Economics 204
Problem Set 4 Solutions

Exercise 1

a) First note that S is a subset of $\mathbf{R}^3$, which is a vector space (over $\mathbf{R}$, with the operations assumed). Hence, all we have to show is that 0 vector is contained in S and that $\forall \alpha, \beta \in \mathbf{R}$, and $x, y \in S$, we have $\alpha x + \beta y \in S$. But this is pretty obvious: $i)$ take $c = 0$ to show that $0 \in S$; $ii)$ if $x = c_1 v$ and $y = c_2 v$, then $\alpha x + \beta y = (\alpha c_1 + \beta c_2)v$ and if we let $c = \alpha c_1 + \beta c_2$ then it follows that $\alpha x + \beta y \in S$.The space is one dimensional, and $\{v\}$ is a basis for S.

b) Same argument applies here: $i)$ 0 vector is obviously in S. $ii)$ Now take $\alpha, \beta \in \mathbf{R}$ and $x, y \in S$; let $z := \alpha x + \beta y$, then $z_1 + z_2 + z_3 = (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) + (\ \alpha x_3 + \beta y_3) = \alpha(x_1 + x_2 + x_3) + \beta(y_1 + y_2 + y_3) = \alpha 0 + \beta 0 = 0$; and $z_1 + 2z_2 = (\alpha x_1 + \beta y_1) + 2(\alpha x_2 + \beta y_2) = (\alpha x_1 + 2\alpha x_2) + (\alpha y_1 + 2\beta y_2) = \alpha(x_1 + 2x_2) + \beta(y_1 + 2y_2) = 0$. The space is again one dimensional (Note that if we fix x_2, then x_1 and x_3 are determined). $\{(1, -1, 0)\}$ is a basis for S.

c) S is not a vector space since it does not contain 0 vector.

d) not a vector space since not all additive inverses of continuous functions are in S.

Exercise 2

a) $x \in Ker(g) => g(x) = 0 => (f \circ g)(x) = f(g(x)) = f(0) = 0$ since f is a linear tranformation. Thus $x \in Ker(f \circ g)$ and thus $\dim Ker(g) \leq \dim Ker(f \circ g)$. We have assumed that $\dim Z = \dim V = \dim W = n$. Since $\dim \operatorname{Im}(h) + \dim Ker(h) = n$ for any linear transformation $h : Z \to U$ (U a vector space), we can conclude that $\dim \operatorname{Im}(g) \geq \dim \operatorname{Im}(f \circ g)$.

b) ($\Rightarrow$) Since f is a linear tranformation, $f(0) = 0$ and since f is one to one, it follows that $Ker(f) = \{0\}$. ($\Leftarrow$) Suppose $f(x) = f(y)$, then we have that $f(x) - f(y) = 0$ and so $f(x - y) = 0$. But $Ker(f) = \{0\}$; thus we must have that $x = y$ and therefore f is one to one.

d) That composition of two linear maps is linear is shown in the R. Anderson's lecture notes. To prove that $f \circ g$ is one to one, suppose that $(f \circ g)(x)$

$= (f \circ g)(y)$ then $f(g(x)) = f(g(y))$. Since f is one to one, it follows that $g(x) = g(y)$, and since g is one to one we have $x = y$. To prove that $f \circ g$ is onto, let $y \in V$. since f is onto there is $x \in V$ such that $f(x) = y$ and since g is onto there is $z \in V$, such that $g(z) = x$. Hence given z, $(f \circ g)(z) = y$. Thus $f \circ g$ is an automorphism of V.

Exercise 3

a) Consider $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$; $Ae_1 = 0$ and $Ae_2 = -e_1$ where $e_1 = (1,0)$ and $e_2 = (0,1)$. Thus $a_{11} = a_{22} = a_{21} = 0$ and $a_{12} = -1$.

b) Projecting onto the x-axis followed by projection onto the y-axis maps every vector to 0. The matrix representing this tranformation is the 0 matrix.

c) *i*) The tranformation maps every vector $(x,y,z) \in \mathbf{R}^3$ to $(x,y,0)$. The matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

ii) The transformation maps every vector $(x,y,z) \in \mathbf{R}^3$ to $(x,y,-z)$. The matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Exercise 4

$Ker(T)$ is the set of 2 by 2 matrices such that $b_{11} = b_{12}$, which is a three dimensional space with

$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ forming a basis for $Ker(T)$; finally $rank(T) = 1$ as $rank(T) + dimKer(T) = 4$. T is not one to one since $Ker(T)$ is non-trivial; T is not onto since $rank(T) = 1$ and not 4.

Exercise 5

$A^n = (P^{-1}BP)(P^{-1}BP)...(P^{-1}BP) = P^{-1}B(PP^{-1})B(PP^{-1})BP...P^{-1}BP = P^{-1}B^nP$. Since $Tr(AB) = \sum_{i=1}^{n}(\sum_{j=1}^{n} a_{ij}b_{ji}) = \sum_{j=1}^{n}(\sum_{i=1}^{n} b_{ji}a_{ij}) = Tr(BA)$ and since B is diagonal, $Tr(A^n) = Tr(P^{-1}B^nP) = Tr((P^{-1}B^n)P) = Tr(P(P^{-1}B^n)) = Tr((PP^{-1})B^n) = Tr(B^n) = \sum_{i=1}^{m} b_{ii}^n$ and $Det(A^n) = \prod_{i=1}^{m} b_{ii}^n$ where m is the number of columns/rows of A and B.

Exercise 6

a) False, since W may be a much larger vector space than V. Let $V = \mathbf{R}$ and $W = \mathbf{R}^2$. Any non-zero transformation $T{:}V \to W$ will have a trivial Kernel but only one-dimensional image. Thus, $\{Tv_\theta\}_{\theta\in\Theta}$ cannot span W and therefore $\{w_\gamma\}_{\gamma\in\Gamma} \nsubseteq Span\{Tv_\theta\}_{\theta\in\Theta}$. The statement would be true if the spaces had the same dimension.

b) True. Since T is an isomorphism, $W = \operatorname{Im} T$. Thus, $span\{Tv_\theta\}_{\theta\in\boldsymbol{\Theta}} = W$. $\{Tv_\theta\}_{\theta\in\boldsymbol{\Theta}}$ is a set of independent vectors: for any linear combination such that $0 = \sum_{i=1}^{n} \alpha_i(Tv_{\theta_i}) = \sum_{i=1}^{n} T(\alpha_i v_{\theta_i}) = T(\sum_{i=1}^{n} \alpha_i v_{\theta_i})$ we have $\sum_{i=1}^{n} \alpha_i v_{\theta_i} = 0$ as T is an isomorphism; and since $\{v_\theta\}_{\theta\in\boldsymbol{\Theta}}$ are independent, $\alpha_i = 0$ for all i. Therefore $\{Tv_\theta\}_{\theta\in\boldsymbol{\Theta}}$ is a basis for W (the set is linearly independent and spans W). $\{Tv_\theta\}_{\theta\in\boldsymbol{\Theta}}$ and $\{w_\gamma\}_{\gamma\in\Gamma}$ are thus numerically equivalent by Theorem 4 Lecture 8.

c) False, since V may be a much larger space than W. Let $V = \mathbf{R}^2$ and $W = \mathbf{R}$ and define $T(v_1) = w$ and $T(v_2) = 0$ where v_1, v_2 are the two vectors in the given basis for V and $\{w\}$ is the given basis for W. the bases $\{v_i\}_{i=1}^2$ and $\{w\}$ are not numerically equivalent; nevertheless $\{Tv_i\}_{i=1}^2$ spans $\mathbf{R}$.