Economics 204
Lecture 4–Thursday, July 30, 2009
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Section 2.4, Open and Closed Sets

Definition 1 Let (X, d) be a metric space. A set $A \subseteq X$ is *open* if

$$\forall_{x\in A}\exists_{\varepsilon>0} B_\varepsilon(x) \subseteq A$$

A set $C \subseteq X$ is *closed* if $X \setminus C$ is open.

Example: (a, b) is open in the metric space E^1 ($\mathbf{R}$ with the usual Euclidean metric). Given $x \in (a, b)$, $a < x < b$. Let

$$\varepsilon = \min\{x - a, b - x\} > 0$$

Then

$$\begin{aligned} y \in B_\varepsilon(x) &\Rightarrow y \in (x - \varepsilon, x + \varepsilon) \\ &\subseteq (x - (x - a), x + (b - x)) \\ &= (a, b) \end{aligned}$$

so $B_\varepsilon(x) \subseteq (a, b)$, so (a, b) is open.
Notice that ε depends on x; in particular, ε gets smaller as x nears the boundary of the set.
Example: In E^1, $[a, b]$ is closed. $\mathbf{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is a union of two open sets, which must be open … .
Example: In the metric space $[0, 1]$, $[0, 1]$ is open. With $[0, 1]$ as the underlying metric space, $B_\varepsilon(0) = \{x \in [0, 1] : |x - 0| < \varepsilon =$ $[0, \varepsilon)$. *Thus, openness and closedness depend on the underlying metric space as well as on the set.*
Example: Most sets are neither open nor closed. For example, in

E^1, $[0,1] \cup (2,3)$ is neither open nor closed.
Example: An open set may consist of a single point. For example, if $X = \mathbf{N}$ and $d(m,n) = |m-n|$, then $B_{1/2}(1) = \{m \in \mathbf{N} : |m-1| < 1/2\} = \{1\}$.
Example: In any metric space (X,d) both $\emptyset$ and X are open, and both $\emptyset$ and X are closed. To see that $\emptyset$ is open, note that the statement

$$\forall_{x \in \emptyset} \exists_{\varepsilon > 0} \; B_\varepsilon(x) \subseteq \emptyset$$

is vacuously true since there aren't any $x \in \emptyset$. To see that X is open, note that since $B_\varepsilon(x)$ is by definition $\{z \in X : d(z,x) < \varepsilon\}$, it is trivially contained in X. Since $\emptyset$ is open, X is closed; since X is open, $\emptyset$ is closed.
Example: Open balls are open sets. Suppose $y \in B_\varepsilon(x)$. Then $d(x,y) < \varepsilon$. Let $\delta = \varepsilon - d(x,y) > 0$. If $d(z,y) < \delta$, then

$$\begin{aligned} d(z,x) &\leq d(z,y) + d(y,x) \\ &< \delta + d(x,y) \\ &= \varepsilon - d(x,y) + d(x,y) \\ &= \varepsilon \end{aligned}$$

so $B_\delta(y) \subseteq B_\epsilon(x)$, so $B_\varepsilon(x)$ is open.

Theorem 2 (4.2) *Let (X,d) be a metric space. Then*

1. *$\emptyset$ and X are both open, and both closed.*
2. *The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.*
3. *The intersection of a finite collection of open sets is open.*

Proof:

1. We have already done this.

2. Suppose $\{A_\lambda\}_{\lambda\in\Lambda}$ is a collection of open sets.

$$\begin{aligned} x \in \bigcup_{\lambda\in\Lambda} A_\lambda &\Rightarrow \exists_{\lambda_0\in\Lambda}\ x \in A_{\lambda_0} \\ &\Rightarrow \exists_{\varepsilon>0}\ B_\varepsilon(x) \subseteq A_{\lambda_0} \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda \end{aligned}$$

so $\cup_{\lambda\in\Lambda} A_\lambda$ is open.

