Economics 201b
Spring 2010
Solutions to Problem Set 5
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1. Suppose there are two goods and $\bar{\omega}_1 > 0$ and $\bar{\omega}_2 = 0$. Now suppose all of the agents' marginal utilites from consuming good 2 is infinite when they are consuming 0 amount of good 2 (e.g. $U(x_{1i}, x_{2i}) = \sqrt{x_{1i}} + \sqrt{x_{2i}}$). Now consider any possible equilibrium price $p$. If $p \cdot \bar{\omega} = 0$, then $p_1 = 0$, in which case, by strong monotonicity, the demand for good 1 is empty (everybody would want arbitrarily large amounts of the good). Otherwise $p \cdot \bar{\omega} > 0$ and regardless of the income transfer, somebody would have positive wealth, say agent 1. Then because of agent 1's marginal utility for good 2, she would always want to allocate some nonzero amount of her wealth in purchasing good 2. Thus there cannot be any equilibrium with transfers.

2. Define $K = F^{-1}(\{0\})$. Since $\{0\}$ is closed, by the continuity of $F$ we have $K$ is closed. To show that $E$ has a closed graph, we need to show for a sequence of endowments $\{\omega_n\}_{n=1,2,\ldots}$ and a sequence of normalized prices $\{\hat{p}_n\}_{n=1,2,\ldots}$ satisfying:

$$\hat{p}_n \in E(\omega_n) \qquad n = 1, 2, \ldots \tag{1}$$

$$\omega_n \to \omega \in \mathbb{R}^{LI}_{+} \qquad \text{and} \qquad \hat{p}_n \to \hat{p} \in \mathbb{R}^{L-1}_{++} \tag{2}$$

that $\hat{p} = E(\omega)$.

By assumption (1), we have $F(\hat{p}_n, \omega_n) = 0$, so by definition of $K$

$$(\hat{p}_n, \omega_n) \in K$$

By assumption (2), we have

$$(\hat{p}_n, \omega_n) \to (\hat{p}, \omega) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}^{LI}_{+}$$

Since $K$ is closed, we have

$$(\hat{p}, \omega) \in K \Rightarrow \hat{p} \in E(\omega)$$

3a. See figure below.

b. Since $p_1$ and $p_2$ are the horizontal and vertical components of $p$ we have

$$p_1 = |p| \cos(\theta(p)) \quad \text{and} \quad p_2 = |p| \sin(\theta(p))$$

and since $p_1 + p_2 = 1$, we have

$$p_1 = \frac{\cos(\theta(p))}{\cos(\theta(p)) + \sin(\theta(p))} \quad \text{and} \quad p_2 = \frac{\sin(\theta(p))}{\cos(\theta(p)) + \sin(\theta(p))}$$

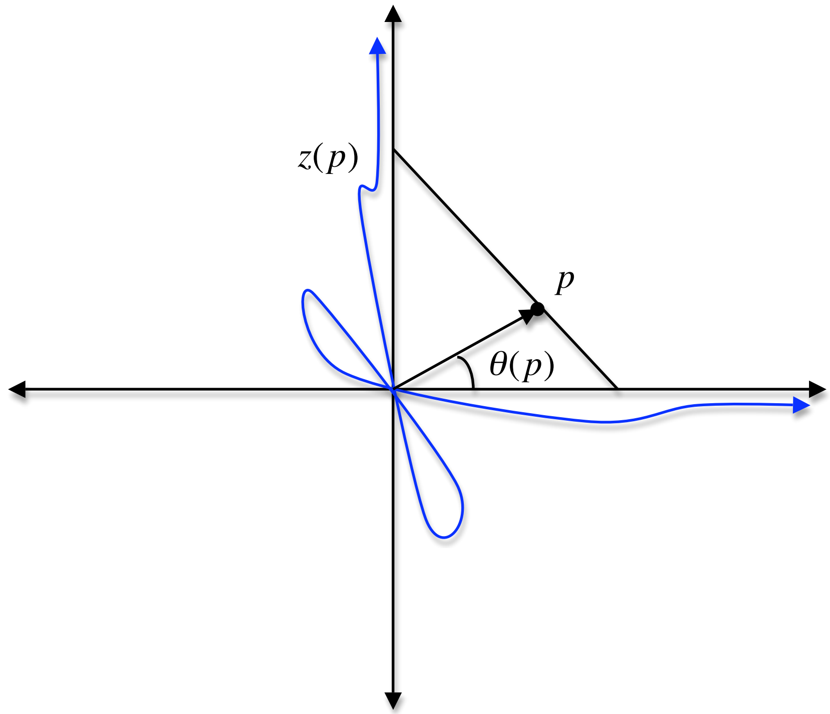


Figure 1: $z(p)$ and $\theta(p)$

c.

