Economics 204

Lecture 13–Wednesday, August 12, 2009

Section 5.5 (Cont.) Transversality Theorem

The Transversality Theorem is a particularly convenient formulation of Sard's Theorem for our purposes:

Theorem 1 (2.5', Transversality Theorem) *Let*

$$\begin{aligned} X \times \Omega &\subseteq \mathbf{R}^{n+p} \textit{be open} \\ F : X \times \Omega &\rightarrow \mathbf{R}^m \in C^r \\ &\qquad \textit{with } r \geq 1 + \max\{0, n-m\} \end{aligned}$$

Suppose that

$$F(x, \omega) = 0 \Rightarrow DF(x, \omega) \textit{ has rank } m$$

Then there is a set $\Omega_0 \subseteq \Omega$ *such that* $\Omega \setminus \Omega_0$ *has Lebesgue measure zero such that*

$$\omega \in \Omega_0,\ F(x, \omega) = 0 \Rightarrow D_x F(x, \omega) \textit{ has rank } m$$

If $m = n$ *and* $\omega_0 \in \Omega_0$*,*

- *there is a local implicit function*

  $$x^*(\omega)$$

  characterized by

  $$F(x^*(\omega), \omega) = 0$$

  where x^* *is a* C^r *function of* ω*.*

- *the correspondence*

  $$\omega \rightarrow \{x : F(x, \omega) = 0\}$$

  is lower hemicontinuous at ω_0*.*

Interpretation of Tranversality Theorem

- Ω: a set of parameters (agents' endowments and preferences, or players' payoff functions).

- X: a set of variables (price vectors, or strategies).

- $\mathbf{R}^m$ is the range of F (excess demand, or best-response strategies).

- $F(x, \omega) = 0$ is equilibrium condition, given parameter ω.

- Rank $DF(x, \omega) = m$ says that, by adjusting either the variables or parameters, it is possible to move F in any direction.

- When $m = n$, Rank $D_x F(x, \omega) = m$ says $\det D_x F(x, \omega) \neq 0$, which says the economy is regular and is the hypothesis of the Implicit Function Theorem; this tells us that the equilibrium correspondence is lower hemicontinuous. Economic correspondences like $\omega \to \{x : F(x, \omega) = 0\}$ are generally upper hemicontinuous, so regularity in fact tells us the correspondence is continuous. You will see in 201B that regularity, plus a property of demand functions, tell us that the equilibrium prices are given by a finite number of implicit functions of the parameters (endowments).

- Parameters of any given economy are fixed. However, we want to study the *set* of parameters for which the resulting economy is well-behaved.

- Theorem says the following:

  > "If, whenever $F(x, \omega) = 0$, it is possible by perturbing the parameters and variables to move F in any direction, then for almost all parameter values, all equilibria

are regular, the equilibria are implicitly defined C^r functions of the parameters, and the equilibrium correspondence is lower hemicontinuous." You will see in 201B that the regularity of the equilibria plus a property of demand functions implies that there are only finitely many equilibria.

- If $n < m$, Rank $D_x F(x,\omega) \leq \min\{m,n\} = n < m$. Therefore,

$$\begin{aligned} & (F(x,\omega) = 0 \Rightarrow DF(x,\omega) \text{ has rank m}) \\ \Rightarrow \quad & \text{for all } \omega \text{ except for a set of Lebesgue measure zero} \\ & F(x,\omega) = 0 \text{ has no solution} \end{aligned}$$

- *Why is it true?* Sard's Theorem says the set of critical values of F is a set of Lebesgue measure zero. As long as you have the freedom to move F away from zero in every direction, then you can make zero not be a critical value and hence make the economy regular.

