Econ 204
Problem Set 5 Solutions

Exercise 1

C is already diagonal.

B can be diagonalized: it has eigenvalues 1 and 2 and a pair of eigenvectors $(1,0)$ and $(1,1)$ corresponding to these eigenvalues.

Let $V = \{(1,0), (1,1)\}$ be the set with the two eigenvectors. It is also a basis for $\mathbf{R}^2$. Let W be the standard basis and consider the change of basis matrix $Mtx_{V,W}(id) = Mtx_{W,V}(id)^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$. Thus, B can be written as $B = Mtx_W(B) = Mtx_{W,V}(id) \times Mtx_V(B) \times Mtx_{V,W}(id)$ where $Mtx_V(B)$ is precisely the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (since $(1,0)$ and $(1,1)$ are both the eigenvectors of B and the basis vectors in V).

To check that this works, $Mtx_{W,V}(id) \times Mtx_V(B) \times Mtx_{V,W}(id) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = B$.

Eigenvalues for matrix A are $1, -1$, and 2 with eigenvectors $(1,0,-1)$, $(0,0,1)$, and $(2,1,-2)$. Let M the matrix formed by these eigenvectors and K the diagonal matrix with eigenvalues on the diagonal. You can proceed similarly for matrix A and verify that $A = MKM^{-1}$; M^{-1} is thus again the change of basis matrix from the standard basis to the basis formed by the above eigenvectors.

Exercise 2

If A is positive semidefinite and B is an n by m matrix, then $B^T AB$ is indeed positive semidefinite. Let x be a vector in $\mathbf{R}^m$. Then Bx is a vector in $\mathbf{R}^n$ and since A is positive semidefinite, $x^T B^T ABx = (Bx)^T A(Bx) \geq 0$, so that $B^T AB$ is positive semidefinite.

Now suppose A is positive definite. Previous reasoning shows that $x^T B^T ABx \geq 0$; however we need the inequality to be strict for $x \neq 0$. Since A is positive definite, $x^T B^T ABx \neq 0$ for $x \neq 0$ if and only if $Bx \neq 0$ for $x \neq 0$; and $Bx \neq 0$ for $x \neq 0$ if and only if $Ker(B) = \{0\}$ This can only hold if $m \leq n$ since $m = rank(B) + \dim Ker(B) = rank(B) + 0$. We have $rank(B) \leq n$ since B is n by m.

If $m > n$, $B^T AB$ can never be positive definite.

Exercise 3

a) First result is $z \cdot v = 0$. (Picture vectors u and v in a plane; then the shortest distance from vector u to vector v must be along the ray perpendicular to vector v; vector $\alpha^* v$ is formed by connecting the origin to the point at which the ray perpendicular to v hits vector v; vector $z = u - \alpha^* v$ is parallel to that ray). From that we can find an expression for α^*: $0 = z \cdot v = (u - \alpha^* v) \cdot v$ and thus $\alpha^* = \frac{u \cdot v}{v \cdot v} = \frac{u \cdot v}{\|v\|^2}$.

b) The expressions for γ and β are simpler: $\gamma = \frac{u \cdot v_1}{\|v_1\|^2} = \frac{u \cdot v_1}{1} = av_1 \cdot v_1 = a$ and similar computation yields $\beta = b$. When v_1 and v_2 are orthogonal, the coefficients on the two vectors are found by projecting vector u on each of the two vectors *separately.*

Exercise 4

Here, we can use The Inverse Function Theorem (IFT). Thus, we need to check that the function is $C^1(\mathbf{R}^3)$ and find points in $\mathbf{R}^3$ such that $\det(Df(x_0, y_0, z_0))$ is non-zero. By Theorem 4 in Lecture 11, to get differentiability of f, it is enough to check that the partial derivatives exist and are continuous: partials are either the 0 function or $2x, 2y$, or $2z$, all of which are continuous on $\mathbf{R}^3$.

The Jacobian of f is $Df(x_0, y_0, z_0) = \begin{pmatrix} 2x_0 & 0 & 0 \\ 0 & 2y_0 & 0 \\ 0 & 0 & 2z_0 \end{pmatrix}$, which is invertible if and only if $x_0, y_0,$ and z_0 are all non-zero.

At all such points, $(Df^{-1})(f(x_0, y_0, z_0)) = \begin{pmatrix} 1/(2x_0) & 0 & 0 \\ 0 & 1/(2y_0) & 0 \\ 0 & 0 & 1/(2z_0) \end{pmatrix} =$

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/x_0 \\ 1/y_0 \\ 1/z_0 \end{pmatrix} = \begin{pmatrix} 1/(2x_0) & 0 & 0 \\ 0 & 1/(2y_0) & 0 \\ 0 & 0 & 1/(2z_0) \end{pmatrix} \begin{pmatrix} 1/(\pm 2 \times \sqrt{f_1(x_0, y_0, z_0)}) \\ 1/[2 \times (1 \pm \sqrt{f_2(x_0, y_0, z_0)})] \\ 1/(\pm 2 \times \sqrt{f_3(x_0, y_0, z_0)}) \end{pmatrix},$$

where f_i denotes the i^{th} component of f evaluated at (x_0, y_0, z_0). If x_0 is negative then we take $-\sqrt{f_1(x_0, y_0, z_0)}$ to be negative, and if x_0 is positive we take it positive. We do the same for y_0 and z_0.

