Supplemental Appendix for "GMM 'equivalence' for semiparametric missing data models," by Bryan S. Graham.

This supplemental Appendix provides details of some of the more tedious calculations used to prove Theorems 5.1 and 6.1 of the paper "GMM 'equivalence' for semiparametric missing data models," by Bryan S. Graham. It also details the calculation of the variance bound for the ATE when the two potential outcomes have partially linear CEFs and homoscedastic variances.

B Bound for ATE under Assumption 1.5

To calculate the variance bound for the average treatment effect (under homoscedasticity) let $e(X_2) = \Pr(D=1|X_2)$ be the marginal propensity score and define

$$\psi_1(Y_1, X, \beta) = Y_1, \qquad \psi_0(Y_0, X, \beta) = Y_1 + \beta, \qquad q_1(X) = X_1'\delta_1 + h_1(X_2), \qquad q_0(X) = X_1'\delta_0 + h_0(X_2) + \beta.$$

This gives

$$\begin{aligned}
\mathbb{E}[\Gamma(X)] &= -1, \qquad \mathbb{E}\left[(q_1(X) - q_0(X))(q_1(X) - q_0(X))'\right] = Var\left(X_1'(\delta_1 - \delta_0) + h_1(X_2) - h_0(X_2)\right) \\
\Delta_{h_1}(X_2) &= 1, \qquad \Delta_{\delta_1}(X_2) = \mathbb{E}\left[X_1' | X_2\right] \\
\Upsilon_{h_1}(X_2) &= \frac{e(X_2)}{\sigma_1^2}, \qquad \Upsilon_{\delta_1}(X_2) = \frac{1}{\sigma_1^2}\mathbb{E}\left[DX_1X_1' | X_2\right], \qquad \Upsilon_{h_1\delta_1}(X_2) = \frac{1}{\sigma_1^2}\mathbb{E}\left[DX_1' | X_2\right] \\
\Delta_{h_0}(X_2) &= 1, \qquad \Delta_{\delta_0}(X_2) = \mathbb{E}\left[X_1' | X_2\right] \\
\Upsilon_{h_0}(X_2) &= \frac{1 - e(X_2)}{\sigma_0^2}, \qquad \Upsilon_{\delta_0}(X_2) = \frac{1}{\sigma_0^2}\mathbb{E}\left[(1-D)X_1X_1' | X_2\right], \qquad \Upsilon_{h_0\delta_0}(X_2) = \frac{1}{\sigma_0^2}\mathbb{E}\left[(1-D)X_1' | X_2\right],
\end{aligned}$$

and hence implies that

$$\begin{aligned}
\mathbb{E}[A_1(X_2)] &= \sigma_1^2 \mathbb{E}\left[\frac{1}{e(X_2)}\right] \\
\mathbb{E}[B_1(X_2)] &= \mathbb{E}\left[\mathbb{E}\left[X_1' | X_2\right] - \frac{\mathbb{E}\left[DX_1' | X_2\right]}{e(X_2)}\right] = -\mathbb{E}\left[\frac{\mathbb{C}\left(D, X_1' | X_2\right)}{e(X_2)}\right] \\
\mathbb{E}[C_1(X_2)]^{-1} &= \sigma_1^2 \mathbb{E}\left[\mathbb{E}\left[DX_1X_1' | X_2\right] - \frac{\mathbb{E}\left[DX_1' | X_2\right]' \mathbb{E}\left[DX_1' | X_2\right]}{e(X_2)}\right]^{-1} \\
&= \sigma_1^2 \mathbb{E}\left[e(X_2)\mathbb{E}\left[X_1X_1' | X_2, D=1\right] - e(X_2)\mathbb{E}\left[X_1' | X_2, D=1\right]' \mathbb{E}\left[X_1' | X_2, D=1\right]\right]^{-1} \\
&= \sigma_1^2 \mathbb{E}\left[e(X_2) Var(X_1 | X_2, D=1)\right]^{-1},
\end{aligned}$$

since $\mathbb{E}\left[DX_1X_1' | X_2\right] = e(X_2)\mathbb{E}\left[X_1X_1' | X_2, D=1\right]$ and $\mathbb{E}\left[DX_1' | X_2\right] = e(X_2)\mathbb{E}\left[X_1' | X_2, D=1\right]$. The expressions for $\mathbb{E}[A_0(X_2)]$, $\mathbb{E}[B_0(X_2)]$ and $\mathbb{E}[C_0(X_2)]$ are analogous; plugging into $\mathcal{I}^f(\beta)^{-1}$ gives the result.