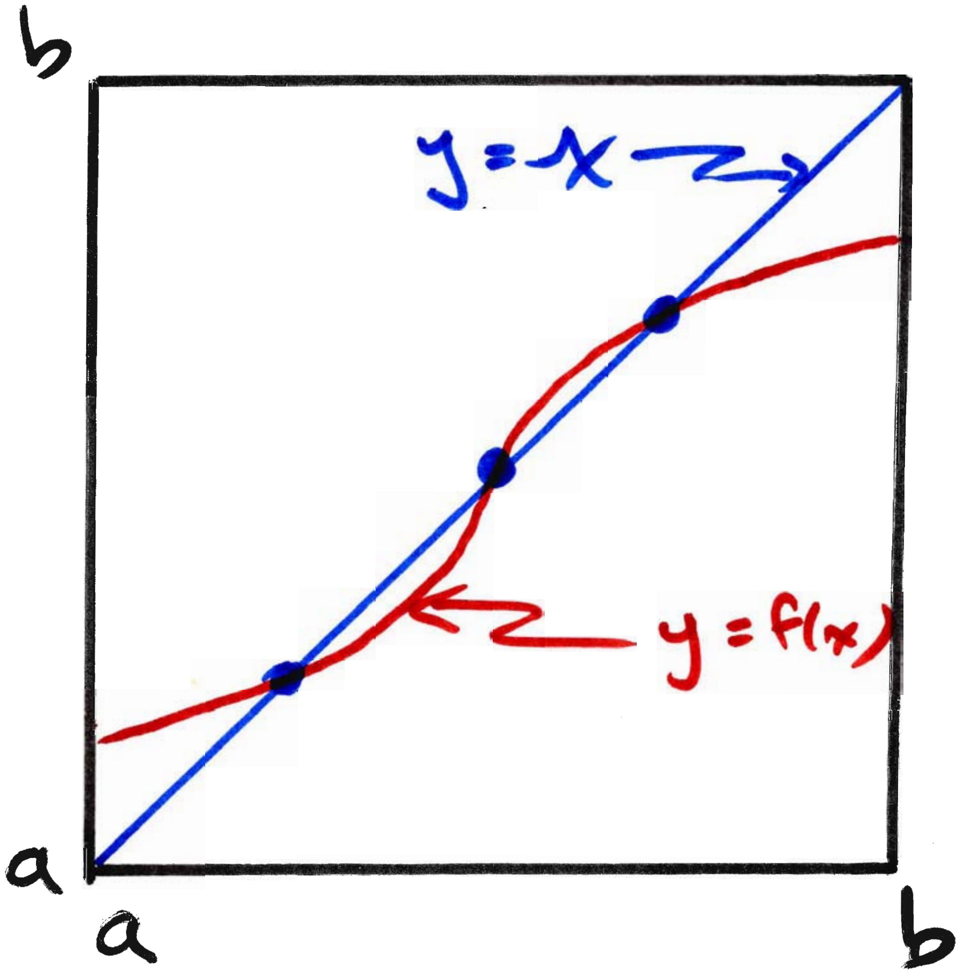
Section 5.3, Brouwer's and Kakutani's Fixed Point Theorems

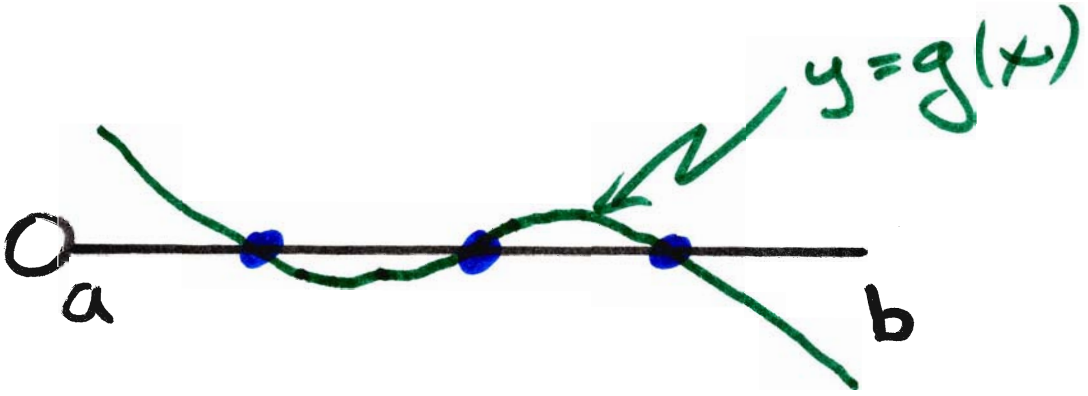
Theorem 2 (3.2, Brouwer's Fixed Point Theorem) *Let $X \subseteq \mathbf{R}^n$ be nonempty, compact, convex and let $f : X \to X$ be continuous. Then*

$$\exists_{x^* \in X} f(x^*) = x^*$$

i.e. f has a fixed point.

Proof: (in very special case $n = 1$): A is a closed interval $[a,b]$. Let


b
y = x
y = f(x)
a
a
b


y = g(x)
0
a
b

$$
\begin{aligned}
g(x) &= f(x) - x \\
g(a) &= f(a) - a \geq 0 \\
g(b) &= f(b) - b \leq 0
\end{aligned}
$$

By the Intermediate Value Theorem, there exists $x^* \in [a, b]$ such that $g(x^*) = 0$. Then

$$f(x^*) = g(x^*) + x^* = x^*$$

■

General Case: Much harder, but a wonderful result due to Scarf gives an efficient algorithm to find approximate fixed points:

$$\forall \varepsilon > 0 \ \exists_{x_\varepsilon^*} \ \left| f\left(x_\varepsilon^*\right) - x_\varepsilon^* \right| < \varepsilon$$

Sketch of Idea of Scarf Algorithm:

