Econ 204
Supplement to Section 2.3
Lim Sup and Lim Inf

Definition 1 We extend the definition of sup and inf to unbounded sets as follows:

$$\begin{aligned} \sup S &= +\infty \text{ if } S \text{ is not bounded above} \\ \inf S &= -\infty \text{ if } S \text{ is not bounded below} \end{aligned}$$

Definition 2 [Definition 3.7 in de La Fuente] If $\{x_n\}$ is a sequence of real numbers, we say that $\{x_n\}$ tends to infinity (written $\{x_n\} \to \infty$ or $\lim_{n\to\infty} x_n = \infty$) if

$$\forall_{K\in\mathbf{R}}\ \exists_{N(K)}\ n > N(K) \Rightarrow x_n > K$$

Similarly, we say $\lim_{n\to\infty} x_n = -\infty$ if

$$\forall_{K\in\mathbf{R}}\ \exists_{N(K)}\ n > N(K) \Rightarrow x_n < K$$

Definition 3 Consider a sequence $\{x_n\}$ of real numbers. Let

$$\begin{aligned} \alpha_n &= \sup\{x_k : k \geq n\} \\ &= \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} \\ \beta_n &= \inf\{x_k : k \geq n\} \end{aligned}$$

Notice that either $\alpha_n = \infty$ for all n; or α_n is a decreasing sequence of real numbers, in which case α_n tends to a limit (either a real number or $-\infty$) by Theorem 3.1 and Definition 3.7 ; similarly, either $\beta_n = -\infty$ for all n; or β_n is a increasing sequence of real numbers; in which case β_n tends to a limit (either a real number or ∞). Thus, we define

$$\begin{aligned} \limsup_{n\to\infty} x_n &= \begin{cases} +\infty & \text{if } \alpha_n = +\infty \text{ for all } n \\ \lim_{n\to\infty} \alpha_n & \text{otherwise} \end{cases} \\ \liminf_{n\to\infty} x_n &= \begin{cases} -\infty & \text{if } \beta_n = -\infty \text{ for all } n \\ \lim_{n\to\infty} \beta_n & \text{otherwise} \end{cases} \end{aligned}$$

Theorem 4 *Let* $\{x_n\}$ *be a sequence of real numbers. Then*

$$\lim_{n\to\infty} x_n = x \in \mathbf{R} \cup \{-\infty, \infty\}$$

if and only if

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$$