C Details of calculations used in proof of Theorem 5.1

In order to calculate the bound the inverse of $\begin{pmatrix} M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0} & M_{2\lambda_0}' V_{22}^{-1} M_{2\delta_0} \\ M_{2\delta_0}' V_{22}^{-1} M_{2\lambda_0} & M_{2\delta_0}' V_{22}^{-1} M_{2\delta_0} \end{pmatrix}$ is required. This inverse evaluates to

$$\begin{pmatrix} \left[\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0}\right) - \left(M_{2\lambda_0}' V_{22}^{-1} M_{2\delta_0}\right)\left(M_{2\delta_0}' V_{22}^{-1} M_{2\delta_0}\right)^{-1}\left(M_{2\delta_0}' V_{22}^{-1} M_{2\lambda_0}\right)\right]^{-1} \\ -\left[\left(M_{2\delta_0}' V_{22}^{-1} M_{2\delta_0}\right) - \left(M_{2\delta_0}' V_{22}^{-1} M_{2\lambda_0}\right)\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0}\right)^{-1}\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\delta_0}\right)\right]^{-1} \\ \times\left(M_{2\delta_0}' V_{22}^{-1} M_{2\lambda_0}\right)\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0}\right)^{-1} \\ -\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0}\right)^{-1}\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\delta_0}\right) \\ \times\left[\left(M_{2\delta_0}' V_{22}^{-1} M_{2\delta_0}\right) - \left(M_{2\delta_0}' V_{22}^{-1} M_{2\lambda_0}\right)\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0}\right)^{-1}\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\delta_0}\right)\right]^{-1} \\ \left[\left(M_{2\delta_0}' V_{22}^{-1} M_{2\delta_0}\right) - \left(M_{2\delta_0}' V_{22}^{-1} M_{2\lambda_0}\right)\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0}\right)^{-1}\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\delta_0}\right)\right]^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \left[M_{2\lambda_0}'\left(V_{22}^{-1} - V_{22}^{-1} M_{2\delta_0}\left(M_{2\delta_0}' V_{22}^{-1} M_{2\delta_0}\right)^{-1} M_{2\delta_0}' V_{22}^{-1}\right) M_{2\lambda_0}\right]^{-1} \\ -\left[M_{2\delta_0}'\left(V_{22}^{-1} - V_{22}^{-1} M_{2\lambda_0}\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0}\right)^{-1} M_{2\lambda_0}' V_{22}^{-1}\right) M_{2\delta_0}\right]^{-1} \\ \times\left(M_{2\delta_0}' V_{22}^{-1} M_{2\lambda_0}\right)\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0}\right)^{-1} \\ -\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0}\right)^{-1}\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\delta_0}\right) \\ \times\left[M_{2\delta_0}'\left(V_{22}^{-1} - V_{22}^{-1} M_{2\lambda_0}\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0}\right)^{-1} M_{2\lambda_0}' V_{22}^{-1}\right) M_{2\delta_0}\right]^{-1} \\ \left[M_{2\delta_0}'\left[V_{22}^{-1} - V_{22}^{-1} M_{2\lambda_0}\left(M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0}\right)^{-1} M_{2\lambda_0}' V_{22}^{-1}\right] M_{2\delta_0}\right]^{-1}. \end{pmatrix}.$$

We then evaluate

$$\begin{pmatrix} (\iota_L \otimes I_K)' M_{2\lambda_0} & (\iota_L \otimes I_K)' M_{2\delta_0} \end{pmatrix} \begin{pmatrix} M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0} & M_{2\lambda_0}' V_{22}^{-1} M_{2\delta_0} \\ M_{2\delta_0}' V_{22}^{-1} M_{2\lambda_0} & M_{2\delta_0}' V_{22}^{-1} M_{2\delta_0} \end{pmatrix}^{-1} \begin{pmatrix} M_{2\lambda_0}' (\iota_L \otimes I_K) \\ M_{2\delta_0}' (\iota_L \otimes I_K) \end{pmatrix}$$

as

$$\begin{pmatrix} (\iota_L \otimes I_K)' M_{2\lambda_0} \left[M'_{2\lambda_0} \left(V_{22}^{-1} - V_{22}^{-1} M_{2\delta_0} \left(M'_{2\delta_0} V_{22}^{-1} M_{2\delta_0} \right)^{-1} M'_{2\delta_0} V_{22}^{-1} \right) M_{2\lambda_0} \right]^{-1} \\ - (\iota_L \otimes I_K)' M_{2\delta_0} \left[M'_{2\delta_0} \left(V_{22}^{-1} - V_{22}^{-1} M_{2\lambda_0} \left(M'_{2\lambda_0} V_{22}^{-1} M_{2\lambda_0} \right)^{-1} M'_{2\lambda_0} V_{22}^{-1} \right) M_{2\delta_0} \right]^{-1} \\ \times \left(M'_{2\delta_0} V_{22}^{-1} M_{2\lambda_0} \right) \left(M'_{2\lambda_0} V_{22}^{-1} M_{2\lambda_0} \right)^{-1} \\ - (\iota_L \otimes I_K)' M_{2\lambda_0} \left(M'_{2\lambda_0} V_{22}^{-1} M_{2\lambda_0} \right)^{-1} \left(M'_{2\lambda_0} V_{22}^{-1} M_{2\delta_0} \right) \\ \times \left[M'_{2\delta_0} \left(V_{22}^{-1} - V_{22}^{-1} M_{2\lambda_0} \left(M'_{2\lambda_0} V_{22}^{-1} M_{2\lambda_0} \right)^{-1} M'_{2\lambda_0} V_{22}^{-1} \right) M_{2\delta_0} \right]^{-1} \\ (\iota_L \otimes I_K)' M_{2\delta_0} \left[M'_{2\delta_0} \left[V_{22}^{-1} - V_{22}^{-1} M_{2\lambda_0} \left(M'_{2\lambda_0} V_{22}^{-1} M_{2\lambda_0} \right)^{-1} M'_{2\lambda_0} V_{22}^{-1} \right] M_{2\delta_0} \right]^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} M'_{2\lambda_0} (\iota_L \otimes I_K) \\ M'_{2\delta_0} (\iota_L \otimes I_K) \end{pmatrix}$$

