Section 7
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Key Words

Compactness, Open Cover, Sequentially Compact, Totally Bounded

Section 7.1 Two Definitions of Compactness

- **Lecture 6 Definition 1:** A collection of sets $U = \{U_\lambda : \lambda \in \Lambda\}$ in a metric space (X, d) is an **open cover** of A if U_λ is open for all $\lambda \in \Lambda$ and $\cup_{\lambda\in\Lambda} U_\lambda \supseteq A$.
- **Lecture 6 Definition 1:** A set A in a metric space is **compact** if every open cover of A contains a finite subcover of A. In other words, if $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of A, there exist $n \in \mathbf{N}$ and $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $A \subseteq U_{\lambda_1} \cup \ldots \cup U_{\lambda_n}$.
- **Lecture 6 Definition 4:** A set A in a metric space (X, d) is **sequentially compact** if every sequence of elements of A contains a convergent subsequence whose limit lies in A.

Example 7.1.1 Exhibit an open cover of $(0, 1)$ with no finite sub-cover.
Solution:
Consider the cover $\{(0, 1 - 1/n)\}$. This covers $(0, 1)$ but has no finite subcover. Observe that the sequence of intervals is increasing, hence any finite cover is equivalent to a cover of the form $(0, 1 - 1/N)$, for some N. This interval is clearly a proper subset of $(0, 1)$.

Example 7.1.2 Show that $\mathbf{Q} \cap [\mathbf{0}, \mathbf{2}]$ is not compact.
Solution:
We can find a counter-example. Let $U_n = (-1, \sqrt{2} - \frac{1}{n}) \cup (\sqrt{2} + \frac{1}{n}, 3)$. The collection of sets given by $\cup_{n\in\mathbf{N}} U_n = (-1, \sqrt{2}) \cup (\sqrt{2}, 3)$ is an open cover of $\mathbf{Q} \cap [\mathbf{0}, \mathbf{2}]$. Since $\mathbf{Q}$ is dense, for every finite number N, there exists a rational number $q \in \mathbf{Q} \cap [\sqrt{2} - \frac{1}{N}, \sqrt{2} + \frac{1}{N}]$. So any finite subcollection does not cover $\mathbf{Q} \cap [\mathbf{0}, \mathbf{2}]$.

Example 7.1.3 Show that a finite union of compact sets is compact.
Solution:
Let $A_1, \ldots, A_n$ be compact sets and consider any open cover of $A_1 \cup \cdots \cup A_n$. This open cover must cover each A_i individually, and because each A_i is compact, there must be a finite subcover of each A_i. The union of these n subcovers is finite, and clearly it covers $A_1 \cup \cdots \cup A_n$. Therefore every open cover of $A_1 \cup \cdots \cup A_n$ has a finite subcover, so $A_1 \cup \cdots \cup A_n$ is compact.

Example 7.1.4 (Cantor's Intersection Theorem) Use the open cover definition of compactness to prove a decreasing sequence of nonempty compact subsets $A_1 \supset A_2 \supset \cdots$ of a metric space (X, d) has nonempty intersection.
Solution:
By contradiction. Suppose their intersection in empty: $A_1 \cap A_2 \cap \cdots = \phi$. Since $A_1 \supset A_2 \supset \cdots$ and they are nonempty sets, $A_2 \cap A_3 \cap \cdots = \phi$. Let $U = X\backslash(A_2 \cap A_3 \cap \cdots) = X\backslash A_2 \cup X\backslash A_3 \cup \cdots$ and it is open, so it constructs an open cover for A_1. Because A_1 is compact, there exists a finite subcover $X\backslash A_2 \cup X\backslash A_3 \cup \cdots \cup X\backslash A_N \supset A_1$. Then its complement $X\backslash(X\backslash A_2 \cup X\backslash A_3 \cup \cdots \cup X\backslash A_N) = A_2 \cap A_3 \cap \cdots \cap A_N$ has no common element with $A_1 \Leftrightarrow A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_N = \phi$. But we know $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_N = A_N$. Contradiction.

