Econ 204 Summer 2009
Problem Set 1 Solutions

1. **Cardinality**

   For each pair of set A and set B, show that A and B are numerically equivalent. (Hint: Show that there exists a bijection $f : A \to B$, i.e. f is one to one and onto.)

   (a) $A = (-1, 1)$ $B = (-\infty, +\infty)$

   (b) $A = [0, 1]$ $B = (0, 1)$

   (c) A is an infinite uncountable set, $B = A \cup C$ where C is an infinite countable set.

   Solution:

   (a) $f(x) = \tan \frac{\pi}{2} x$, $x \in (-1, 1)$

   (b) $f(x) = \begin{cases} \frac{1}{2} & if\ x = 0 \\ \frac{1}{n+2} & if\ x = \frac{1}{n}, n = 1, 2, ...,, \ x \in [0,1] \\ x & otherwise \end{cases}$

   (c) Since A is an infinite set, we can obtain an infiite sequence $\{a_1, a_2, \ldots\}$ from A. Let $A_1 = \{a_1, a_2, \ldots\}$. $A_1 \subseteq A$ and $A \backslash A_1 \neq \emptyset$ as A is uncountablė.
   There are three cases:
   Case 1: $C \cap A_1 = \emptyset$
   Since C is an infinite countable set, let $C = \{c_1, c_2, \ldots\}$.

   $$f(x) = \begin{cases} a_i & if\ x = a_{2i}, i = 1, 2, ..., \\ c_i & if\ x = a_{2i-1}, i = 1, 2, ..., \\ x & if\ x \in A \backslash A_1 \end{cases}, \ x \in A$$

   Case 2: $C \cap A_1 \neq \emptyset$ and $C \backslash A_1$ is a finite set.
   Let $C \backslash A_1 = \{k_1, k_2, \ldots, k_m\}$ where m is a natural number.

   $$f(x) = \begin{cases} a_i & if\ x = a_{i+m}, i = 1, 2, ..., \\ k_i & if\ x = a_i, i = 1, 2, ..., m \\ x & if\ x \in A \backslash A_1 \end{cases}, \ x \in A$$

   Case 3: $C \cap A_1 \neq \emptyset$ and $C \backslash A_1$ is a infinite countable set.
   Let $C \backslash A_1 = \{s_1, s_2, \ldots\}$

   $$f(x) = \begin{cases} a_i & if\ x = a_{2i}, i = 1, 2, ... \\ s_i & if\ x = a_{2i-1}, i = 1, 2, ... \\ x & if\ x \in A \backslash A_1 \end{cases}, \ x \in A$$

2. **Induction**

   Using mathematical induction, show the following: $n = 1, 2, 3, \ldots$

   (a) $\sum_{i=1}^{n} k^{-i} = \frac{1-\frac{1}{k^n}}{k-1}$, $k \neq 1$.

   (b) $\sum_{i=n}^{\infty} (k-1) k^{-i} = k^{1-n}$, $k > 1$.

   (c) $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$

   Solution:

   (a) For $n = 1$, $\frac{1}{k} = \frac{1-\frac{1}{k}}{k-1}$.

   Suppose for $n = m$, $\sum_{i=1}^{m} k^{-i} = \frac{1-\frac{1}{k^m}}{k-1}$ holds.

   For $n = m+1$, $\sum_{i=1}^{m+1} k^{-i} = \sum_{i=1}^{m} k^{-i} + \frac{1}{k^{m+1}} = \frac{1-\frac{1}{k^m}}{k-1} + \frac{k-1}{k^{m+1}(k-1)} = \frac{1-\frac{1}{k^m}}{k-1} + \frac{\frac{1}{k^m} - \frac{1}{k^{m+1}}}{k-1} = \frac{1-\frac{1}{k^{m+1}}}{k-1}$.

(b) For $n=1$, $\sum_{i=1}^{\infty}(k-1)k^{-i} = (k-1)\cdot\sum_{i=1}^{\infty}k^{-i} = (k-1)\cdot\lim_{n\to\infty}\frac{1-\frac{1}{k^n}}{k-1} = 1$

Suppose for $n=m$, $\sum_{i=m}^{\infty}(k-1)k^{-i} = k^{1-m}$ holds.