$$\begin{aligned} = \; & (\iota_L \otimes I_K)' M_{2\lambda_0} \left[M'_{2\lambda_0} \left(V_{22}^{-1} - V_{22}^{-1} M_{2\delta_0} \left(M'_{2\delta_0} V_{22}^{-1} M_{2\delta_0} \right)^{-1} M'_{2\delta_0} V_{22}^{-1} \right) M_{2\lambda_0} \right]^{-1} M'_{2\lambda_0} (\iota_L \otimes I_K) \\ & - (\iota_L \otimes I_K)' M_{2\delta_0} \left[M'_{2\delta_0} \left(V_{22}^{-1} - V_{22}^{-1} M_{2\lambda_0} \left(M'_{2\lambda_0} V_{22}^{-1} M_{2\lambda_0} \right)^{-1} M'_{2\lambda_0} V_{22}^{-1} \right) M_{2\delta_0} \right]^{-1} \\ & \times \left(M'_{2\delta_0} V_{22}^{-1} M_{2\lambda_0} \right) \left(M'_{2\lambda_0} V_{22}^{-1} M_{2\lambda_0} \right)^{-1} M'_{2\lambda_0} (\iota_L \otimes I_K) \\ & - (\iota_L \otimes I_K)' M_{2\lambda_0} \left(M'_{2\lambda_0} V_{22}^{-1} M_{2\lambda_0} \right)^{-1} \left(M'_{2\lambda_0} V_{22}^{-1} M_{2\delta_0} \right) \\ & \times \left[M'_{2\delta_0} \left(V_{22}^{-1} - V_{22}^{-1} M_{2\lambda_0} \left(M'_{2\lambda_0} V_{22}^{-1} M_{2\lambda_0} \right)^{-1} M'_{2\lambda_0} V_{22}^{-1} \right) M_{2\delta_0} \right]^{-1} M'_{2\delta_0} (\iota_L \otimes I_K) \\ & + (\iota_L \otimes I_K)' M_{2\delta_0} \left[M'_{2\delta_0} \left[V_{22}^{-1} - V_{22}^{-1} M_{2\lambda_0} \left(M'_{2\lambda_0} V_{22}^{-1} M_{2\lambda_0} \right)^{-1} M'_{2\lambda_0} V_{22}^{-1} \right] M_{2\delta_0} \right]^{-1} M'_{2\delta_0} (\iota_L \otimes I_K) \, . \end{aligned}$$

This, and similar calculations, give the penultimate expression for $\mathcal{I}^f(\beta)$ given in the proof.

The task is now to use the expression for M and V given in the proof to evaluate $\mathcal{I}^f(\beta)$. We begin by evaluating