Example 7.1.5 Let $\{x_n\}$ be a convergent sequence in a metric space with limit l. Show that the set $\{l, x_1, x_2, x_3, x_4, ...\}$ is compact.
Solution:

If $\{x_n\}$ converges, then it is bounded and it has exactly one limit point. Let$\cup_{i\in I}A_i$ be an open cover of the set $X = \{l, x_1, x_2, \dots x_n \dots\}$. This means that for all $x \in X$, $\exists i \in I$ such that $x \in A_i$. Let take one $i \in I$ such that $l \in A_i$ and label this i as i_0 . Because l is a limit point of $\{x_n\}$, A_{i_0} must contain infinitely many elements of $\{x_n\}$. Also, because $x_n \to l$, $X \cap A^c_{i_0}$ is finite. To see this, we recall from the definition of open sets that there exists an $\varepsilon > 0$ such that $B_\varepsilon(l) \subset A_{i_0}$. From the definition of convergence, $\exists N \in \mathbf{N}$ such that $n > N \Rightarrow x_n \in B_\varepsilon(l)$. So $n > N \Rightarrow x_n \notin B_\varepsilon(l)^c$ and there are at most N elements of $\{x_n\}$ in $B_\varepsilon(l)^c$. Because $A^c_{i_0} \subset B_\varepsilon(l)^c$, this also means that there are only finitely many elements of X not in A_{i_0} . For each $n \leq N$, take one $i \in I$ such that $x_n \in A_i$ and relabel it with i_n. Then $\cup_{n=0,1,\dots N}A_{i_n}$ is a finite subcover (not necessary minimal, in the sense that some sets may be dispensable). This proves that X is compact.

Section 7.2 Closed, Totally Bounded and Compact

- Lecture 6 Theorem 2: Every closed subset A of a compact metric space (X, d) is compact.
- Lecture 6 Theorem 3: If A is a compact subset of the metric space (X, d), then A is closed.

- Lecture 6 Definition 6**:** A set A in a metric space (X, d) is **totally bounded** if, for every $\varepsilon > 0$, $\exists x_1, \dots, x_n \in A$ $A \subseteq \cup_{i=1}^n B_\varepsilon(x_i)$.
- Lecture 6 Theorem 7: Let A be a subset of a metric space (X, d). Then A is compact if and only if it is complete and totally bounded.
- Lecture 6 Corollary 8: Let A be a subset of a complete metric space (X, d). Then A is compact if and only if it is closed and totally bounded.
- Lecture 6 Theorem 10 (**Heine-Borel**): If $A \subseteq E_n$, then A is compact if and only if A is closed and bounded.

Example 7.2.1 Show that R is not compact
Solution 1:
We can find a counter-example. Let $U_n = (n-1, n+1)$. The collection of sets given by $\cup_{n\in\mathbf{N}}U_n$ is an open cover of $\mathbf{R}$. Any finite subcollection does not cover $\mathbf{R}$.
Solution 2:
By Heine Borel theorem, since $\mathbf{R}$ is not bounded, the result follows immediately.
Example 7.2.2 Consider metric space (X, d). Let $X = [0, 1] \subseteq \mathbf{R}$. d is defined as

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Is (X, d) closed? Is (X, d) totally bounded? Is (X, d) complete? Is (X, d) compact?
Solution:
We know that under discrete metric d, every set is both open and closed. And any open ball

$$B_\varepsilon(x) = \begin{cases} [0,1] & \varepsilon \geq 1 \\ x & \varepsilon < 1 \end{cases}$$

When $\varepsilon < 1$, the open ball around any point is just the point itself, in this case, since there are infinitely many points in $[0, 1]$, any union of finitely many points can not cover $[0, 1]$. Therefore it is not totally bounded. It is complete since for a Cauchy sequence, the points could be arbitrarily close – in the case of the discrete metric, the points are the same. Thus, any Cauchy sequence, under the discrete metric, converges. The set is complete. Since it is not totally bounded, it is not compact. Alternatively, you can show a open cover with no finite subcover: $\cup_{all\ x\in[0,1]}B_{\frac{1}{2}}(x)$.

Example 7.2.3 Let $U_1 \subseteq U_2 \subseteq U_3 \subseteq ....$ be open subsets of $\mathbf{R}$ with non-empty and bounded complement. Prove that $\bigcup_{j=0}^\infty U_j \neq \mathbf{R}$.