3. Suppose $A_1, \ldots, A_n \subseteq X$ are open sets. If $x \in \cap_{i=1}^n A_i$, then

$$x \in A_1, x \in A_2, \ldots, x \in A_n$$

so

$$\exists_{\varepsilon_1>0,\ldots,\varepsilon_n>0}\ B_{\varepsilon_1}(x) \subseteq A_1, \ldots, B_{\varepsilon_n}(x) \subseteq A_n$$

Let

$$\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$$

(*Aside: this is where we need the fact that we are taking a finite **intersection. The infimum of an infinite set of positive numbers could be zero.*)
Then

$$B_\varepsilon(x) \subseteq B_{\varepsilon_1}(x) \subseteq A_1, \ldots, B_\varepsilon(x) \subseteq B_{\varepsilon_n}(x) \subseteq A_n$$

so

$$B_\varepsilon(x) \subseteq \bigcap_{i=1}^n A_i$$

which proves that $\cap_{i=1}^n A_i$ is open.

$\blacksquare$

Definition 3 • int A: the *interior* of A, the largest open set contained in A (the union of all open sets contained in A)

- $\bar{A}$: the *closure* of A, the smallest closed set containing A (the intersection of all closed sets containing A)

- $\operatorname{ext} A$: the *exterior* of A, the largest open set contained in $X \setminus A$.
- ∂A: the *boundary* of A, $\overline{(X \setminus A)} \cap \bar{A}$

Theorem 4 (4.13) *A set A in a metric space (X, d) is closed if and only if*

$$\{x_n\} \subset A, \{x_n\} \to x \in X \Rightarrow x \in A$$

Proof: (*This is different from the proof in de la Fuente: he puts the meat of the proof into Theorem 4.12*)
Suppose A is closed. Then $X \setminus A$ is open. Consider a convergent sequence $\{x_n\} \to x \in X$, with $x_n \in A$ for all n. If $x \notin A$, $x \in X \setminus A$, so there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq X \setminus A$. Since $x_n \to x$, there exists $N(\varepsilon)$ such that

$$\begin{aligned} n > N(\varepsilon) \ &\Rightarrow \ x_n \in B_\varepsilon(x) \\ &\Rightarrow \ x_n \in X \setminus A \\ &\Rightarrow \ x_n \notin A \end{aligned}$$

contradiction. Therefore,

$$\{x_n\} \subset A, \{x_n\} \to x \in X \Rightarrow x \in A$$

Conversely, suppose

$$\{x_n\} \subset A, \{x_n\} \to x \in X \Rightarrow x \in A$$

We need to show that A is closed, i.e. $X \setminus A$ is open. Suppose not, so $X \setminus A$ is not open. Then there exists $x \in X \setminus A$ such that for every $\varepsilon > 0$,

$$B_\varepsilon(x) \not\subseteq X \setminus A$$

so there exists $y \in B_\varepsilon(x)$ such that $y \notin X \setminus A$, so $y \in A$ so

$$B_\varepsilon(x) \cap A \neq \emptyset$$

Construct a sequence $\{x_n\}$ as follows: for each n, choose $x_n \in B_{\frac{1}{n}}(x) \cap A$. Given $\varepsilon > 0$, we can find $N(\varepsilon)$ such that $N(\varepsilon) > \frac{1}{\varepsilon}$ by the Archimedean Property, so $n > N(\varepsilon) \Rightarrow \frac{1}{n} < \frac{1}{N(\varepsilon)} < \varepsilon$, so $x_n \to x$. Then $\{x_n\} \subseteq A$, $\{x_n\} \to x$, so $x \in A$, contradiction. Therefore, $X \setminus A$ is open, so A is closed. ■

Section 2.5: Limits of Functions *Read this on your own.* Note that we may have $\lim_{x \to a} f(x) = y$ even though

- f is not defined at a; or
- f is defined at a but $f(a) \neq y$.

The existence and value of the limit depends on values of f *near* a but not *at* a.

Section 2.6: Continuity in Metric Spaces

Definition 5 Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$. f is *continuous at a point* $x_0 \in X$ if

$$\forall_{\varepsilon>0} \exists_{\delta(x_0,\varepsilon)>0} \; d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

f is *continuous* if it is continuous at every element of its domain.

