Math 104–Spring 2005–Anderson
Lecture Notes on Metrics on $\mathbf{R}^k$

Lemma 13.1$\frac{1}{4}$ (Cauchy-Schwarz Inequality):

$$\left|\sum_{j=1}^{k} x_j y_j\right| \leq \left(\sum_{j=1}^{k} x_j^2\right)^{1/2} \left(\sum_{j=1}^{k} y_j^2\right)^{1/2}$$

Proof:

$$\begin{aligned}
0 &\leq \sum_{j=1}^{k} \left(x_j - \frac{\sum_{i=1}^{k} x_i y_i}{\sum_{i=1}^{k} y_i^2} y_j\right)^2 \\
&= \sum_{j=1}^{k} x_j^2 - 2\sum_{j=1}^{k} \frac{\sum_{i=1}^{k} x_i y_i}{\sum_{i=1}^{k} y_i^2} x_j y_j + \sum_{j=1}^{k} \frac{\left(\sum_{i=1}^{k} x_i y_i\right)^2}{\left(\sum_{i=1}^{k} y_i^2\right)^2} y_j^2 \\
&= \sum_{j=1}^{k} x_j^2 - 2\frac{\sum_{i=1}^{k} x_i y_i}{\sum_{i=1}^{k} y_i^2} \sum_{j=1}^{k} x_j y_j + \frac{\left(\sum_{i=1}^{k} x_i y_i\right)^2}{\left(\sum_{i=1}^{k} y_i^2\right)^2} \sum_{j=1}^{k} y_j^2 \\
&= \sum_{j=1}^{k} x_j^2 - 2\frac{\left(\sum_{j=1}^{k} x_j y_j\right)^2}{\sum_{j=1}^{k} y_j^2} + \frac{\left(\sum_{j=1}^{k} x_j y_j\right)^2}{\sum_{j=1}^{k} y_j^2} \\
&= \sum_{j=1}^{k} x_j^2 - \frac{\left(\sum_{j=1}^{k} x_j y_j\right)^2}{\sum_{j=1}^{k} y_j^2}
\end{aligned}$$

so

$$\begin{aligned}
\left(\sum_{j=1}^{k} x_j y_j\right)^2 &\leq \left(\sum_{j=1}^{k} x_j^2\right)\left(\sum_{j=1}^{k} y_j^2\right) \\
\left|\sum_{j=1}^{k} x_j y_j\right| &\leq \left(\sum_{j=1}^{k} x_j^2\right)^{1/2}\left(\sum_{j=1}^{k} y_j^2\right)^{1/2}
\end{aligned}$$

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Definition 13.1$\frac{1}{2}$ (Metrics on $\mathbf{R}^k$) We define the following three metrics on $\mathbf{R}^k$:

1. $d_1(x,y) = \Sigma_{j=1}^k |x_j - y_j|$

2. $d_2(x,y) = |x-y| = \left(\Sigma_{j=1}^k (x_j - y_j)^2\right)^{1/2}$

3. $d_\infty(x,y) = \max\{|x_j - y_j| : j = 1, \ldots, k\}$.

Theorem 13.1$\frac{3}{4}$: d_1, d_2 and d_∞ are all metrics on $\mathbf{R}^k$.

Proof: The proofs for d_1 and d_∞ are contained in Problem 13.1, so we consider only d_2. The proofs of the first two properties of a metric are obvious, so we only need to prove the triangle inequality.

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\begin{aligned}
(d_2(x,z))^2 &= |x-z|^2 \\
&= (x-z)\cdot(x-z) \\
&= ((x-y)+(y-x))\cdot((x-y)+(y-z)) \\
&= (x-y)\cdot(x-y) + 2(x-y)\cdot(y-z) + (y-z)\cdot(y-z) \\
&= |x-y|^2 + 2(x-y)\cdot(y-z) + |y-z|^2 \\
&\leq |x-y|^2 + 2|x-y||y-z| + |y-z|^2 \\
&= (|x-y| + |y-z|)^2 \\
&= (d_2(x,y) + d_2(y,z))^2 \\
d_2(x,z) &\leq d_2(x,y) + d_2(y,z)
\end{aligned}
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