$\underset{M\times M}{M_{2\lambda_0}' V_{22}^{-1} M_{2\lambda_0}}$ as

$$
\begin{aligned}
& \begin{pmatrix} \tau_{11}\nabla_{h_0} q_{0,11}' & \cdots & 0 & \cdots & \tau_{I1}\nabla_{h_0} q_{0,I1}' & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{1M}\nabla_{h_0} q_{0,1M}' & \cdots & 0 & \cdots & \tau_{IM}\nabla_{h_0} q_{0,IM}' \end{pmatrix} \\
& \times \begin{pmatrix} \frac{1}{\tau_{11}}\left\{\frac{\Sigma_{0,11}}{1-\rho_{11}}\right\}^{-1} & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & & & & \vdots \\ 0 & & \frac{1}{\tau_{1M}}\left\{\frac{\Sigma_{0,1M}}{1-\rho_{1M}}\right\}^{-1} & & & & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & & & & \frac{1}{\tau_{I1}}\left\{\frac{\Sigma_{0,I1}}{1-\rho_{I1}}\right\}^{-1} & & 0 \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & \frac{1}{\tau_{IM}}\left\{\frac{\Sigma_{0,IM}}{1-\rho_{IM}}\right\}^{-1} \end{pmatrix} \\
& \times \begin{pmatrix} \tau_{11}\nabla_{h_0} q_{0,11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{1M}\nabla_{h_0} q_{0,1M} \\ \vdots & & \vdots \\ \tau_{I1}\nabla_{h_0} q_{0,I1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{IM}\nabla_{h_0} q_{0,IM} \end{pmatrix} \\
= & \begin{pmatrix} \nabla_{h_0} q_{0,11}'\left\{\frac{\Sigma_{0,11}}{1-\rho_{11}}\right\}^{-1} & \cdots & 0 & \cdots & \nabla_{h_0} q_{0,I1}'\left\{\frac{\Sigma_{0,I1}}{1-\rho_{I1}}\right\}^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & \nabla_{h_0} q_{0,1M}'\left\{\frac{\Sigma_{0,1M}}{1-\rho_{1M}}\right\}^{-1} & \cdots & 0 & \cdots & \nabla_{h_0} q_{0,IM}'\left\{\frac{\Sigma_{0,IM}}{1-\rho_{IM}}\right\}^{-1} \end{pmatrix} \\
= & \times \begin{pmatrix} \tau_{11}\nabla_{h_0} q_{0,11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{1M}\nabla_{h_0} q_{0,1M} \\ \vdots & & \vdots \\ \tau_{I1}\nabla_{h_0} q_{0,I1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{IM}\nabla_{h_0} q_{0,IM} \end{pmatrix} \\
= & \begin{pmatrix} \sum_{i=1}^{I}\tau_{i1}\nabla_{h_0} q_{0,i1}'\left\{\frac{\Sigma_{0,i1}}{1-\rho_{i1}}\right\}^{-1}\nabla_{h_0} q_{0,i1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{i=1}^{I}\tau_{iM}\nabla_{h_0} q_{0,IM}'\left\{\frac{\Sigma_{0,IM}}{1-\rho_{IM}}\right\}^{-1}\nabla_{h_0} q_{0,IM} \end{pmatrix} \\
= & \begin{pmatrix} \varsigma_1\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left[\frac{\Sigma_0}{1-p}\right]^{-1}\left(\frac{\partial q_0}{\partial h_0'}\right)\middle| X_2 = x_{2,1}\right] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varsigma_M\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left[\frac{\Sigma_0}{1-p}\right]^{-1}\left(\frac{\partial q_0}{\partial h_0'}\right)\middle| X_2 = x_{2,M}\right] \end{pmatrix}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\underset{K\times M}{(\iota_L \otimes I_K)'} M_{0,2\lambda} &= -\begin{pmatrix} I_K & \cdots & I_K \end{pmatrix}' \begin{pmatrix} H_1 \\ \vdots \\ H_I \end{pmatrix} \\
&= -\sum_{i=1}^{I} \begin{pmatrix} \tau_{i1}\nabla_{h_0} q_{0,i1} & \cdots & \tau_{iM}\nabla_{h_0} q_{0,iM} \end{pmatrix} \\
&= -\left(\varsigma_1 \mathbb{E}\left[\left(\tfrac{\partial q_0}{\partial h_0'}\right)' \middle| X_2 = x_{2,1} \right] \quad \cdots \quad \varsigma_M \mathbb{E}\left[\left(\tfrac{\partial q_0}{\partial h_0'}\right)' \middle| X_2 = x_{2,M} \right] \right),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\underset{K\times J}{(\iota_L \otimes I_K)'} M_{0,2\delta} &= -\begin{pmatrix} I_K & \cdots & I_K \end{pmatrix}' \begin{pmatrix} \tau_1 \nabla_{\delta_0} q_{0,1} \\ \vdots \\ \tau_L \nabla_{\delta 0} q_{0,L} \end{pmatrix} = -\sum_{i=1}^{I}\sum_{m=1}^{M} \tau_{im} \nabla_{\delta} q_{0,im} \\
&= -\mathbb{E}\left[\mathbb{E}\left[\left(\frac{\partial q_0}{\partial \delta_0'} \right) \middle| X_2 \right] \right] \\
&= \mathbb{E}\left[\Delta_{\delta_0}(X_2) \right].
\end{aligned}
$$

Putting these three expressions together gives $(\iota_L \otimes I_K)' M_{0,2\lambda} \underset{K\times K}{\left(M_{0,2\lambda}' V_{22}^{-1} M_{0,2\lambda} \right)} M_{0,2\lambda}' (\iota_L \otimes I_K)$ equal to

$$
\begin{aligned}
& \sum_{m=1}^{M} \varsigma_m \mathbb{E}\left[\frac{\partial q_0}{\partial h_0'} \middle| X_2 = x_{2,m} \right] \mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'} \right)' \left[\frac{\Sigma_0}{1-p} \right]^{-1} \left(\frac{\partial q_0}{\partial h_0'} \right) \middle| X_2 = x_{2,m} \right]^{-1} \mathbb{E}\left[\frac{\partial q_0}{\partial h_0'} \middle| X_2 = x_{2,m} \right]' \\
= \; & \mathbb{E}\left[\mathbb{E}\left[\frac{\partial q_0}{\partial h_0'} \middle| X_2 \right] \mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'} \right)' \left[\frac{\Sigma_0}{1-p} \right]^{-1} \left(\frac{\partial q_0}{\partial h_0'} \right) \middle| X_2 \right]^{-1} \mathbb{E}\left[\frac{\partial q_0}{\partial h_0'} \middle| X_2 \right]' \right] \\
= \; & \mathbb{E}\left[\Delta_{h_0}(X_2)\, \Upsilon_{h_0}(X_2)^{-1}\, \Delta_{h_0}(X_2)' \right] \\
= \; & \mathbb{E}\left[A_0(X_2) \right].
\end{aligned}
$$