Note: δ depends on x_0, ε. This is a straightforward generalization of the definition of continuity in $\mathbf{R}$. Continuity at x_0 requires:

- $f(x_0)$ is defined; and
- either
  - x_0 is an isolated point of X, i.e. $\exists_{\varepsilon>0} B_\varepsilon(x) = \{x\}$; or
  - $\lim_{x \to x_0} f(x)$ exists and equals $f(x_0)$

(*We will go out of order.*)

**Suppose $f : X \to Y$ and $A \subseteq Y$. Define $f^{-1}(A) = \{x \in X : f(x) \in A\}$.

Theorem 6 (6.14) *Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$. Then f is continuous if and only if*

$$\forall_{A \subseteq Y}\ A \textit{ open in } Y \Rightarrow f^{-1}(A) \textit{ is open in } X$$

Proof: (*We give a direct proof; de la Fuente works via closed sets*)

Suppose f is continuous. Given $A \subseteq Y$, A open, we must show that $f^{-1}(A)$ is open in X. Suppose $x_0 \in f^{-1}(A)$. Let $y_0 = f(x_0) \in A$. Since A is open, we can find $\varepsilon > 0$ such that $B_\varepsilon(y_0) \subseteq A$. Since f is continuous, there exists $\delta > 0$ such that

$$\begin{aligned} d(x, x_0) < \delta &\Rightarrow \rho(f(x), f(x_0)) < \varepsilon \\ &\Rightarrow f(x) \in B_\varepsilon(y_0) \\ &\Rightarrow f(x) \in A \\ &\Rightarrow x \in f^{-1}(A) \end{aligned}$$

so $B_\delta(x_0) \subseteq f^{-1}(A)$, so $f^{-1}(A)$ is open.

Conversely, suppose

$$\forall_{A \subseteq Y}\ A \text{ open in } Y \Rightarrow f^{-1}(A) \text{ is open in } X$$

We need to show that f is continuous. Let $x_0 \in X$, $\varepsilon > 0$. Let $A = B_\varepsilon(f(x_0))$. A is an open ball, hence an open set, so $f^{-1}(A)$ is open in X. $x_0 \in f^{-1}(A)$, so there exists $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(A)$.

$$\begin{aligned} d(x, x_0) < \delta &\Rightarrow x \in B_\delta(x_0) \\ &\Rightarrow x \in f^{-1}(A) \\ &\Rightarrow f(x) \in A \\ &\Rightarrow \rho(f(x), f(x_0)) < \varepsilon \end{aligned}$$

Thus, we have shown that f is continuous at x_0; since x_0 is an arbitrary point in X, f is continuous.■

Theorem 7 (Slightly weaker version of 6.10) *Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

Proof: Suppose $A \subseteq Z$ is open. Since g is continuous, $g^{-1}(A)$ is open in Y; since f is continuous, $f^{-1}(g^{-1}(A))$ is open in X.

We claim that

$$f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$$

Observe

$$\begin{aligned} x \in f^{-1}(g^{-1}(A)) &\Leftrightarrow f(x) \in g^{-1}(A) \\ &\Leftrightarrow g(f(x)) \in A \\ &\Leftrightarrow (g \circ f)(x) \in A \\ &\Leftrightarrow x \in (g \circ f)^{-1}(A) \end{aligned}$$

which establishes the claim. This shows that $(g \circ f)^{-1}(A)$ is open in X, so $g \circ f$ is continuous. ∎

Definition 8 [Uniform Continuity] (*Important*) Suppose $f : (X, d) \to (Y, \rho)$. f is continuous means

$$\forall_{x_0 \in X, \varepsilon > 0} \exists_{\delta(x_0, \varepsilon) > 0} \; d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

f is uniformly continuous means

$$\forall_{\varepsilon > 0} \exists_{\delta(\varepsilon) > 0} \forall_{x_0 \in X} \; d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

Example: Consider

$$f(x) = \frac{1}{x}, \quad x \in (0, 1]$$

f is continuous from Math 1A. We will show that f is not uniformly continuous. Fix $\varepsilon > 0$ and $x_0 \in (0,1]$. If $x = \frac{x_0}{1+\varepsilon x_0}$, then

$$\begin{aligned} 1+\varepsilon x_0 &> 1 \\ x = \frac{x_0}{1+\varepsilon x_0} &< x_0 \\ \frac{1}{x} - \frac{1}{x_0} &> 0 \\ |f(x) - f(x_0)| &= \left|\frac{1}{x} - \frac{1}{x_0}\right| \\ &= \frac{1}{x} - \frac{1}{x_0} \\ &= \frac{1+\varepsilon x_0}{x_0} - \frac{1}{x_0} \\ &= \frac{\varepsilon x_0}{x_0} \\ &= \varepsilon \end{aligned}$$

Thus, $\delta(x_0, \varepsilon)$ must be chosen small enough so that

$$\left|\frac{x_0}{1+\varepsilon x_0} - x_0\right| \geq \delta(x_0, \varepsilon)$$

$$\begin{aligned} \delta(x_0, \varepsilon) &\leq x_0 - \frac{x_0}{1+\varepsilon x_0} \\ &= \frac{\varepsilon (x_0)^2}{1+\varepsilon x_0} \\ &< \varepsilon (x_0)^2 \end{aligned}$$

which converges to zero as $x_0 \to 0$, so there is no $\delta(\varepsilon)$ which will work for all $x_0 \in (0,1]$.

