Econ 204
Problem Set 6
Due Monday, August 17th

Exercise 1

Consider $f : \mathbf{R}^2 \to \mathbf{R}^2$ such that $f \in C^3(\mathbf{R}^2)$. Now let $F(x, y, w, z) = f(x, y) - (w, z)$ and suppose that $F(x, y, w, z) = 0$ has solutions in $\mathbf{R}^4$. Let $S \subset \mathbf{R}^4$ be the set of solutions to this system. Show that there exists a set B such that B^c has measure zero and for $(x, y, w, z) \in S$ where $(w, z) \in B$, there is a local implicit function $h : W \subset \mathbf{R}^2 \to \mathbf{R}^2$ (W open) such that $F(h(w, z), w, z) = 0$ for all $(w, z) \in W$ and $h \in C^3(\mathbf{R}^2)$.

Exercise 2

Let $f : [0, 1] \to [0, 1]$ be a correspondence defined as $f(x) = \{0, 1/(x + 1)\}$ for $x \neq 0$ and $f(0) = \{1/2\}$. Does f have a fixed point? If yes, find the point(s). Does any of the fixed point theorems you have learned apply here? Explain. Answer the same questions for $f(x) = [\epsilon, 1/(x+1)]$ for all $x \in [0, 1]$ where $0 < \epsilon$ $< 1/2$.

Exercise 3

We say that a relation R on X is convex if whenever xRy and zRy then $(\alpha x + (1 - \alpha)z)Ry$ for all $\alpha \in (0, 1)$. (if x and y are in $\mathbf{R}$, $\geq$ is an example of such relation). Let R_i be a convex relation on $\mathbf{R}^n$ for $i = 1, 2, ...m$, fix $x \in \mathbf{R}^n$ and let $B_i = \{y - x : yR_i x,\ y \in R^n\}$. Show that B_i is convex.

Let $B = \sum_{i=1}^{m} B_i := \{z_1 + z_2 + ... + z_m;$ such that $z_i \in B_i$ for all $i\}$. Show that B is convex. In the case where R is a "preference" relation (you will learn this later in $Econ201B$), $0 \notin B$ is equivalent to x being a Pareto optimal allocation. Show that in the case where $0 \notin B$, there exists $p \neq 0$ such that $inf(p \cdot B) \geq 0$. This is how we construct prices in $Econ201B$.

Exercise 4

Show that if $B \subset R^n$ is open and convex, then $B = \cap_{i \in I} S_i$, where $\{S_i, i \in I\}$ is the set of all open half-spaces containing B (an open half-space in $\mathbf{R}^n$ is a set $S = \{y \in R^n : p \cdot y < c\}$ for some $p \in \mathbf{R}^n, c \in \mathbf{R}$.

Exercise 5

State whether the following functions are Lipschitz and prove your claim:

a) $f(x) = \ln(x)$ for $x > 0$;

b) $f(x) = \cos(x)$ for $x \in \mathbf{R}$;

c) $f : \mathbf{R} \to \mathbf{R}$, f differentiable, such that $\left|\frac{df}{dx}\right| \leq M$ for some $M \in \mathbf{R}$;

If any of the functions above is not Lipschitz, what can you change to make them Lipschitz?

Consider the differential equation $\frac{dy}{dt} = \frac{3}{2}y(t)^{1/3}$ defined for all $t \geq 0$ and $y(t_0) = 0$. Does this differential equation have a solution? Is that solution unique? If yes, prove it. If not, explain why not and then modify the problem to make the solution unique.

Try to find a solution if it exists.

Exercise 6

Consider the following system of first order differential equations:

$$\begin{aligned} x'(t) &= x^2 - y \\ y'(t) &= y(y-1) \end{aligned}$$

a) Plot the $x'(t) = 0$ and $y'(t) = 0$ curves on the $x - y$ coordinate axes. Find the stationary point corresponding to $x, y > 0$.

b) Linearize the system using Taylor-series expansion around the $x, y > 0$ steady state. Write down the linearized equations.

c) Describe the behavior of the system and write down the general solution.