Next evaluate $M'_{2\lambda} \underset{M\times J}{V_{22}^{-1}} M_{2\delta}$ as

$$\begin{pmatrix} \tau_{11}\nabla_{h_0}q'_{0,11} & \cdots & 0 & \cdots & \tau_{I1}\nabla_{h_0}q'_{0,I1} & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{1M}\nabla_{h_0}q'_{0,1M} & \cdots & 0 & \cdots & \tau_{IM}\nabla_{h_0}q'_{0,IM} \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\tau_{11}}\left\{\frac{\Sigma_{0,11}}{1-\rho_{11}}\right\}^{-1} & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & & & & \vdots \\ 0 & & \frac{1}{\tau_{1M}}\left\{\frac{\Sigma_{0,1M}}{1-\rho_{1M}}\right\}^{-1} & & & & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & & & & \frac{1}{\tau_{I1}}\left\{\frac{\Sigma_{0,I1}}{1-\rho_{I1}}\right\}^{-1} & & 0 \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & \frac{1}{\tau_{IM}}\left\{\frac{\Sigma_{0,IM}}{1-\rho_{IM}}\right\}^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} \tau_1\nabla_{\delta_0}q_{0,1} \\ \vdots \\ \tau_L\nabla_{\delta_0}q_{0,L} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{I}\tau_{i1}\nabla_{h_0}q'_{0,i1}\left\{\frac{\Sigma_{0,i1}}{1-\rho_{i1}}\right\}^{-1}\nabla_{\delta_0}q_{0,i1} \\ \vdots \\ \sum_{i=1}^{I}\tau_{iM}\nabla_{h_0}q'_{0,iM}\left\{\frac{\Sigma_{0,iM}}{1-\rho_{iM}}\right\}^{-1}\nabla_{\delta_0}q_{0,iM} \end{pmatrix}$$

$$= \begin{pmatrix} \varsigma_1\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h'_0}\right)'\left\{\frac{\Sigma_0}{1-p}\right\}^{-1}\left(\frac{\partial q_0}{\partial \delta'_0}\right)\middle| X_2 = x_{2,1}\right] \\ \vdots \\ \varsigma_M\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h'_0}\right)'\left\{\frac{\Sigma_0}{1-p}\right\}^{-1}\left(\frac{\partial q_0}{\partial \delta'_0}\right)\middle| X_2 = x_{2,M}\right] \end{pmatrix}$$

and hence we evaluate $(\iota_L \otimes I_K)' \underset{K\times J}{\left(M_{2\delta} - M_{2\lambda} \left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda} \right)^{-1} M_{2\lambda}' V_{22}^{-1} M_{2\delta} \right)}$ equal to

$$
\begin{aligned}
& -\left[\mathbb{E}\left[\mathbb{E}\left[\frac{\partial q_0}{\partial \delta_0'} \middle| X_2 \right] \right] \right. \\
& -\left(\begin{array}{ccc} \varsigma_1 \mathbb{E}\left[\frac{\partial q_0}{\partial h_0'} \middle| X_2 = x_{2,1} \right] & \cdots & \varsigma_M \mathbb{E}\left[\frac{\partial q_0}{\partial h_0'} \middle| X_2 = x_{2,M} \right] \end{array} \right) \\
& \times diag \left\{ \begin{array}{ccc} \varsigma_1 \mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'} \right)' \left[\frac{\Sigma_0}{1-p} \right]^{-1} \left(\frac{\partial q_0}{\partial h_0'} \right) \middle| X_2 = x_{2,1} \right] & \cdots & \varsigma_M \mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'} \right)' \left[\frac{\Sigma_0}{1-p} \right]^{-1} \left(\frac{\partial q_0}{\partial h_0'} \right) \middle| X_2 = x_{2,M} \right] \end{array} \right\}^{-1} \\
& \times \left(\begin{array}{c} \varsigma_1 \mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'} \right)' \left\{ \frac{\Sigma_0}{1-p} \right\}^{-1} \left(\frac{\partial q_0}{\partial \delta_0'} \right) \middle| X_2 = x_{2,1} \right] \\ \vdots \\ \varsigma_M \mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'} \right)' \left\{ \frac{\Sigma_0}{1-p} \right\}^{-1} \left(\frac{\partial q_0}{\partial \delta_0'} \right) \middle| X_2 = x_{2,M} \right] \end{array} \right) \\
= \; & -\mathbb{E}\left[\mathbb{E}\left[\frac{\partial q_0}{\partial \delta_0'} \middle| X_2 \right] \right. \\
& \left. -\mathbb{E}\left[\frac{\partial q_0}{\partial h_0'} \middle| X_2 \right] \mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'} \right)' \left[\frac{\Sigma_0}{1-p} \right]^{-1} \left(\frac{\partial q_0}{\partial h_0'} \right) \middle| X_2 \right]^{-1} \mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'} \right)' \left\{ \frac{\Sigma_0}{1-p} \right\}^{-1} \left(\frac{\partial q_0}{\partial \delta_0'} \right) \middle| X_2 \right] \right] \\
= \; & -\mathbb{E}\left[\Delta_{\delta_0}(X_2) - \Delta_{h_0}(X_2) \Upsilon_{h_0}(X_2)^{-1} \Upsilon_{h_0\delta_0}(X_2) \right] \\
= \; & -\mathbb{E}\left[B_0(X_2) \right].
\end{aligned}
$$