- Suppose X is $n-1$ dimensional. Let X be the price simplex

$$X = \left\{ p \in \mathbf{R}_+^n : \sum_{\ell=1}^n p_\ell = 1 \right\}$$

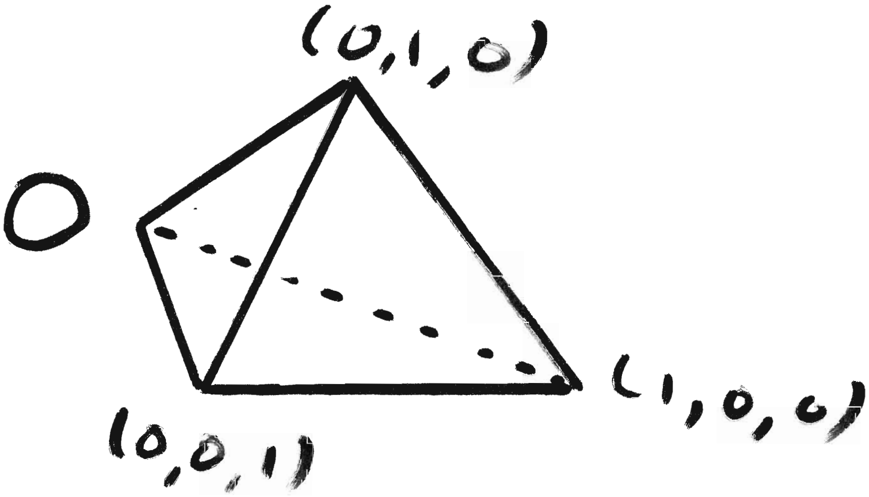
- Triangulate X, i.e. divide X into a set of simplices such that the intersection of any two simplices is either empty or a whole face of both.

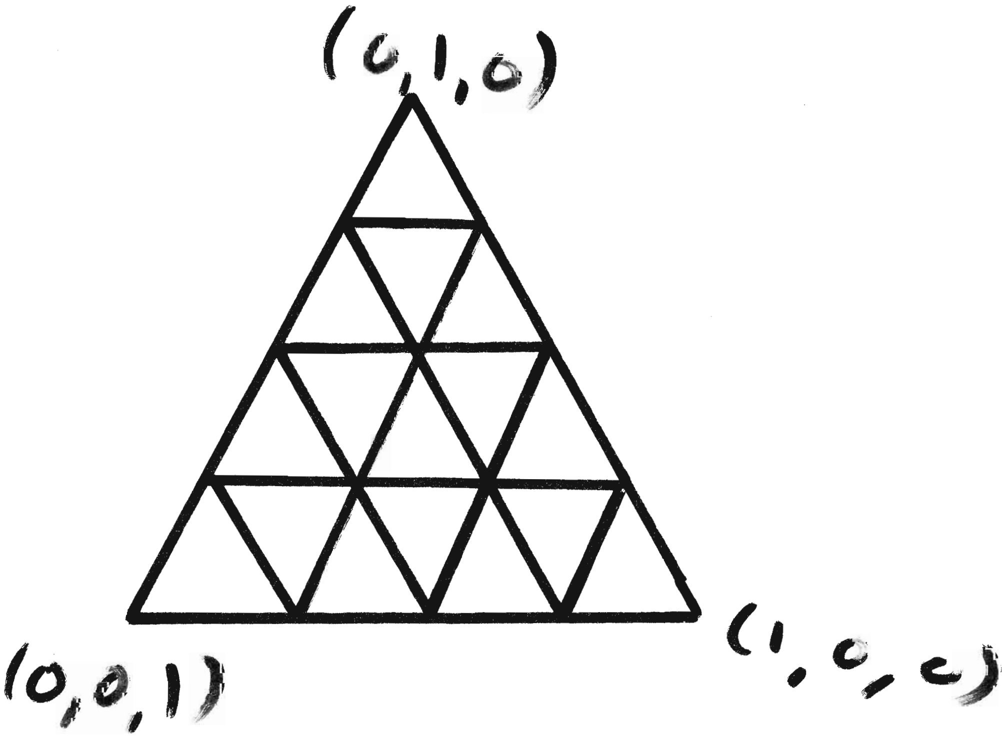
- Label each vertex in the triangulation by

$$L(x) = \min \left\{ \ell : f(x)_\ell < x_\ell \right\}$$

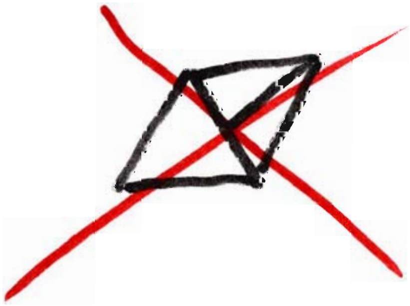
- Each simplex in the triangulation has n vertices. A simplex is *completely labelled* if its vertices carry each of the labels $1, \ldots, n$ exactly once; it is *almost completely labelled* if its vertices carry the labels $1, \ldots, n-1$ with exactly one of these labels repeated.

L = 3





Triangulation



- A simplex which is almost completely labelled has two "doors," the faces opposite the two vertices with repeated labels. Algorithm pivots from one simplex to another by going in one door and alway leaving by the other door. The new simplex must either be completely labelled, in which case the algorithm stops, or it is almost completely labelled and the algorithm continues.

- One can show that one can never visit the same simplex twice: if you did, there would be a first simplex visited a second time, but then you had to enter it through a door, and you previously used both doors, so some other simplex must be the first simplex visited a second time, contradiction.

- One can show that one cannot exit through a face of the the large simplex. Since there are only finitely many simplices, and you visit each one at most once, you must stop after a finite number of steps at a completely labelled simplex.

