can both be supported by the price vector (p_1,p_2) where $p_1 = p_2 = q$. We will now show that no other Pareto Optimal allocation can be supported as an equilibrium with transfers. So let's consider one of the non corner Pareto Optimal allocations in, say, Case 1:

$$\{(x_{11},x_{21}),(1-x_{11},\gamma-x_{21})\}$$

where x_{11} is some number in $(\gamma, 1-\gamma)$ and x_{21} is some number in $[0,\gamma]$.

The total wealth of the of exchange economy is $p_1 + p_2\gamma$. If there is a good that is strictly cheaper (say good i), both agents would use all their wealth to purchase that good. So the total demand for good i is $\frac{p_1+p_2\gamma}{p_i} > 1+\gamma > \max\{1, \ \gamma\} \geq$ social endowment of good i. Thus $p_1 = p_2 = q$. In this case, each agent would still choose one good and use their entire wealth to purchase this good. Since $x_{11} > \gamma$, for the consumption bundle (x_{11}, x_{21}) to be even feasible for agent 1, it must be the case that his wealth $w_1 \geq qx_{11} > q\gamma_1$, which means for there to be an equilibrium he must choose to only buy good 1 (if he chose instead to buy only good 2, there wouldn't be enough). Similarly, agent 2 is too wealthy to only buy good 2, which means he must also choose to only buy good 1 but then the total demand for good 1 is too great and the total demand for good 2 is to small.

Thus the Second Welfare Theorem fails for $\gamma \in (0, \frac{1}{2}) \ \cup \ (2, \infty)$.