$$z(p) \cdot p =$$

$$\left(f(\theta(p))\cos(6\theta(p))\cos(\theta(p)+\frac{\pi}{2}), f(\theta(p))\cos(6\theta(p))\sin(\theta(p)+\frac{\pi}{2})\right)\cdot$$

$$\left(\frac{\cos(\theta(p))}{\cos(\theta(p))+\sin(\theta(p))}, \frac{\sin(\theta(p))}{\cos(\theta(p))+\sin(\theta(p))}\right) =$$

$$G(\theta(p))\left[\left(\cos(\theta(p)+\frac{\pi}{2}), \sin(\theta(p)+\frac{\pi}{2})\right)\cdot\left(\cos(\theta(p)), \sin(\theta(p))\right)\right]$$

where

$$G(\theta(p)) = \frac{f(\theta(p))\cos(6\theta(p))}{\cos(\theta(p))+\sin(\theta(p))}$$

So it suffices to show $(\cos(\theta(p)+\frac{\pi}{2}), \sin(\theta(p)+\frac{\pi}{2}))\cdot(\cos(\theta(p)), \sin(\theta(p))) = 0$:

$$\left(\cos(\theta(p)+\frac{\pi}{2}), \sin(\theta(p)+\frac{\pi}{2})\right)\cdot\left(\cos(\theta(p)), \sin(\theta(p))\right) =$$

$$\cos(\theta(p)+\frac{\pi}{2})\cos(\theta(p)) + \sin(\theta(p)+\frac{\pi}{2})\sin(\theta(p)) =$$

$$\cos(\theta(p)+\frac{\pi}{2}-\theta(p)) = \cos(\frac{\pi}{2}) = 0$$

d. Condition 1 is obvious, and condition 2 has been verified. To see that condition 3 is true we break the analysis into 3 cases. First

$$Z(\theta) = \left( f(\theta)\cos(6\theta)\cos(\theta + \frac{\pi}{2}), f(\theta)\cos(6\theta)\sin(\theta + \frac{\pi}{2}) \right)$$

*Case 1:* $[\epsilon, \frac{\pi}{2} - \epsilon]$
Since $[\epsilon, \frac{\pi}{2} - \epsilon]$ is compact, and $Z(\theta)$ is continuous, $Z(\theta)$ is bounded on this interval.

*Case 2:* $(0, \epsilon)$
First consider the second coordinate. Since,

$$\lim_{\theta\to 0} \cos(6\theta) = \lim_{\theta\to 0} \sin(\theta + \frac{\pi}{2}) = 1 > 0 \quad \text{and} \quad \lim_{\theta\to 0} f(\theta) = \infty$$

we have

$$\lim_{\theta\to 0} f(\theta)\cos(6\theta)\sin(\theta + \frac{\pi}{2}) = \infty > 0$$

Now consider the first coordinate. First we have

$$\lim_{\theta\to 0} f(\theta)\cos(6\theta)\cos(\theta + \frac{\pi}{2}) = \lim_{\theta\to 0} f(\theta)\sin(\theta + \pi) = -\lim_{\theta\to 0} f(\theta)\sin(\theta) = -\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}\frac{f(\theta)}{\frac{1}{\theta}} =$$

$$-\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} \lim_{\theta\to 0} \frac{f(\theta)}{\frac{1}{\theta}} \geq -M$$

Thus we have show $Z(\theta)$ is bounded below on this interval.