- Completely labelled simplices are approximate fixed points:

  - Fix $\varepsilon > 0$. Since X is compact and f is continuous, f is uniformly continuous, so we can find a triangulation fine enough so that for every simplex σ in the triangulation,

$$x, y \in \sigma \Rightarrow \left(|x - y| < \frac{\varepsilon}{4n}, \ |f(x) - f(y)| < \frac{\varepsilon}{4n}\right)$$

  - Suppose σ is completely labelled. Let its vertices be $v_1, \ldots, v_n$, and assume WLOG

$$L(v_\ell) = \ell$$

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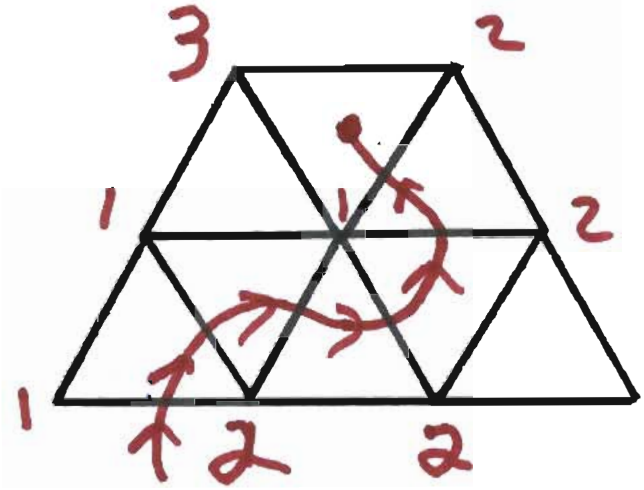
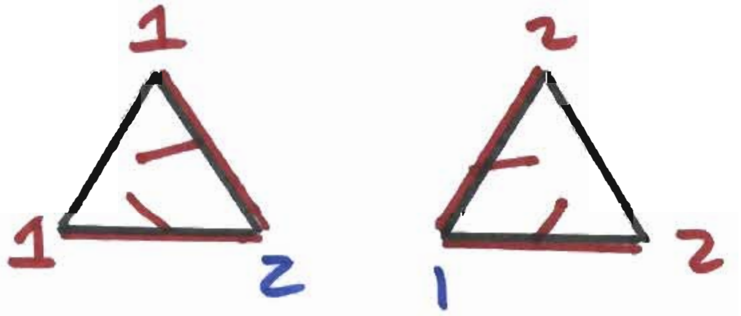
$$L(v_1) = 1 \quad \Rightarrow \quad f(v_1)_1 < (v_1)_1$$





— Doors

$$\begin{aligned}
L(v_2) = 2 \neq 1 \quad &\Rightarrow \quad f(v_2)_1 \geq (v_2)_1 \\
&\Rightarrow \quad \exists_{y_1 \in \sigma}\ f(y_1)_1 = (y_1)_1 \\
L(v_2) = 2 \quad &\Rightarrow \quad f(v_2)_2 < (v_2)_2 \\
L(v_3) = 3 \neq 2 \quad &\Rightarrow \quad f(v_3)_2 \geq (v_3)_2 \\
&\Rightarrow \quad \exists_{y_2 \in \sigma}\ f(y_2)_2 = (y_2)_2 \\
&\vdots \\
L(v_{n-1}) = n-1 \quad &\Rightarrow \quad f(v_{n-1})_{n-1} < (v_{n-1})_{n-1} \\
L(v_n) = n \neq n-1 \quad &\Rightarrow \quad f(v_n)_{n-1} \geq (v_n)_{n-1} \\
&\Rightarrow \quad \exists_{y_{n-1} \in \sigma}\ f(y_{n-1})_{n-1} = (y_{n-1})_{n-1}
\end{aligned}$$

Given any $x \in \sigma$ and any $\ell \in \{1, \ldots, n-1\}$,

$$\begin{aligned}
&|f(x)_\ell - x_\ell| \\
&\quad \leq \quad |f(x)_\ell - f(y_\ell)_\ell| + |f(y_\ell)_\ell - (y_\ell)_\ell| + |(y_\ell)_\ell - x_\ell| \\
&\quad \leq \quad \frac{\varepsilon}{4n} + 0 + \frac{\varepsilon}{4n} \\
&\quad = \quad \frac{\varepsilon}{2n} \\
&|f(x)_n - x_n| \\
&\quad = \quad \left| \left(1 - \sum_{\ell=1}^{n-1} f(x)_\ell\right) - \left(1 - \sum_{\ell=1}^{n-1} x_\ell\right) \right| \\
&\quad = \quad \left| \sum_{\ell=1}^{n-1} (x_\ell - f(x)_\ell) \right| \\
&\quad \leq \quad \sum_{\ell=1}^{n-1} |f(x)_\ell - x_\ell| \\
&\quad \leq \quad (n-1)\frac{\varepsilon}{2n} \\
&\quad < \quad \frac{\varepsilon}{2}
\end{aligned}$$

$$
\begin{aligned}
|f(x)-x| &\leq \|f(x)-x\|_1 \\
&\leq (L-1)\frac{\varepsilon}{2n}+\frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

Theorem 3 (3.4', Kakutani's Fixed Point Theorem)

Suppose $X \subseteq \mathbf{R}^m$ is compact, convex, nonempty, and $\Psi : X \to X$ is a convex-valued, closed-valued, nonempty-valued and and upper hemicontinuous correspondence. Then

$$\exists_{x^* \in X} x^* \in \Psi(x^*)$$

i.e. Ψ has a fixed point.

