Econ 204 Summer 2009
Problem Set 2 Solutions

1. **Boundary, Exterior and Closure**

   Find the boundary, exterior, and closure of the following sets:

   (a) $\{(x,y) \in \mathbf{R}^2 | x^2 + y^2 > 1\}$

   (b) $\{(x,y) \in \mathbf{R}^2 | x - y = 3\}$

   Solution:

   (a) Boundary $\{(x,y) \in \mathbf{R}^2 | x^2 + y^2 = 1\}$. Exterior $\{(x,y) \in \mathbf{R}^2 | x^2 + y^2 < 1\}$. Closure $\{(x,y) \in \mathbf{R}^2 | x^2 + y^2 \geq 1\}$.

   (b) Boundary $\{(x,y) \in \mathbf{R}^2 | x - y = 3\}$. Exterior $\{(x,y) \in \mathbf{R}^2 | x - y \neq 3\}$ .Closure $\{(x,y) \in \mathbf{R}^2 | x - y = 3\}$.

2. **Closed Set**

   Show that $E = \{x \in \mathbf{R}^1 : | x - a | \leq 2\}$ is a closed set where a is a real number.

   Solution:

   We prove that E^c is open. Then E is closed by the definition. Consider $x \in E^c$, where $E^c = \{x \in \mathbf{R}^1 : | x - a | > 2\}$. There are only two cases, $x > a + 2$ and $x < a - 2$. If $x > a + 2$, then there exists $\varepsilon = \frac{x-a-2}{2} > 0$ such that $x - \varepsilon = \frac{x+a+2}{2} > a + 2$ and $x + \varepsilon > x > a + 2$. So $B_\varepsilon(x) \subseteq E^c$. If $x < a - 2$, then there exists $\varepsilon = \frac{a-2-x}{2} > 0$ such that $x + \varepsilon = \frac{a-2+x}{2} < a - 2$ and $x - \varepsilon < x < a - 2$. So $B_\varepsilon(x) \subseteq E^c$. Hence $\forall x \in E^c$, there exists ε such that $B_\varepsilon(x) \subseteq E^c \Rightarrow E^c$ is open. So E is closed.

3. **Intersection of Closed Sets**

   Suppose $\{A_k\}$ is a sequence of non-empty closed sets on $\mathbf{R}^n$ such that $A_1 \supset A_2 \supset A_3 ... \supset A_k \supset$ ...Show that if A_m is bounded for some m, then $\cap_{k=1}^{\infty} A_k \neq \varnothing$.

   Solution:

   Choose any $x_k \in A_k$, $k = 1, 2, ...,$

   Since A_m is bounded, $\{x_k\}$ is bounded. By Bolzano-Weierstrass theorem, there exists x_0 and a subsequence$\{x_{k_i}\}$ of $\{x_k\}$ such that $\{x_{k_i}\} \to x_0$. For every k, if $k_i > k$, $x_{k_i} \in A_{k_i} \subset A_k$. Hence $x_0 = \lim_{i\to\infty} x_{k_i} \in A_k$.Since A_k is closed, $x_0 \in \cap_{k=1}^{\infty} A_k$.

4. **Uniform Continuity in Euclidean Metric Space**

   $(\mathbf{R}^n, d)$ is the n-dimentional Euclidean metric space. Suppose $E \subset \mathbf{R}^n$ is a nonempty set. Define $d(x, E) = \inf\{d(x,y) : y \in E\}$

   (a) Show that E is a closed set if and only if for any $x \in \mathbf{R}^n$, there exists $y \in E$, such that $d(x,y) = d(x,E)$.

   (b) Define function $f : \mathbf{R}^n \to \mathbf{R}^+$ as $f(x) = d(x,E)$. Show that $f(x)$ is uniformly continuous.

   Solution:

   (a) $\Rightarrow$: Suppose E is a closed set. Given $x \in R^n$, since $d(x,E) = \inf_{y\in E} d(x,y)$, $d(x,E) + \frac{1}{n}$ is not a lower bound for $\{d(x,y) : y \in E\}$. Therefore we can find $y_n \in E$ such that $d(x, y_n) < d(x,E) + \frac{1}{n}$and $d(x, y_n) \geq d(x,E)$. Hence $\lim_{n\to\infty} d(x, y_n) = d(x,E)$. So $\{y_n\}$ is bounded and by Bolzano-Weierstrass theorem there exists a subsequence $\{y_{n_i}\} \to y$. $y \in E$ as E is closed. So there exists $y \in E$ $d(x,y) = \lim_{i\to\infty} d(x, y_{n_i}) = d(x,E)$.

   $\Leftarrow$: Consider a convergent sequence $\{x_n\} \subset E$, $\lim_{n\to\infty} x_n = x_0$. There exists $y \in E$ such that $d(x_0, y) = d(x_0, E)$. Since x_0 is a cluster point of E, $d(x_0, E) = 0$, so $x_0 = y \in E$. Therefore $x_0 \in E$, E is a closed set.

(b) Suppose $x, y \in R^n$. Given any $\theta > 0$, there exists $z \in E$, such that $d(x, z) < d(x, E) + \theta$. Hence $f(y) = d(y, E) \leq d(y, z) \leq d(y, x) + d(x, z) < d(x, y) + d(x, E) + \theta = d(x, y) + f(x) + \theta$. Since θ is arbitrary, $f(y) - f(x) \leq d(x, y)$. Similarly we can get $f(x) - f(y) \leq d(x, y)$. Hence $\mid f(y) - f(x) \mid \leq d(x, y)$. So $\forall \varepsilon > 0$ there exists $\delta(\varepsilon) = \varepsilon$ such that $\forall y \in R^n$ $d(x, y) < \delta(\varepsilon) = \varepsilon \Rightarrow \rho(x, y) = \mid f(y) - f(x) \mid \leq d(x, y) < \varepsilon$, which shows that $f(x)$ is uniformly continuous.