Example: If $f'(x)$ is defined and uniformly bounded on an interval $[a,b]$, then $f(x)$ is uniformly continuous on $[a,b]$. However, even a function with an unbounded derivative may be uniformly continuous. Consider

$$f(x) = \sqrt{x}, \;\; x \in [0, 1]$$

f is continuous from Math 1A. We will show that f is uniformly continuous. Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. Then given any $x_0 \in [0, 1]$, $|x - x_0| < \delta$ implies by the Fundamental Theorem of Calculus

$$\begin{aligned}
|f(x) - f(x_0)| &= \left| \int_{x_0}^{x} \frac{1}{2\sqrt{t}}\, dt \right| \\
&\leq \int_0^{|x-x_0|} \frac{1}{2\sqrt{t}}\, dt \\
&= \sqrt{|x - x_0|} \\
&< \sqrt{\delta} \\
&= \sqrt{\varepsilon^2} \\
&= \varepsilon
\end{aligned}$$

Thus, $f(x)$ is uniformly continuous on $[0, 1]$, even though $f'(x) \to \infty$ as $x \to 0$.

Definition 9 [Lipschitz Functions] Let X, Y be normed vector space, $E \subseteq X$. $f : X \to Y$ is *Lipschitz on E* if

$$\exists_{K>0} \forall_{x,z \in E} \; \|f(x) - f(z)\|_Y \leq K \|x - z\|_X$$

f is *locally Lipschitz* on E if

$$\forall_{x_0 \in E} \exists_{\varepsilon > 0} \; f \text{ is Lipschitz on } B_\varepsilon(x_0) \cap E$$

** *Remark:* De la Fuente only defines Lipschitz and locally Lipschitz in the context of normed vector spaces. The notions make sense in a metric space: Let (X, d) and (Y, ρ) be metric spaces, $E \subseteq X$. $f : X \to Y$ is *Lipschitz on E* if

$$\exists_{K>0} \forall_{x,z \in E} \; \rho(f(x), f(z)) \leq K d(x, z)$$

f is *locally Lipschitz* on E if

$$\forall_{x_0 \in E} \exists_{\varepsilon > 0} \ f \text{ is Lipschitz on } B_\varepsilon(x_0) \cap E$$

However, my sense is that Lipschitz and locally Lipschitz are more useful in the context of normed spaces, because these notions interact with the vector space structure present in the normed space, but absent in the metric space. However, in both the normed vector space and metric space contexts, we have

$$\begin{aligned} \text{locally Lipschitz} &\Rightarrow \text{continuous} \\ \text{Lipschitz} &\Rightarrow \text{uniformly continuous} \end{aligned}$$

A function $f : \mathbf{R}^m \to \mathbf{R}^n$ is said to be C^1 if all its first partial derivatives exist and are continuous. A C^1 function is locally Lipschitz.**

Homeomorphisms:

Definition 10 Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is called a *homeomorphism* if it is one-to-one and continuous, and its inverse function is continuous on $f(X)$.

(*Aside: this is not the standard definition; most texts also require that the function be onto. See the Corrections handout for a correction to Theorem 6.21*)

Now suppose that f is a homeomorphism and $U \subset X$.

$$\begin{aligned} y \in \left(f^{-1}\right)^{-1}(U) &\Leftrightarrow f^{-1}(y) \in U \\ &\Leftrightarrow y \in f(U) \\ U \text{ open in } X &\Rightarrow \left(f^{-1}\right)^{-1}(U) \text{ is open in } (f(X), \rho) \\ &\Rightarrow f(U) \text{ is open in } (f(X), \rho) \end{aligned}$$

This says that X and $\big(f(X), \rho|_{f(X)}\big)$ are identical in terms of properties that can be characterized solely in terms of open sets; such properties are called "topological properties."