*Case 3:* $(\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$
The method is similar to Case 2. The analysis of the first coordinate is like the analysis of the second coordinate in Case 2. To show that the second coordinate is also bounded below:

$$\lim_{\theta\to\frac{\pi}{2}} f(\theta)\cos(6\theta)\sin(\theta + \frac{\pi}{2}) = -\lim_{\theta\to\frac{\pi}{2}} f(\theta)\sin(\theta + \frac{\pi}{2}) = \lim_{\theta\to\frac{\pi}{2}} f(\theta)\sin(\theta - \frac{\pi}{2}) =$$

$$\lim_{\theta\to\frac{\pi}{2}} \frac{\sin(\theta - \frac{\pi}{2})}{\theta - \frac{\pi}{2}} \frac{f(\theta)}{\frac{1}{\theta - \frac{\pi}{2}}} = -\lim_{\theta\to\frac{\pi}{2}} \frac{\sin(\theta - \frac{\pi}{2})}{\theta - \frac{\pi}{2}} \frac{f(\theta)}{\frac{1}{\frac{\pi}{2} - \theta}} \geq -M$$

Thus $Z(\theta)$ is bounded below on this interval.

Putting everything together, we have $Z(\theta)$ is bounded below.

To show that condition 4 is satisfied, again it suffices to show that

$$\lim_{\theta\to 0} |Z(\theta)| = \lim_{\theta\to\frac{\pi}{2}} |Z(\theta)| = \infty$$

Take a look at the analysis of the second coordinate of Case 2 notice it implies

$$\lim_{\theta\to 0} |Z(\theta)| = \infty$$

Similarly, an analysis of the first coordinate of Case 3 would imply

$$\lim_{\theta \to \frac{\pi}{2}} |Z(\theta)| = \infty$$

Thus all the condition of the D-G-K-N Lemma are satisfied.

e. The solutions to $Z(\theta) = 0$ are simply the solutions to $\cos(6\theta) = 0$ on $(0, \frac{\pi}{2})$:

$$\theta_1^* = \frac{\pi}{12} + \frac{2\pi}{6} \quad \theta_2^* = \frac{\pi}{12} + \frac{\pi}{6} \quad \theta_3^* = \frac{\pi}{12}$$

So the equilibrium prices are

$$p_1^* = \left( \frac{\cos(\frac{\pi}{12} + \frac{2\pi}{6})}{\cos(\frac{\pi}{12} + \frac{2\pi}{6}) + \sin(\frac{\pi}{12} + \frac{2\pi}{6})}, \frac{\sin(\frac{\pi}{12} + \frac{2\pi}{6})}{\cos(\frac{\pi}{12} + \frac{2\pi}{6}) + \sin(\frac{\pi}{12} + \frac{2\pi}{6})} \right)$$

$$p_2^* = \left( \frac{\cos(\frac{\pi}{12} + \frac{\pi}{6})}{\cos(\frac{\pi}{12} + \frac{\pi}{6}) + \sin(\frac{\pi}{12} + \frac{\pi}{6})}, \frac{\sin(\frac{\pi}{12} + \frac{\pi}{6})}{\cos(\frac{\pi}{12} + \frac{\pi}{6}) + \sin(\frac{\pi}{12} + \frac{\pi}{6})} \right)$$

$$p_3^* = \left( \frac{\cos(\frac{\pi}{12})}{\cos(\frac{\pi}{12}) + \sin(\frac{\pi}{12})}, \frac{\sin(\frac{\pi}{12})}{\cos(\frac{\pi}{12}) + \sin(\frac{\pi}{12})} \right)$$

The normalized versions of these prices are

$$(\hat{p}_1^*, 1) = \big(\cot(\frac{\pi}{12} + \frac{2\pi}{6}), 1\big) \quad (\hat{p}_2^*, 1) = \big(\cot(\frac{\pi}{12} + \frac{\pi}{6}), 1\big) \quad (\hat{p}_3^*, 1) = \big(\cot(\frac{\pi}{12}), 1\big)$$

f. As $\hat{p}$ increases, $\phi(\hat{p})$ moves down the line $\Delta^o$, which means the angle $\theta$ is decreasing. So the sign is -1.