Similarly we have

$$
\underset{J\times J}{M_{2\delta}' V_{22}^{-1} M_{2\delta}^{-1}} = \mathbb{E}\left[\mathbb{E}\left[\left(\frac{\partial q_0}{\partial \delta_0'} \right)' \left\{ \frac{\Sigma_0}{1-p} \right\}^{-1} \left(\frac{\partial q_0}{\partial \delta_0'} \right) \middle| X_2 \right] \right] = \mathbb{E}\left[\Upsilon_{\delta_0}(X_2) \right]
$$

and hence $M'_{2\delta}V_{22}^{-1}M_{2\lambda}\underbrace{\left(M'_{2\lambda}V_{22}^{-1}M_{2\lambda}\right)^{-1}}_{J\times J}M'_{2\lambda}V_{22}^{-1}M_{2\delta}^{-1}$ equal to

$$
\begin{aligned}
& \left(\begin{array}{ccc} \varsigma_1\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left\{\frac{\Sigma_0}{1-p}\right\}^{-1}\left(\frac{\partial q_0}{\partial \delta_0'}\right)\middle| X_2=x_{2,1}\right]' & \cdots & \varsigma_M\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left\{\frac{\Sigma_0}{p_0}\right\}^{-1}\left(\frac{\partial q_0}{\partial \delta_0'}\right)\middle| X_2=x_{2,M}\right]' \end{array}\right) \\
& \times \left(\begin{array}{ccc} \varsigma_1\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left[\frac{\Sigma_0}{1-p}\right]^{-1}\left(\frac{\partial q_0}{\partial h_0'}\right)\middle| X_2=x_{2,1}\right] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varsigma_M\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left[\frac{\Sigma_0}{1-p}\right]^{-1}\left(\frac{\partial q_0}{\partial h_0'}\right)\middle| X_2=x_{2,M}\right] \end{array}\right)^{-1} \\
& \times \left(\begin{array}{c} \varsigma_1\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left\{\frac{\Sigma_0}{1-p}\right\}^{-1}\left(\frac{\partial q_0}{\partial \delta_0'}\right)\middle| X_2=x_{2,1}\right] \\ \vdots \\ \varsigma_M\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left\{\frac{\Sigma_0}{1-p}\right\}^{-1}\left(\frac{\partial q_0}{\partial \delta_0'}\right)\middle| X_2=x_{2,M}\right] \end{array}\right) \\
= \; & \mathbb{E}\left[\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left\{\frac{\Sigma_0}{1-p}\right\}^{-1}\left(\frac{\partial q_0}{\partial \delta_0'}\right)\middle| X_2\right]'\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left[\frac{\Sigma_0}{1-p}\right]^{-1}\left(\frac{\partial q_0}{\partial h_0'}\right)\middle| X_2\right]^{-1}\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left\{\frac{\Sigma_0}{1-p}\right\}^{-1}\left(\frac{\partial q_0}{\partial \delta_0'}\right)\middle| X_2\right]\right] \\
= \; & \mathbb{E}\left[\Upsilon_{h_0\delta_0}(X_2)'\Upsilon_{h_0}(X_2)^{-1}\Upsilon_{h_0\delta_0}(X_2)\right],
\end{aligned}
$$

which then gives $M'_{2\delta}V_{22}^{-1}M_{2\delta}^{-1}-M'_{2\delta}V_{22}^{-1}M_{2\lambda}\left(M'_{2\lambda}V_{22}^{-1}M_{2\lambda}\right)^{-1}M'_{2\lambda}V_{22}^{-1}M_{2\delta}^{-1}$ equal to