5. **Continuous Function in Euclidean Metric Space**

$(\mathbf{R}^n, d)$ is the n-dimentional Euclidean metric space. $f : \mathbf{R}^n \to \mathbf{R}^1$ is a function. Show that f is countinuous if and only if for every $c \in \mathbf{R}^1$, A_c and B_c are closed sets where $A_c = \{x \in \mathbf{R}^n : f(x) \geq c\}$ and $B_c = \{x \in \mathbf{R}^n : f(x) \leq c\}$.

Solution:

$\Rightarrow$: Consider a convergent sequence $\{x_n\} \subset A_c$, $\lim_{n\to\infty} x_n = x_0$. Since $f(x_n) \geq c$ and $f(x)$ is continuous, $f(x_0) = \lim_{n\to\infty} f(x_n) \geq c$. So $x_0 \in A_c$, A_c is a closed set. Similarly we can show that B_c is closed.

$\Leftarrow$: If $f(x)$ is not continuous at x_0, then there exists $\varepsilon_0 > 0$ and $\{x_n\} \to x_0$ such that $f(x_n) \geq f(x_0) + \varepsilon_0$ or $f(x_n) \leq f(x_0) - \varepsilon_0$. If $f(x_k) \geq f(x_0) + \varepsilon_0$, let $c = f(x_0) + \epsilon_0$,then $\{x_n\} \subset A_c$,$x_0 \notin A_c$, contradiction with that A_c is closed. If $f(x_n) \leq f(x_0) - \varepsilon_0$, let $c = f(x_0) - \epsilon_0$, then contradition with that B_c is closed.

6. **Lipschitz Equivalent**

Theorem 10.8 on page 107 of de la Fuente says that all norms on $\mathbf{R}^n$ are Lipschitz-equivalent to the Euclidean norm. The Theorem is correct, but is the proof correct?

(a) Suppose $\parallel \cdot \parallel : (\mathbf{R}^n, d) \to (\mathbf{R}_+, \rho)$ is a norm on $\mathbf{R}^n$. d is the metric generated by the norm, $d(x, y) = \parallel x - y \parallel$. ρ is the Euclidean metric. Show that $\parallel \cdot \parallel$ is a continuous function. (Hint: Use the triangle inequality.)

(b) Now consider the Euclidean norm $\parallel \cdot \parallel_E : \mathbf{R}^n \to \mathbf{R}_+$. The unit circle on $\mathbf{R}^n$ is defined as $C = \{x \in \mathbf{R}^n : \parallel x \parallel_E = 1\}$. Show that C is compact. (Hint: Show that C is closed and bounded.)

(c) Can we use the result of part a and the extreme-value theorem to prove that that $\parallel \cdot \parallel$ attains a minimum and a maximum in the set C definded in part b?

Solution:

(a) By $d(\cdot, \cdot) \geq 0$ and triangle inequality we have $\forall x, y, z \in \mathbf{R}^n$, $d(x, z) \leq d(x, y) + d(z, y)$ $\Rightarrow d(x, y) \geq d(x, z) - d(z, y)$ and $d(z, y) \leq d(x, z) + d(x, y) \Rightarrow d(x, y) \geq d(z, y) - d(x, z)$. Hence $d(x, y) \geq \mid d(x, z) - d(z, y) \mid$.

Suppose $\parallel \cdot \parallel$ is a norm in a $\mathbf{R}^n$. Then $\parallel x - y \parallel$ is a valid metric on $\mathbf{R}^n$. Take any pair $x, x_0 \in \mathbf{R}^n$. We have

$|\parallel x - 0 \parallel - \parallel x_0 - 0 \parallel| \leq \parallel x - x_0 \parallel \Rightarrow |\parallel x \parallel - \parallel x_0 \parallel| \leq \parallel x - x_0 \parallel$. For any $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$. If $\parallel x - x_0 \parallel < \delta$, then $|\parallel x \parallel - \parallel x_0 \parallel| \leq \delta = \frac{\varepsilon}{2} < \varepsilon$. Therefore $\parallel \cdot \parallel$ is a continuous function.

(b) For any convergent sequence $\{x_n\} \in C$, $\{x_n\} \to x$, suppose $\parallel x \parallel_E > 1$, then by continuity of $\parallel \cdot \parallel_E$, there exists $m \in N$ such that $\parallel x_n \parallel_E > 1$ for all $n > m$ which is contradiction with that $\{x_n\} \in C$. Similarly we can show that $\parallel x \parallel_E < 1$ is contradiction with $\{x_n\} \in C$. Hence $\parallel x \parallel_E = 1$, $x \in C$. So C is closed. C is bounded by definition. Therefore C is compact.

(c) No. What the extreme value theorem tells us is that

Let C be a compact set in a metric space (X, d) and $f : (C, d) \to \mathbf{E}^1$ a continuous function. Then f attains both its maximum and its minimum in the set. The compact set C and the continuous function f should be with the same metric.

In part a we show that $\parallel \cdot \parallel$ is a continuous function in metric space $(\mathbf{R}^n, d)$. The metric d is generated by a norm which is not Euclidean. In part b we show that C is a compact set in the Euclidean metric space. As d is not Euclidean metric, we can not use the extreme-value theorem.