Outline of Proof:

- If we could find a continuous selection g from Ψ, i.e.

  $$g : X \to X \text{ continuous, } \forall_{a \in X}\ g(a) \in \Psi(a)$$

  we could apply Brouwer: there exists $a^* \in X$ such that $g(a^*) = a^*$, so $a^* \in \Psi(a^*)$ and we would be done. Unfortunately, we cannot in general find such a selection.

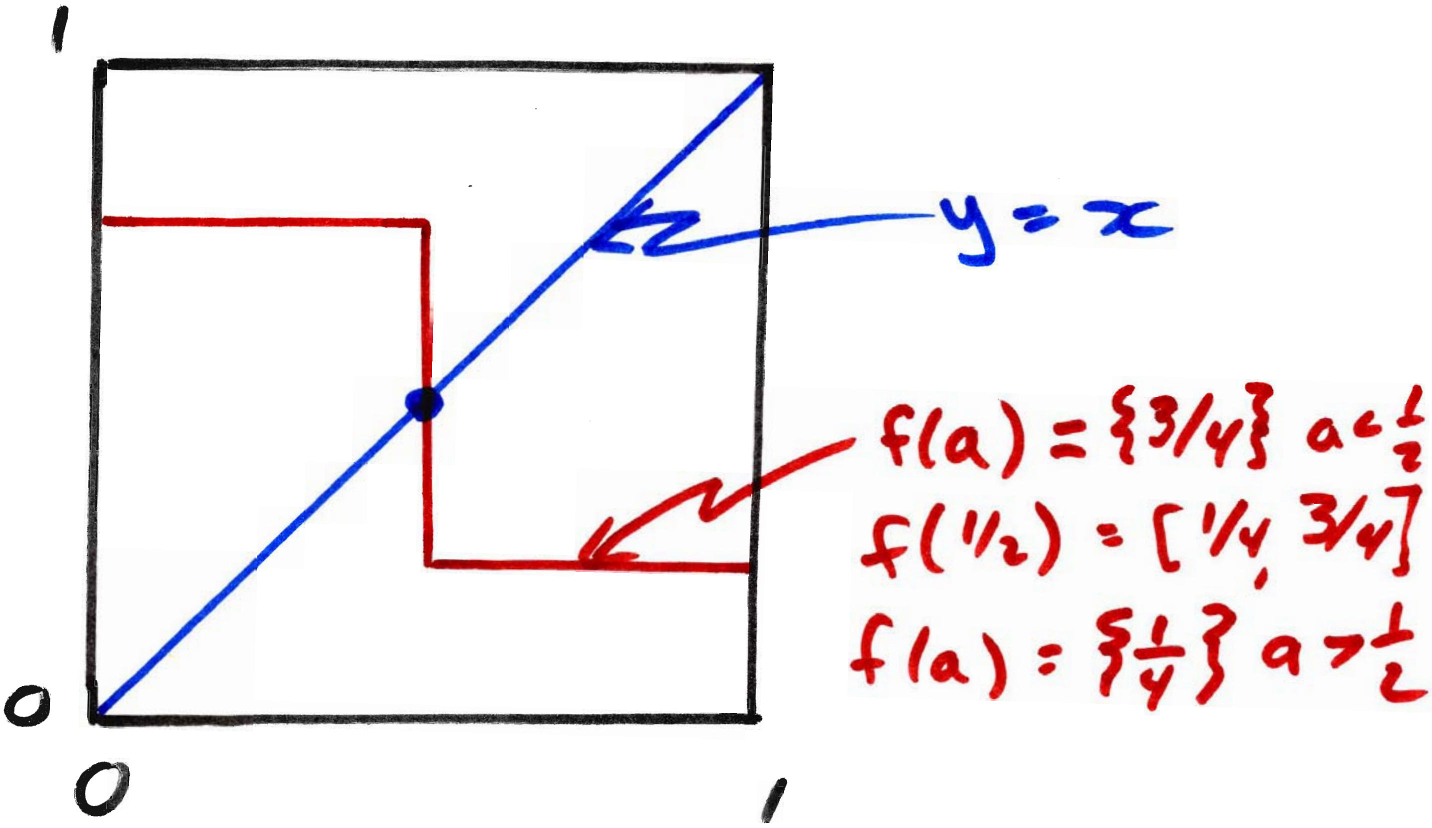
- For each $n \in \mathbf{N}$, find a continuous function g_n whose graph is within $\frac{1}{n}$ of the graph of Ψ.