$$
\begin{aligned}
& \mathbb{E}\left[\mathbb{E}\left[\left(\frac{\partial q_0}{\partial \delta_0'}\right)'\left\{\frac{\Sigma_0}{1-p}\right\}^{-1}\left(\frac{\partial q_0}{\partial \delta_0'}\right)\middle| X_2\right]\right. \\
& \left. -\,\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left\{\frac{\Sigma_0}{1-p}\right\}^{-1}\left(\frac{\partial q_0}{\partial \delta_0'}\right)\middle| X_2\right]'\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left[\frac{\Sigma_0}{1-p}\right]^{-1}\left(\frac{\partial q_0}{\partial h_0'}\right)\middle| X_2\right]^{-1}\mathbb{E}\left[\left(\frac{\partial q_0}{\partial h_0'}\right)'\left\{\frac{\Sigma_0}{1-p}\right\}^{-1}\left(\frac{\partial q_0}{\partial \delta_0'}\right)\middle| X_2\right]\right] \\
= \; & \mathbb{E}\left[\Upsilon_{\delta_0}(X_2)-\Upsilon_{h_0\delta_0}(X_2)'\Upsilon_{h_0}(X_2)^{-1}\Upsilon_{h_0\delta_0}(X_2)\right] \\
= \; & \mathbb{E}\left[C_0(X_2)\right].
\end{aligned}
$$

The expressions for $A_1(X_2)$, $B_1(X_2)$ and $C_1(X_2)$ can be derived analogously.

D Details of calculations used to in proof of Theorem 6.1

Multiplying $M^{-1}VM^{-1\prime}$ gives

$$
\left(\begin{array}{cc} M_{1\rho}^{-1}V_{11}M_{1\rho}^{-1\prime} & -M_{1\rho}^{-1}\left[V_{11}M_{1\rho}^{-1\prime}M'_{2\rho}+V_{12}\right]M_{2\beta}^{-1\prime} \\ -M_{2\beta}^{-1}\left[M_{2\rho}M_{1\rho}^{-1}V_{11}+V'_{12}\right]M_{1\rho}^{-1\prime} & M_{2\beta}^{-1}\left[M_{2\rho}M_{1\rho}^{-1}V_{11}M_{1\rho}^{-1\prime}M'_{2\rho}-V'_{12}M_{1\rho}^{-1\prime}M'_{2\rho}-M_{2\rho}M_{1\rho}^{-1}V_{12}+V_{22}\right]M_{2\beta}^{-1\prime} \end{array}\right).
$$

The variance bound for β is given by the lower right-hand block of this matrix. Using the V and M components given in the main paper we have

$$\begin{aligned} M_{2\rho}M_{1\rho}^{-1}V_{11}M_{1\rho}^{-1\prime} &= -\frac{1}{Q_0}\begin{pmatrix} \tau_1\frac{q_{0,1}}{1-\rho_1} & \cdots & \tau_L\frac{q_{0,L}}{1-\rho_L} \end{pmatrix} \\ &\quad\times\begin{pmatrix} \frac{\tau_1}{\rho_1} & & 0 \\ & \ddots & \\ 0 & & \frac{\tau_L}{\rho_L} \end{pmatrix}^{-1}\times\begin{pmatrix} \tau_1\frac{1-\rho_1}{\rho_1} & & 0 \\ & \ddots & \\ 0 & & \tau_L\frac{1-\rho_L}{\rho_L} \end{pmatrix}\times\begin{pmatrix} \frac{\tau_1}{\rho_1} & & 0 \\ & \ddots & \\ 0 & & \frac{\tau_L}{\rho_L} \end{pmatrix}^{-1} \\ &= -\frac{1}{Q}\begin{pmatrix} \rho_1 q_{0,1} & \cdots & \rho_L q_{0,L} \end{pmatrix}, \end{aligned}$$

and hence

$$\begin{aligned} M_{2\rho}M_{1\rho}^{-1}V_{11}M_{1\rho}^{-1\prime}M_{2\rho}' &= \frac{1}{Q^2}\begin{pmatrix} \rho_1 q_{0,1} & \cdots & \rho_L q_{0,L} \end{pmatrix}\times\begin{pmatrix} \tau_1\frac{q_{0,1'}}{1-\rho_1} \\ \\ \tau_L\frac{q_{0,L}'}{1-\rho_L} \end{pmatrix} \\ &= \frac{1}{Q^2}\sum\nolimits_{l=1}^{L}\tau_l\frac{\rho_l}{1-\rho_l}q_{0,l}q_{0,l}' \\ &= \frac{1}{Q^2}\mathbb{E}\left[\frac{p(X)}{1-p(X)}q_0(X)\,q_0(X)'\right]. \end{aligned}$$