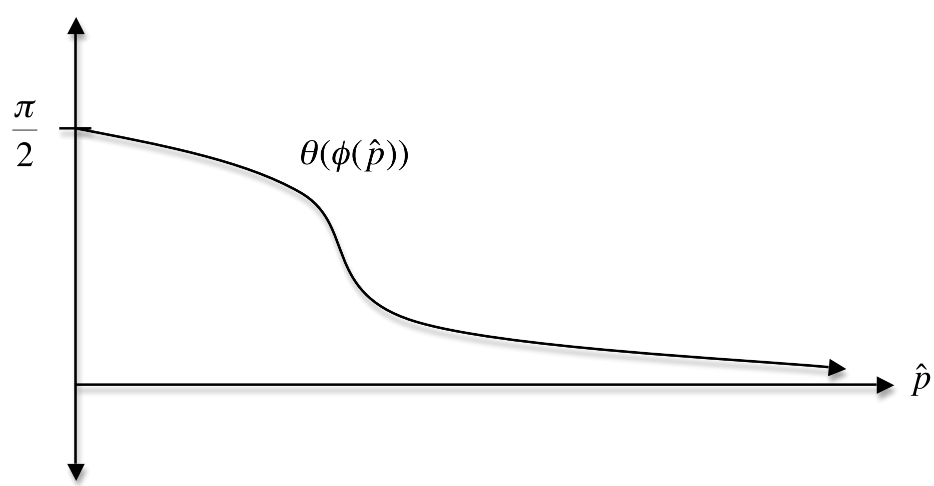


Figure 2: Downward sloping $\theta(\phi(\hat{p}))$

g.

$$\hat{z}(\hat{p}) = z_1(\phi(\hat{p})) =$$

$$f(\theta(\phi(\hat{p}))) \cos(6\theta(\phi(\hat{p}))) \cos(\theta(\phi(\hat{p})) + \frac{\pi}{2})$$

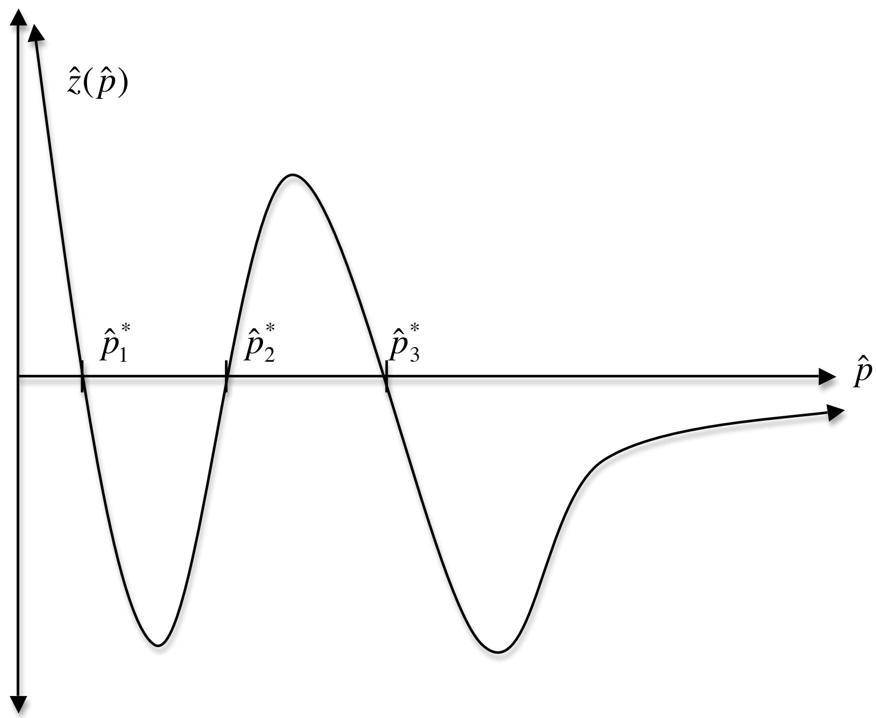


Figure 3: The Normalized Excess Demand Function

By the chain rule

$$-\mathrm{sgn}\frac{d}{d\hat{p}}\hat{z}(\hat{p})\Big|_{\hat{p}=\hat{p}_1^*} =$$

$$-\mathrm{sgn}\left[\frac{d}{d\theta}f(\theta)\cos(6\theta)\cos(\theta + \frac{\pi}{2})\Big|_{\theta=\frac{\pi}{12}+\frac{2\pi}{6}}\frac{d}{d\hat{p}}\theta(\phi(\hat{p}))\Big|_{\hat{p}=\hat{p}_1^*}\right] =$$

$$-\mathrm{sgn}\frac{d}{d\theta}f(\theta)\cos(6\theta)\cos(\theta + \frac{\pi}{2})\Big|_{\theta=\frac{\pi}{12}+\frac{2\pi}{6}}\mathrm{sgn}\frac{d}{d\hat{p}}\theta(\phi(\hat{p}))\Big|_{\hat{p}=\hat{p}_1^*} =$$

$$\mathrm{sgn}\frac{d}{d\theta}f(\theta)\cos(6\theta)\cos(\theta + \frac{\pi}{2})\Big|_{\theta=\frac{\pi}{12}+\frac{2\pi}{6}} = \mathrm{sgn}\big[-6f(\theta)\sin(6\theta)\cos(\theta + \frac{\pi}{2})\Big|_{\theta=\frac{\pi}{12}+\frac{2\pi}{6}}\big] = 1$$