For $n=m+1$, $\sum_{i=m+1}^{\infty}(k-1)k^{-i} = \sum_{i=m}^{\infty}(k-1)k^{-i} - (k-1)k^{-m} = \frac{k-(k-1)}{k^m} = k^{(1-(m+1))}$.

(c) .For $n=1$, $\frac{1}{\sqrt{1}} \geq \sqrt{1}$.

Suppose for $n=m$, $\sum_{i=1}^{m}\frac{1}{\sqrt{i}} \geq \sqrt{m}$ holds.

For $n=m+1$, $\sum_{i=1}^{m+1}\frac{1}{\sqrt{i}} = \sum_{i=1}^{m}\frac{1}{\sqrt{i}} + \frac{1}{\sqrt{m+1}} \geq \sqrt{m} + \frac{1}{\sqrt{m+1}}$.

Since $\sqrt{m+1}-\sqrt{m} = \frac{1}{\sqrt{m+1}+\sqrt{m}} \leq \frac{1}{\sqrt{m+1}}$, we have $\sum_{i=1}^{m+1}\frac{1}{\sqrt{i}} \geq \sqrt{m+1}$.

3. **Bijection**

Suppose $f: X \to Y$ is a bijection, i.e. f is one to one and onto. Show that for any $A, B \subset X$, $f(A\cap B) = f(A)\cap f(B)$.

Solution:

For any $A, B \subset X$, if $y \in f(A\cap B)$,then there exists $x \in A\cap B$ such that $f(x) = y$, so $y \in f(A)\cap f(B)$. Hence $f(A\cap B) \subset f(A)\cap f(B)$

If $y \in f(A)\cap f(B)$,then there exists $a \in A$, $b \in B$ such that $f(a) = f(b) = y$. Since f is one to one, $a = b$, so $y \in f(A\cap B)$. Hence $f(A)\cap f(B) \subset f(A\cap B)$. So we have $f(A)\cap f(B) = f(A\cap B)$.

4. **Supremum Property and Completeness Axiom**

Use the Completeness Axiom to prove that every nonempty set of real numbers which is bounded below has an infimum.

Solution:

Assume the Completeness Axiom. Let $X \subset \mathbf{R}$ be a nonempty set which is bounded below. Let U be the set of all lower bounds for X. Since X is bounded below, $U \neq \varnothing$. If $x \in X$ and $u \in U$, $x \geq u$ since u is a lower bound for X. So for any $x \in X$, $u \in U$, $x \geq u$. By the Completeness Axiom, there exists $\alpha \in \mathbf{R}$, for any $x \in X$, $u \in U$, $x \geq \alpha \geq u$. Hence α is a lower bound for X, and it is larger than or equal to every other lower bound for X, so it is the largest lower bound for X, so $\inf X = \alpha \in \mathbf{R}$.

5. **Limit of Decreasing Sequence**

Show that every decreasing sequence of real numbers that is bounded below converges to its infimum. (Hint: you can directly use the result of question 4)

Solution:

Suppose $\{x_n\}$ is a decreasing sequence of real numbers and assume it is bounded below. By the supremum property, $\{x_n\}$ has a infimum that is denoted as y. For some $\varepsilon > 0$, by the definition of infimum, $x_n \geq y$ for all n and $y+\varepsilon$ is not an lower bound of $\{x_n\}$, so there exists some $N(\varepsilon) \in N$ such that $x_{N(\varepsilon)} < y+\varepsilon$. Since $\{x_n\}$ is decreasing, we have $x_n < y+\varepsilon$ for all $n > N(\varepsilon)$ and $x_n \geq y$ for all n. Since ε is arbitrary, $\{x_n\} \to y$.

6. **Metric Space**

(a) $\rho(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$, prove whether or not it is a metric on $\mathbf{R}^n$.

(b) $\rho(x,y) = \sum_{i=1}^{n}|x_i - y_i|$, prove whether or not it is a metric on $\mathbf{R}^n$.