Solution:
The ${U_j}^c$ are closed and bounded, hence compact by the Heine-Borel Theorem. Note that since ${U_j}^c \supseteq {U_{j+1}}^c$ and each $U_j \neq \emptyset, \mathbf{R} \Rightarrow$ for each n, $\bigcap_{j=1}^{n} \{U_j\}^c \neq \emptyset$. For each n, choose an element of $\bigcap_{j=1}^{n} {U_j}^c$ and label it x_n. Note that $\{x_n\} \subseteq \{U_1\}^c$. By compactness, this sequence has a convergent subsequence with limit $L \in \{U_1\}^c$. We claim that, in fact, $L \in \bigcap_{j=1}^{\infty} \{U_j\}^c$. To see this, denote the convergent subsequence by $\{x_{n_k}\}$. From the construction of the sequence it follows that $\{x_{n_k}\}_{k=1}^{\infty} \subseteq \{U_{n_1}\}^c$, $\{x_{n_k}\}_{k=2}^{\infty} \subseteq \{U_{n_2}\}^c$, $\{x_{n_k}\}_{k=3}^{\infty} \subseteq \{U_{n_3}\}^c$, *etc.*. Note that $L \in \{U_{n_k}\}^c\ \forall k$, by compactness. This implies that $L \in \bigcap_{k=1}^{\infty} \{U_{n_k}\}^c = \bigcap_{j=1}^{\infty} \{U_j\}^c$. The latter equality follows because the $\{U_j\}^c$ form a decreasing sequence. Hence, given any $\{U_j\}^c\ \exists\, n_k$ *such that* $\{U_{n_k}\}^c \supseteq \{U_j\}^c$. Similarly, given any $\{U_{n_k}\}^c$, $\exists\, j$ *such that* $\{U_{n_k}\}^c \subseteq \{U_j\}^c$. Consequently, $\bigcap_{j=1}^{\infty} \{U_j\}^c \neq \emptyset$ $\Rightarrow \bigcup_{j=1}^{\infty} U_j \neq \mathbf{R}$.

Example 7.2.4 Suppose that for some $\varepsilon > 0$, the closure of every ε-open ball in X is compact (Every ε-cloesd ball in X is compact). Show that X is complete.
Solution:
Let $\{x_n\}$ be a Cauchy sequence in X. We know that there exists some M such that $\forall m, n > M, d(x_m, x_n) < \varepsilon$. Let $N = M + 1$. Consider $B_\varepsilon(x_N)$ and its closure, $B_\varepsilon[x_N]$. Since $d(x_m, x_N) < \varepsilon$ for all $m \geq N$, we have $x_n \in B_\varepsilon(x_N) \subset B_\varepsilon[x_N]$ for all $n \geq N$. The subsequence of $\{x_n\}$ consisting of all x_n such that $n \geq N$ is clearly also a Cauchy sequence and it is contained entirely in $B_\varepsilon[x_N]$, which, by hypothesis, is compact. We know that a sequence in a compact set must have a convergent subsequence, and Theorem 7.8 in de la Fuente establishes that a Cauchy sequence with a convergent subsequence must itself converge. Thus, the sequence $\{x_n, n \geq N\}$ must converge and naturally the $\{x_n\}$ must converge as well. Therefore, X is complete.

