Economics 204–Final Exam–August 19, 2008, 9am-12pm
Each of the four questions is worth 25% of the total
Please use three *separate* blue/greenbooks, one for each of the three Parts

Part I

1. Prove that if X and Y are vector spaces over the same field F and $\dim X = \dim Y$, then X and Y are isomorphic.

2. Consider the function

$$f(x,y) = 3x^2 + 3y^2 - 2xy + x^4 + y^5$$

   (a) Show that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a critical point of f.

   (b) Determine whether f has a local max, a local min, or neither at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

   (c) Does f have a global max, a global min, or neither at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$?

Part II

3. Consider the Initial Value Problem

$$y'' = -y, \; y(0) = y'(0) = 1 \tag{1}$$

   (a) Write this as a first order linear Initial Value Problem using the variables $y_1 = y$ and $y_2 = y'$.

   (b) Find the eigenvalues of the matrix obtained in part (a).

   (c) Find the unique solution of the Initial Value Problem in Equation (1). *Hint*: you can use the product of three complex matrices if you wish, but there is a simpler approach.

   (d) Now consider the Initial Value Problem

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 - y_1^3/100 \\ -y_1 - y_2^3/100 \end{pmatrix}, \; y_1(0) = y_2(0) = 1 \tag{2}$$

   Show that the unique stationary point for Equation (2) is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Show that the linearized equation corresponding to Equation (2) is the Initial Value Problem you found in part (a). Find a function $G : \mathbf{R}^2 \to \mathbf{R}_+$ such that every solution of the linearized differential equation follows a level set of G.

(e) Now suppose $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is a solution of the nonlinear Initial Value Problem in Equation (2). Compute $\frac{dG(y(t))}{dt}$. What does this tell you about the behavior of $y(t)$ as $t \to \infty$?

Part III

4. Suppose $F : X \times \Omega \to \mathbf{R}$ is continuous, where X is a compact subset of E^m and $\Omega \subseteq E^n$. Define

$$\psi(\omega) = \{x \in X : F(x, \omega) = \sup\{F(z, \omega) : z \in X\}\}$$

(a) Show that for all $\omega \in \Omega$, $\Psi(\omega) \neq \emptyset$.

(b) Show that Ψ is upper hemicontinuous. *Hint*: Suppose not. Negate the definition of upper hemicontinuity and show that this implies the existence of a sequence $\{x_n\}$ with certain properties. Take a convergent subsequence $\{x_{n_k}\}$, and derive a contradiction.