(c) Suppose (S_1, d_1) and (S_2, d_2) are metric spaces. Show that $(S_1 \times S_2, \rho)$ is a metric space, where $\rho((x_1,x_2),(y_1,y_2)) = max\{d_1(x_1,y_1), d_2(x_2,y_2)\}$ for all $x_1, y_1 \in S_1$ and all $x_2, y_2 \in S_2$.

Solution:

(a) To verify that d is a metric, we need to check that

(i) $\rho(x,x) = 0\ \forall x$ (ii) $\rho(x,y) = \rho(y,x)\ \forall x,y,$ and (iii) $\rho(x,y) + \rho(y,z) \leq \rho(x,z)\ \forall x,y,z.$

(i) and (ii) are easily verified. To verify (iii) there are essentially two cases to consider: $x = z$ or $x \neq z$.

Case I: Take $x \neq z$. Then, either $x \neq y$ *or* $y \neq z \Rightarrow \rho(x,y) + \rho(y,z) \geq 1 = \rho(x,z)$.

Case II: Take $x = z$. Then, $\rho(x,y) + \rho(y,z) \geq 0 = \rho(x,z)$.

It follows that (iii) holds and ρ is a metric.

(b) We need to check that

(i) $\rho(x,x) = 0\ \forall x$ (ii) $\rho(x,y) = \rho(y,x)\ \forall x,y,$ and (iii) $\rho(x,y) + \rho(y,z) \leq \rho(x,z)\ \forall x,y,z.$ (i) and (ii) are easily verified. To verify (iii)

$$\rho(x,y)+\rho(y,z) = \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| = \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) \geq \sum_{i=1}^{n} |x_i - z_i| = \rho(x,z)$$

since $|x_i - y_i| + |y_i - z_i| \geq |x_i - y_i + y_i - z_i| = |x_i - z_i|$, (iii) holds and ρ is a metric.

(c) We need to check that

(i) $\rho\left((x_1,x_2),(x_1,x_2)\right) = 0\ \forall\,(x_1,x_2) \in S_1 \times S_2$

(ii)$\rho\left((x_1,x_2),(y_1,y_2)\right) = \rho\left((y_1,y_2),(x_1,x_2)\right)\ \ \forall\,(x_1,x_2),(y_1,y_2) \in S_1 \times S_2$

(iii) $\rho((x_1,x_2),(y_1,y_2))+\rho((y_1,y_2),(z_1,z_2)) \geq \rho((x_1,x_2),(z_1,z_2))\ \forall\,(x_1,x_2),(y_1,y_2),(z_1,z_2) \in S_1 \times S_2$.

(i) and (ii) are easily verified. Our job is to verify (iii):

Since $d_i\left(x_i,y_i\right)$ is a well-defined metric, for $i = 1,2$, we must have $d_i\left(x_i,z_i\right) \leq d_i\left(x_i,y_i\right) + d_i\left(y_i,z_i\right)$ for any $x_i, y_i, z_i \in S_i$.

Then

$$\begin{aligned}
\rho((x_1,x_2),(z_1,z_2)) &= \max\left\{d_1\left(x_1,z_1\right), d_2\left(x_2,z_2\right)\right\} \\
&\leq \max\left\{d_1\left(x_1,y_1\right) + d_1\left(y_1,z_1\right), d_2\left(x_2,y_2\right) + d_2\left(y_2,z_2\right)\right\} \\
&\leq \max\left\{d_1\left(x_1,y_1\right), d_2\left(x_2,y_2\right)\right\} + \max\left\{d_1\left(y_1,z_1\right), d_2\left(y_2,z_2\right)\right\} * \\
&= \rho((x_1,x_2),(y_1,y_2)) + \rho((y_1,y_2),(z_1,z_2))
\end{aligned}$$

* To prove this inequality is equal to show that $\max\left\{a+b, c+d\right\} \leq \max\{a,c\} + \max\{b,d\}$. WLOG, suppose that $a+b \geq c+d$.Hence $\max\left\{a+b, c+d\right\} = a+b$. Since $a \leq \max\{a,c\}$, $b \leq \max\{b,d\}$, $a+b \leq \max\{a,c\} + \max\{b,d\}$. Thus $\max\left\{a+b, c+d\right\} \leq \max\{a,c\} + \max\{b,d\}$.