- By Brouwer's Theorem, we can find a fixed point a_n^* of g_n, so (a_n^*, a_n^*) is in the graph of g_n. Therefore, there exists (x_n, y_n) in the graph of Ψ such that

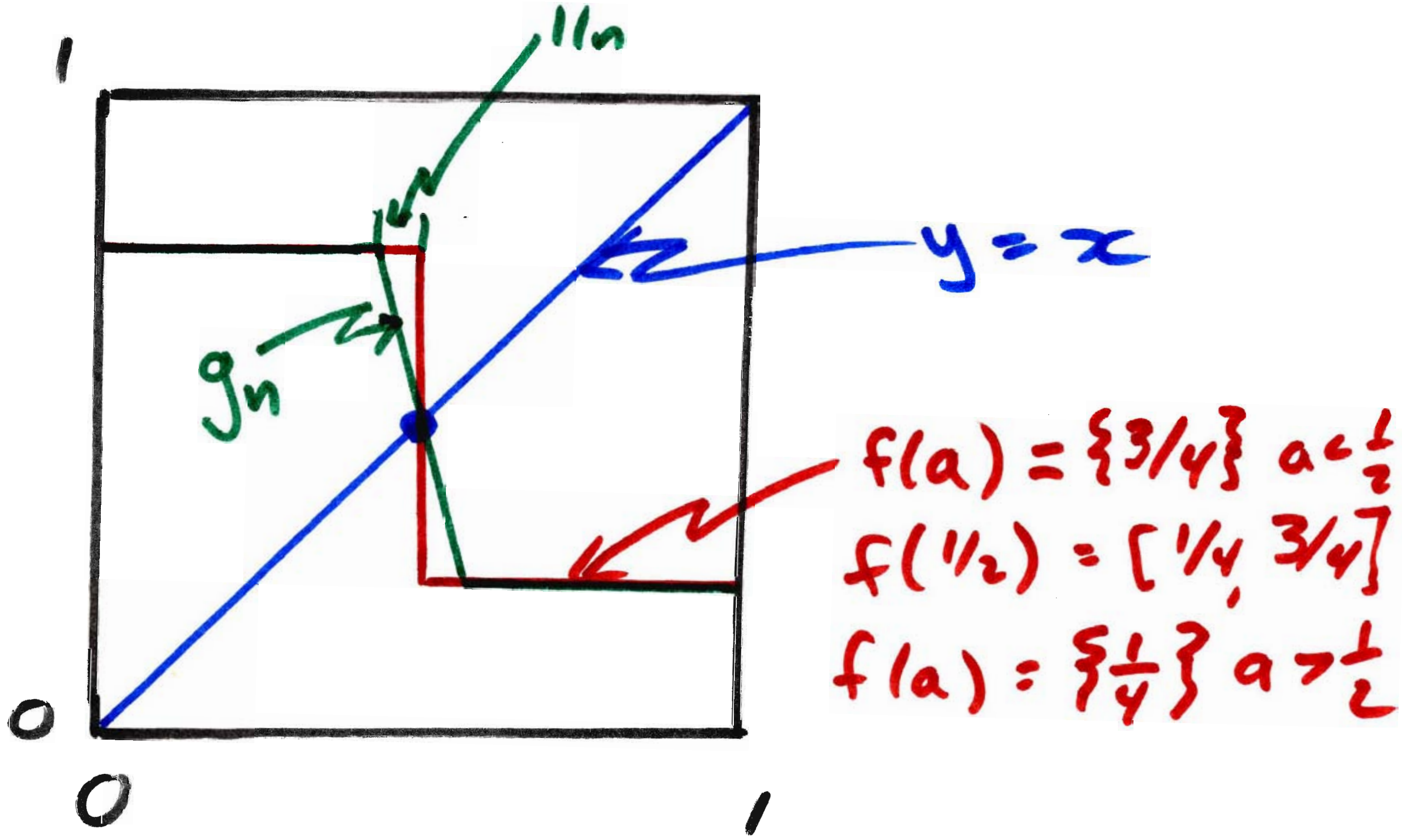
  $$|a_n^* - x_n| < \frac{1}{n},\ |a_n^* - y_n| < \frac{1}{n}$$

- Since X is compact, $\{a_n^*\}$ has a convergent subsequence

  $$a_{n_k}^* \to a^*$$



No continuous selection



No continuous selection

for some $a^* \in X$.

$$\lim_{k\to\infty} x_{n_k} = a^*, \ \lim_{k\to\infty} y_{n_k} = a^*$$

- Since Ψ is closed-valued and upper hemicontinuous, it has closed graph, so

$$(x_{n_k}, y_{n_k}) \to (a^*, a^*)$$

(a^*, a^*) is in the graph of Ψ, so $a^* \in \Psi(a^*)$.

Section 6.1(d): Separating Hyperplane Theorem

The Separating Hyperplane Theorem (also known as Minkowski's Theorem) is used to find establish the existence of prices with specified properties.