Example 7.2.5 Let (X, d) be a metric space, $A, B \subset X$. $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Assume that A is compact, B is closed and $A \cap B = \emptyset$.
(a)Prove that $d(A, B) > 0$
(b)Suppose that B is also compact. Prove that there exist $a \in A$, $b \in B$ such that $d(A, B) = d(a, b)$.
(c)Is (b) true if B is just closed?
Solution:
(a) We prove it by contradiction. Suppose that $d(A, B) = 0$. Since $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$, so $\forall\ n \in \mathbf{N}$, there exist $a_n \in A$, $b_n \in B$ such that $d(a_n, b_n) < 1/n$. Since A is compact, so $\{a_n\}$ contains a convergent subsequence $\{a_{n_k}\}$ whose limit, a, lies in A (from the definition of sequentially compact). $d(b_{n_k}, a) \leq d(b_{n_k}, a_{n_k}) + d(a_{n_k}, a) < 2/n$, so also converges to a. Since B is closed, so $a \in B$, so $a \in A \cap B$, which contradicts the assumption that $A \cap B = \emptyset$
(b) $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$, so $\forall\ n \in \mathbf{N}$, there exists $a_n \in A$, $b_n \in B$, such that $d(A, B) \leq d(a_n, b_n) \leq d(A, B) + 1/n$. A is compact, so $\{a_n\}$ contains a convergent subsequence $\{a_{n_k}\}$ whose limit lies in A; B is compact, so $\{b_n\}$ contains convergent subsequence $\{b_{n_k}\}$ whose limit lies in B. For each n, $d(a, b) \leq d(a, a_n) + d(a_n, b_n) + d(b_n, b)$. The first term converges to zero, the second term converges to $d(A, B)$, and the third term converges to zero, so $d(a, b) \leq d(A, B)$. But $d(A, B) \leq d(a, b)$, so $d(a, b) = d(A, B)$.
(c) If B is merely closed, the result is not true. For example, suppose $A = [0, 1], B = (2, 3]$, and $X = A \cup B$, with the Euclidean metric. B is closed in X, A is compact, and $d(A, B) = 1$, but $d(a, b) > 1$ for every $a \in A, b \in B$.

Section 7.3 Continuous Function and Compactness

- **Lecture 6 Theorem 11:** Let (X, d) and (Y, ρ) be metric spaces. If $f : X \to Y$ is continuous and $C \subseteq X$ is compact in (X, d), then $f(C)$ is compact in (Y, ρ).
- **Lecture 6 Corollary 12 (Extreme Value Theorem):** Let C be a compact set in a metric space (X, d), and suppose$f : C \to \mathbf{R}$ is continuous. Then f is bounded on C and attains its minimum and maximum on C.

- **Lecture 6 Theorem 13:** Let (X, d) and (Y, ρ) be metric spaces, $C \subseteq X$ compact, and $f : C \to Y$ continuous. Then f is uniformly continuous on C.

Example 7.3.1 (Extension of Example 5.2.3) Consider a subset A of an arbitrary metric space (X, d). The distance between a point $x \in X$ and the set A is defined as $d(x, A) = inf_{a \in A} d(x, a)$. Define the function $f : X \to \mathbf{R}$ by $f(x) = d(x, A)$, and let the metric on the range be Euclidean. In Example 5.2.3, we have shown that 1. $d|f(x) - f(y)| \leq d(x, y)\, \forall x, y \in X$ and 2. f is uniformly continuous. Now you are asked to show

3. If A is compact, then $\exists a \in A$ such that $d(x, a) = d(x, A)$.

4. Does the statement of part (c) hold if A is closed but not necessarily compact? If so, prove it. If not, provide a counter-example.

Solution:

3. We can see a similar example in PS2. After we learned compactness, using the definition of sequential compactness would simplify the proof.By the definition of infimum, $\forall n \in N$, $\exists a_n \in A$, s.t. $d(x, A) \leq d(x, a_n) \leq d(x, A) + \frac{1}{n}$.Therefore we have a sequence $\{a_n\}$ of A and $d(x, a_n) \to d(x, A)$. Since A is compact, by the sequential compactness we know that $\exists \{a_{n_k}\}$ of $\{a_n\}$ s.t. $a_{n_k} \to a \in A$. So $d(x, a_{n_k}) \to d(x, a)$. And since $d(x, a_n) \to d(x, A) \Rightarrow d(x, a_{n_k}) \to d(x, A)$ as $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$. By the uniqueness of limit point, we know $d(x, A) = d(x, a)$ where $a \in A$. So we are done.

4. Does the statement of part (c) hold if A is closed but not necessarily compact? If so, prove it. If not, provide a counter-example.

Solution: False. As a counterexample, let $X = \mathbf{R} \setminus \{0\}$, $A = \{\frac{1}{n} : n \in \mathbf{N}\}$. This set is closed because zero is not in the universe, and it is clearly bounded, but it is not compact. This can be established using an open cover such as the innite union of intervals of the form $(\frac{1}{n}, 2)$ for $n \in \mathbf{N}$, which has no finite sub-cover. Now consider any negative real number z and the fact that $f(z) = \left|z\right|$. However, zero is not in A (nor is it in X) so there is no point $a \in A$ such that $d(z, a) = d(z, A) = f(z)$.