Econ 204 Summer 2009
Problem Set 3
Due in Lecture Friday August 7

1. **Cauchy Sequence**

   Suppose $\{x_n\} \in \mathbf{R}^n$ is a Cauchy sequence. It has a subsequence $\{x_{n_k}\}$ such that $\lim_{n_k\to\infty} x_{n_k} = x$. Show that $\lim_{n\to\infty} x_n = x$.

   Solution:

   Consider a Cauchy sequence $\{x_n\} \in \mathbf{R}^n$. For any $\varepsilon > 0$,there exists an $N(\varepsilon)$ such that for any $m,\, n > 0$, $\left|x_m - x_n\right| < \frac{\varepsilon}{2}$. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $\lim_{n_k\to\infty} x_{n_k} = x$, then there exists a $K(\varepsilon)$ such that $k > K(\varepsilon) \Rightarrow \left|x_{n_k} - x\right| < \frac{\varepsilon}{2}$. Choose $M > max\{n_{K(\varepsilon)}, N(\varepsilon)\}$. For every $n > M$, we can find a $n_k > M$. By triangle inequality, $\left|x_n - x\right| \le \left|x_n - x_{n_k}\right| + \left|x_{n_k} - x\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence $\lim_{n\to\infty} x_n = x$.

2. **Compactness**

   Use the open cover definition of compactness to show that the subset $\left\{\frac{n}{n^2+1},\, n = 0, 1, 2 \ldots\right\}$ of $\mathbf{R}$ is compact.

   Solution:

   Denote $A = \left\{\frac{n}{n^2+1},\, n = 0, 1, 2 \ldots\right\}$. Let the collection of sets $\left\{U_\lambda\right\}_{\lambda\in\Lambda}$ be an open cover. Since $0 \in A$, there exists an open set $U_{\lambda_0} \in \left\{U_\lambda\right\}$ such that $0 \in U_{\lambda_0}$. Hence we can find an $\varepsilon > 0$ such that $B_\varepsilon(0) \subset U_{\lambda_0}$. Note that there are only finitely many points of K not included in U_{λ_0} which are those with $\frac{n}{n^2+1} > \varepsilon$. Denote them as $a_1, ..., a_n$ and for each a_j choose a U_{λ_j} from $\left\{U_\lambda\right\}_{\lambda\in\Lambda}$such that $a_j \in U_{\lambda_j}$. Then $\{U_{\lambda_0}, ..., U_{\lambda_n}\}$ is a finite subcover by construction. So A is compact.

3. **Completeness**

   a. Show that $(0, 1)$ is not complete in the Euclidean metric space.

   Solution:

   By Theorem 9 in Lecture 5, $\mathbf{R}$ is complete with Euclidean metric. $\forall\, A \subseteq \mathbf{R}$, by Theorem 11 of Lecture 5, A is complete if and only if A is a closed subset of $\mathbf{R}$. So we only have to show that $(0, 1)$ is not closed. Consider a sequence $\{x_n\}$, $x_n = \frac{1}{n+1}$, $\{x_n\} \subseteq (0, 1)$, $\lim_{n\to\infty} x_n = 0 \notin (0, 1)$. So $(0, 1)$ is not closed. Therefore $(0, 1)$ is not complete in the Euclidean metric space.

   b. Show that $\{\frac{n\sqrt{2}}{m},\, m \neq 0,\, n, m \in N\}$ is not complete in the Euclidean metric space.

   Solution:

   By Theorem 9 in Lecture 5, $\mathbf{R}$ is complete with Euclidean metric. $\forall\, A \subseteq \mathbf{R}$, by Theorem 11 of Lecture 5, A is complete if and only if A is a closed subset of $\mathbf{R}$. So we only have to show that $\{\frac{n\sqrt{2}}{m},\, m \neq 0,\, n, m \in N\}$ is not closed in the Euclidean metric space. Consider a sequence $\{x_n\}$, x_n is the decimal expansion of $\sqrt{2}$ up to the n'th decimal place, i.e $x_1 = 1.4$, $x_2 = 1.41$, $x_3 = 1.414, ...$,So x_n is rational number. Hence $y_n = x_n\sqrt{2} \in \{\frac{n\sqrt{2}}{m},\, m \neq 0,\, n, m \in N\}$. But but $x_n \to \sqrt{2} \Rightarrow y_n \to 2 \notin \{\frac{n\sqrt{2}}{m},\, m \neq 0,\, n, m \in N\}$. So $\{\frac{n\sqrt{2}}{m},\, m \neq 0,\, n, m \in N\}$ is not closed. Thus it's not complete.

4. **Completeness and Compactness**

   $(\mathbf{R}, d)$ is a metric space where d is defined as follows:

$$d(x, y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$$

(a) Show that $(\mathbf{R}, d)$ is complete.