Theorem 4 (1.26, Separating Hyperplane Theorem)

Suppose $X, Y \subseteq \mathbf{R}^n$, $X \neq \emptyset \neq Y$, X, Y *convex,* $X \cap Y = \emptyset$. *Then*

$$\begin{aligned} \exists_{p\in\mathbf{R}^n,\, p\neq 0} \sup p \cdot X &= \sup\{p \cdot x : x \in X\} \\ &\leq \inf\{p \cdot y : y \in Y\} \\ &= \inf p \cdot Y \end{aligned}$$

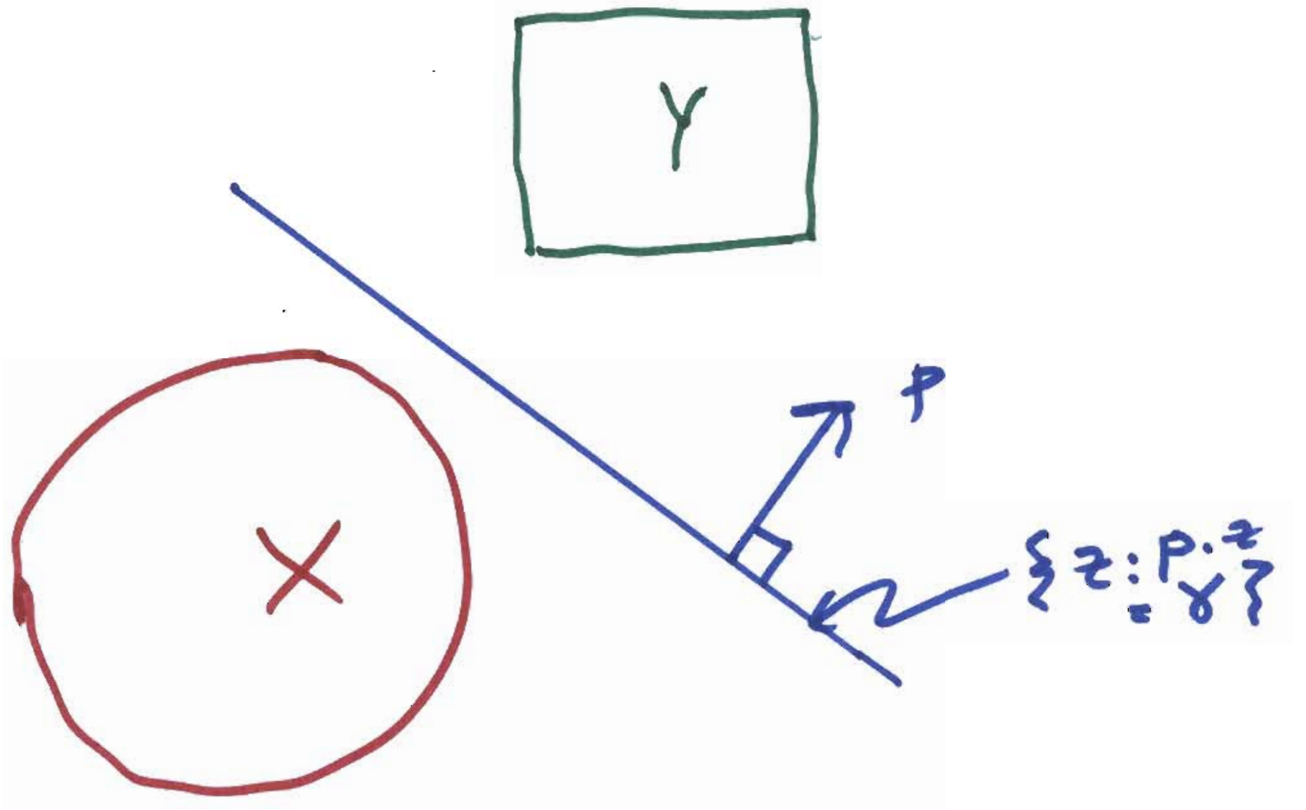
Proof: We sketch the proof in the special case that $X = \{x\}$, Y compact, $x \notin Y$. We will see that we get a stronger conclusion:

$$\exists_{p\in\mathbf{R}^n,\, p\neq 0}\ p \cdot x < \inf p \cdot Y$$

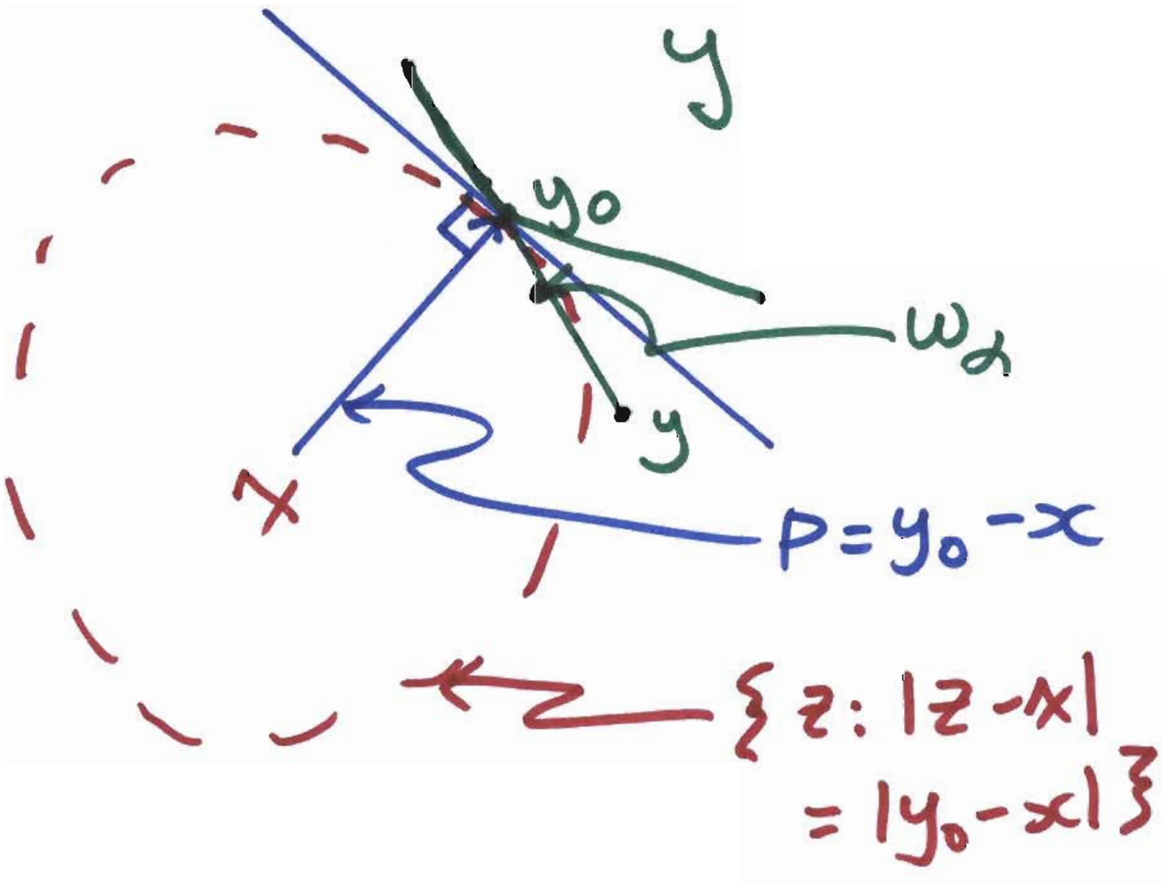
- Choose $y_0 \in Y$ such that $|y_0 - x| = \inf\{|y - x| : y \in Y\}$; such a point exists because Y is compact, so the distance function $g(y) = |y - x|$ assumes its minimum on Y. Since $x \notin Y$, $x \neq y_0$, so $y_0 - x \neq 0$. Let $p = y_0 - x$. The set

$$H = \{z \in \mathbf{R}^n : p \cdot z = p \cdot y_0\}$$

is the hyperplane perpendicular to p through y_0.



Y
X
p
{z : p·z = γ}



y
y₀
w_α
y
x
1
1
p = y₀ − x
{z : |z − x| = |y₀ − x|}

- $$\begin{aligned} p \cdot y_0 &= (y_0 - x) \cdot y_0 \\ &= (y_0 - x) \cdot (y_0 - x + x) \\ &= (y_0 - x) \cdot (y_0 - x) + (y_0 - x) \cdot x \\ &= |y_0 - x|^2 + p \cdot x \\ &> p \cdot x \end{aligned}$$

- We claim that

  $$y \in Y \Rightarrow p \cdot y \geq p \cdot y_0$$

  If not, suppose

  $$y \in Y,\ p \cdot y < p \cdot y_0$$

  Given $\alpha \in (0, 1)$, let

  $$w_\alpha = \alpha y + (1 - \alpha) y_0$$

  Since Y is convex, $w_\alpha \in Y$. We claim that for α sufficiently close to zero,

  $$|w_\alpha - x| < |y_0 - x|$$

  so y_0 is not the closest point in Y to x, contradiction.

  - *Geometric argument:* The hyperplane H is perpendicular to p and goes through y_0; the tangent to the sphere

    $$S = \{z : |z - x| = |y_0 - x|\}$$

    at y_0 is also perpendicular to p, since p is the radius of the sphere. Therefore, H is the tangent to the sphere at y_0. This implies that for α sufficiently close to 0, $|x - w_\alpha| < |y_0 - x|$.

– *Algebraic argument:* If

$$0 < \alpha < \frac{2p \cdot (y_0 - y)}{|y_0 - y|^2}$$

then

$$\begin{aligned}
|x - w_\alpha|^2 &= |x - \alpha y - (1-\alpha) y_0|^2 \\
&= |x - y_0 + \alpha(y_0 - y)|^2 \\
&= |-p + \alpha(y_0 - y)|^2 \\
&= |p|^2 - 2\alpha p \cdot (y_0 - y) + \alpha^2 |y_0 - y|^2 \\
&= |p|^2 + \alpha \left(-2p \cdot (y_0 - y) + \alpha |y_0 - y|^2\right) \\
&< |p|^2 \\
&= |y_0 - x|^2
\end{aligned}$$

■