Similarly,

$$-\mathrm{sgn}\frac{d}{d\hat{p}}\hat{z}(\hat{p})\Big|_{\hat{p}=\hat{p}_2^*} =$$

$$\operatorname{sgn}\Big[-6f(\theta)\sin(6\theta)\cos(\theta+\frac{\pi}{2})\Big|_{\theta=\frac{\pi}{12}+\frac{\pi}{6}}\Big]=-1$$

and

$$-\operatorname{sgn}\frac{d}{d\hat{p}}\hat{z}(\hat{p})\Big|_{\hat{p}=\hat{p}_3^*}=$$

$$\operatorname{sgn}\Big[-6f(\theta)\sin(6\theta)\cos(\theta+\frac{\pi}{2})\Big|_{\theta=\frac{\pi}{12}}\Big]=1$$

The economy is indeed regular since all of the derivatives are nonzero.

h. First note $Z(x)=0$, so $\hat{p}^*(x)=\cot(x)$ is a normalized equilibrium price. Because $f\geq 0$ and $f$ is differentiable, we have $f(x)=0\Rightarrow f'(x)=0$. We can now show

$$\frac{d}{d\theta}f(\theta)\cos(6\theta)\cos(\theta+\frac{\pi}{2})\Big|_{\theta=x}=$$

$$f'(x)\cos(6x)\cos(x+\frac{\pi}{2})-6f(x)\sin(6x)\cos(x+\frac{\pi}{2})-f(x)\cos(6x)\sin(x+\frac{\pi}{2})=0$$

$$\Rightarrow\frac{d}{d\hat{p}}\hat{z}(\hat{p})\Big|_{\hat{p}=\hat{p}^*(x)}=0$$

Thus the index of $\hat{p}^*(x)$ is 0 regardless of $x$.

If $x\neq\theta_i^*$ for all $i$, then $\hat{p}^*(x)$ is an extra equilibrium price. However, since the index at this price is 0, the Index equation is still equal to 1.

If $x=\theta_i^*$ where $i=1$ or 3, then there is no new equilibrium price. However, since the index of $\hat{p}_i^*=\hat{p}^*(x)$ is now 0 instead of 1, so the Index equation equals 0.

If $x=\theta_2^*$, then there is no new equilibrium price. However, since the index of $\hat{p}_2^*=\hat{p}^*(x)$ is now 0 instead of -1, so the Index equation equals 2.

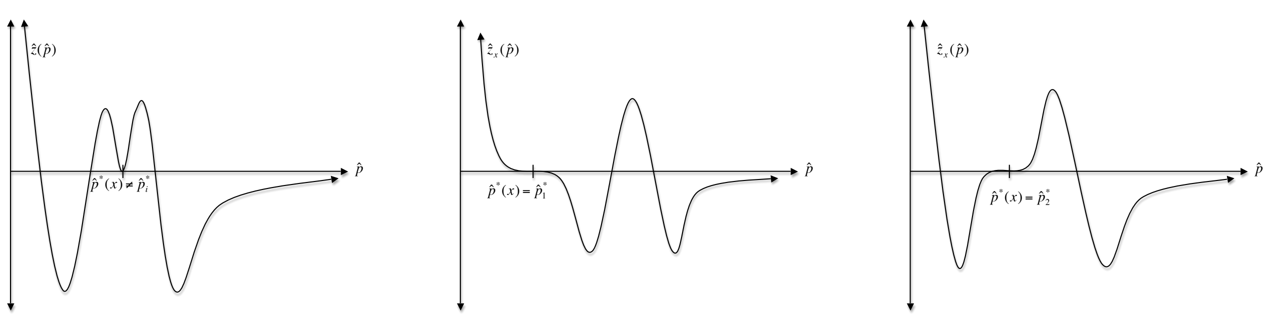


Figure 4: Graphical Representation of the Different Index Equation Values: 1, 0, and 2