a)

The 2nd order Taylor expansion is:

$f(x, y) = f(x_0, y_0) + Df(x_0, y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x - x_0 & y - y_0 \end{pmatrix} D^2 f(x_0, y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} +$ $O(|x - x_0|^3)$ ($f \in C^3$), where $Df(x_0, y_0) = \begin{pmatrix} 14x_0 - 10 + 2y_0 & 22y_0 + 2x_0 + 3 \end{pmatrix}$ and $D^2 f(x_0, y_0) = \begin{pmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 14 & 2 \\ 2 & 22 \end{pmatrix}$

Thus, $f(x, y) = \{11y_0^2 + 7x_0^2 - 10x_0 + (2x_0 + 3)y_0\} + \{14x_0 - 10 + 2y_0\}(x - x_0) + \{22y_0 + 2x_0 + 3\}(y - y_0) + \frac{1}{2}14(x - x_0)^2 + \frac{1}{2}22(y - y_0)^2 + 2(x - x_0)(y - y_0) + O(|x - x_0|^3)$

b) Matrix A takes the form $A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, with eigenvalues: $2, 6$, and 3. Thus f has a global minimum at 0 (Lecture 10, Corollary 5).

Exercise 5

When $F(x, y, z) = x^2yz^3 - 3$, $F(x, y, z) = 0$ implies $x^2yz^3 = 3$, so that when y and z are non-zero, $x^2 = 3/(yz^3)$. Thus as long as x_0, y_0, z_0 are all non-zero and satisfy the previous equation, there is unique local solution $x(z, y)$, such that $x(y, z)^2 = 3/(yz^3)$. Note that we could not have found a globally unique solution $x(y, z)$ since $x^2 = 3/(yz^3)$ implies $x = \pm(3/(yz^3))^{1/2}$. (you could also use ImpFT to answer this question).

For $F(x, y, z) = x^2yz^3 - 3x^{10}$ we can use ImpFT (though we don't have to). First we need to find $D_xF(x, y, z)$. $D_xF(x, y, z) = 2xyz^3 - 30x^9$. We need to find (x_0, y_0, z_0) such that $2xyz^3 - 30x^9 \neq 0$. Thus we must have $x(yz^3 - 15x^8) \neq 0$, i.e. we must have both $x \neq 0$ and $yz^3 - 15x^8 \neq 0$; however we need these two conditions to hold at solutions to the system $x^2yz^3 - 3x^{10} = 0$ which is equivalent to $x^2(yz^3 - 3x^8) = 0$, i.e. $x = 0$ or $(yz^3 - 3x^8) = 0$. If $x = 0$, the Jacobian is not invertible. Thus we must have $yz^3 - 3x^8 = 0$. Since we concluded that x cannot be 0, it follows that $(0, 0, 0)$ is not a point where we can apply ImpFT. Thus we can write $yz^3 - 3x^8 = 0 \Leftrightarrow y = 3x^8/z^3$ (1). This set of points combined with the set of points $x \neq 0$ (2) and $yz^3 - 15x^8 \neq 0$ (3) is the region where we can use the ImpFT to solve locally for x in terms of y and z, i.e. for each point satisfying conditions (1), (2), and (3) we can find open sets U and V with $x_0 \in U$ and $(y_0, z_0) \in V$ and an implicit function $x(y, z) : U \to V$, such that $F(x(y, z), y, z) = 0$; $x(y, z) \in C^1$ and a formula for it's derivative is given in the lecture note 12.

Exercise 6

Now suppose the assumptions of the InvFT hold: i.e. suppose $X \subseteq R^n$ is open, $f : X \to R^n$, $f \in C^1(X)$, $x_0 \in X$ and $det(Df(x_0)) \neq 0$.

Let $y_0 = f(x_0)$, and $F(x, y) = f(x) - y$, so that then $F(x_0, y_0) = 0$ and $D_xF(x_0, y_0) = Df(x_0) \neq 0$. By ImpFT, there are neigboorhoods U and W of x_0 and y_0 respectively, such that for all $y \in W$ there is a unique $x \in U$ such that $F(x, y) = 0$; thus we construct a function $g : W \to U$ uniquely which by the ImpFT is C^1 on W. Thus we have $f(g(y)) = y$ for $y \in W$ and since g is one-to-one, we also have $f(x) = g^{-1}(x)$ which proves that f is invertible on W and that $g = f^{-1}$.

Finally, $(Df^{-1})(f(x_0)) = Dg(y_0) = -[D_xF(x_0, y_0)]^{-1}[D_yF(x_0, y_0)] = [Df(x_0)]^{-1}I_n = [Df(x_0)]^{-1}$ and finally $f \in C^n \implies F \in C^n \implies g \in C^n \implies f^{-1} \in C^n$ and the result follows.