Solution:

To see completeness, note that the only Cauchy sequences are those that are constant after some (finite) point. Otherwise, for every $x_m \neq x_n \Rightarrow d(x_m, x_n) = 1$. As constant sequences trivially converge to their constant value, it follows that the given metric space is complete.

(b) Is $(\mathbf{R}, d)$ bounded? Is $(\mathbf{R}, d)$ compact? Prove your answer.

Solution:

$(\mathbf{R}, d)$ is bounded but not compact. By the definition of d, $(\mathbf{R}, d)$ is a bounded (and complete) metric space, but it is not compact. Since every point is open under the topology induced by this metric, it follows that the cover $\{x_\alpha\}_{\alpha\in\mathbf{R}}$, whose elements are singletons, has no finite sub-cover.

5. **Continuous Function**

Let $f : X \to Y$ be a continuous function (X and Y are metric spaces).

a. Prove that if X is compact then $f(X)$, the image of f, is compact.

Solution:

Let $\bigcup_{\alpha\in\Lambda} U_\alpha$ be an open cover of $f(X)$. Continuity implies that $f^{-1}(U_\alpha)$ is open. By compactness of X, $\exists$ indices $\alpha_1, \alpha_2, ..., \alpha_n$ such that $X = f^{-1}(U_{\alpha_1}) \cup f^{-1}(U_{\alpha_2}) \cup ... \cup f^{-1}(U_{\alpha_n}) \Rightarrow f(X) \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup ... \cup U_{\alpha_n}$. So $\{U_{\alpha_1}, ..., U_{\alpha_n}\}$ is a finite subcover of $f(X)$. Therefore, $f(X)$ is compact.

b. Prove that if X is connected then $f(X)$, the image of f, is connected.

Solution:

We prove the contrapositive: if $f(C)$ is not connected, then C is not connected. Suppose $f(C)$ is not connected. Find $P \neq \emptyset \neq Q$, $f(C) = P \cup Q$, $\overline{P} \cap Q = P \cap \overline{Q} = \emptyset$. Let $A = f^{-1}(P) \cap C$, $B = f^{-1}(Q) \cap C$. So $A \neq \emptyset$ and $B \neq \emptyset$. Then we have $A \cup B = (f^{-1}(P) \cap C) \cup (f^{-1}(Q) \cap C) = (f^{-1}(P) \cup f^{-1}(Q)) \cap C = f^{-1}(f(C)) \cap C = C$. Since $A = f^{-1}(P) \cap C \subseteq f^{-1}(P) \subseteq f^{-1}(\overline{P})$ and $f^{-1}(\overline{P})$ is closed. Hence $\overline{A} \subseteq f^{-1}(\overline{P})$. Similarly, $\overline{B} \subseteq f^{-1}(\overline{Q})$. So $\overline{A} \cap B \subseteq f^{-1}(\overline{P}) \cap f^{-1}(Q) = f^{-1}(\overline{P} \cap Q) = f^{-1}(\emptyset) = \emptyset$. $A \cap \overline{B} \subseteq f^{-1}(P) \cap f^{-1}(\overline{Q}) = f^{-1}(P \cap \overline{Q}) = f^{-1}(\emptyset) = \emptyset$. So C is not connected.

6. **Upper Hemicontinuous**

Let $F : C \times \mathbf{R}^p \to \mathbf{R}^1$ be a continuous function, where $C \subseteq \mathbf{R}^1$. Let $\Psi(\omega) = \{x \in \mathbf{R}^n : F(x, \omega) = 0\}$ be a correspondence. Show directly from the definition that if C is compact, then Ψ is an upper hemicontinuous correspondence. (Hint: The proof is by contradiction. Suppose that Ψ is not upper hemicontinuous at some ω_0; this tells you that there is a sequence $\omega_n \to \omega_0$ with certain properties.)

Solution:

Suppose that Ψ is not an upper hemicontinuous at some ω_0. Then there exists an open set V such that $V \supseteq \Psi(\omega_0)$ and we can find a sequence $\omega_n \to \omega_0$ and $x_n \in \Psi(\omega_n)$ such that $x_n \notin V$. Since C is compact, so we can find a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{n_k\to\infty} x_{n_k} = x'$.Note that $\lim_{n_k\to\infty} \omega_{n_k} = \omega_0$. Since F is a continuous function, $F(x', \omega_0) = \lim_{n_k\to\infty} F(x_{n_k}, \omega_{n_k}) = 0$. So $x' \in \Psi(\omega_0)$ thus $x' \in V$. Since V is an open set, there exists an $\varepsilon > 0$ such that $B_\varepsilon(x') \subseteq V$. Hence $x_{n_k} \in V$ for n_k sufficiently large, a contradiction. Thus Ψ is an upper hemicontinuous.