Similarly we have

$$\begin{aligned} V_{12}'M_{1\rho}^{-1\prime} &= -\frac{1}{Q}\begin{pmatrix} \tau_1(1-\rho_1)\,q_{1,1}+\tau_1\rho_1 q_{0,1} & \cdots & \tau_L(1-\rho_L)\,q_{1,L}+\tau_L\rho_L q_{0,L} \end{pmatrix}\times\begin{pmatrix} \frac{\tau_1}{\rho_1} & & 0 \\ & \ddots & \\ 0 & & \frac{\tau_L}{\rho_L} \end{pmatrix}^{-1} \\ &= -\frac{1}{Q}\begin{pmatrix} \rho_1(1-\rho_1)\,q_{1,1}+\rho_1^2 q_{0,1} & \cdots & \rho_L(1-\rho_L)\,q_{1,L}+\rho_L^2 q_{0,L} \end{pmatrix}, \end{aligned}$$

and hence

$$\begin{aligned} V_{12}'M_{1\rho}^{-1\prime}M_{2\rho}' &= \frac{1}{Q^2}\begin{pmatrix} \rho_1(1-\rho_1)\,q_{1,1}+\rho_1^2 q_{0,1} & \cdots & \rho_L(1-\rho_L)\,q_{1,L}+\rho_L^2 q_{0,L} \end{pmatrix}\times\begin{pmatrix} \tau_1\frac{q_{0,1'}}{1-\rho_1} \\ \\ \tau_L\frac{q_{0,L}'}{1-\rho_L} \end{pmatrix} \\ &= \frac{1}{Q^2}\sum\nolimits_{l=1}^{L}\tau_l\left[\rho_l q_{1,l}q_{0,l}'+\frac{\rho_l^2}{1-\rho_l}q_{0,l}q_{0,l}'\right] \\ &= \frac{1}{Q^2}\mathbb{E}\left[p(X)\,q_1(X)\,q_0(X)'+\frac{p(X)^2}{1-p(X)}q_0(X)\,q_0(X)'\right]. \end{aligned}$$

Putting these calculations together gives

$$
\begin{aligned}
&= M_{2\rho}M_{1\rho}^{-1}V_{11}M_{1\rho}^{-1\prime}M_{2\rho}' - V_{12}'M_{1\rho}^{-1\prime}M_{2\rho}' - M_{2\rho}M_{1\rho}^{-1}V_{12} + V_{22} \\
&= \frac{1}{Q^2}\sum\nolimits_{l=1}^{L}\tau_l\frac{\rho_l}{1-\rho_l}q_{0,l}q_{0,l}' - \frac{1}{Q_0^2}\sum\nolimits_{l=1}^{L}\tau_l\left[\rho_l q_{1,l}q_{0,l}' + \frac{\rho_l^2}{1-\rho_l}q_{0,l}q_{0,l}'\right] \\
&\quad -\frac{1}{Q^2}\sum\nolimits_{l=1}^{L}\tau_l\left[\rho_l q_{0,l}q_{1,l}' + \frac{\rho_l^2}{1-\rho_l}q_{0,l}q_{0,l}'\right] \\
&\quad +\sum\nolimits_{l=1}^{L}\tau_l\frac{\rho_l^2}{Q_0}\left[\frac{\Sigma_{1,l}}{\rho_l} + \frac{1-\rho_l}{\rho_l}q_{1,l}q_{1,l}' + q_{1,l}q_{1,l}' + \frac{\Sigma_{0,l}}{1-\rho_l} + \frac{\rho_l}{1-\rho_l}q_{0,l}q_{0,l}' + q_{0,l}q_{0,l}'\right] \\
&= \sum\nolimits_{l=1}^{L}\tau_l\frac{\rho_l^2}{Q_0^2}\left\{\frac{q_{0,l}q_{0,l}'}{\rho_l(1-\rho_l)} - \frac{q_{1,l}q_{0,l}'}{\rho_l} - \frac{q_{0,l}q_{0,l}'}{1-\rho_l} - \frac{q_{0,l}q_{1,l}'}{\rho_l} - \frac{q_{0,l}q_{0,l}'}{1-\rho_l}\right. \\
&\quad \left. +\frac{\Sigma_{1,l}}{\rho_l} + \frac{1-\rho_l}{\rho_l}q_{1,l}q_{1,l}' + q_{1,l}q_{1,l}' + \frac{\Sigma_{0,l}}{1-\rho_l} + \frac{\rho_l}{1-\rho_l}q_{0,l}q_{0,l}' + q_{0,l}q_{0,l}'\right\} \\
&= \sum\nolimits_{l=1}^{L}\tau_l\frac{\rho_l^2}{Q_0^2}\left\{\frac{q_{0,l}q_{0,l}'}{\rho_l(1-\rho_l)} - \frac{q_{1,l}q_{0,l}'}{\rho_l} - \frac{q_{0,l}q_{1,l}'}{\rho_l} - \frac{q_{0,l}q_{0,l}'}{1-\rho_l} + \frac{q_{1,l}q_{1,l}'}{\rho_l}\right. \\
&\quad \left. +\frac{\Sigma_{1,l}}{\rho_l} + \frac{\Sigma_{0,l}}{1-\rho_l}\right\} \\
&= \sum\nolimits_{l=1}^{L}\tau_l\frac{\rho_l^2}{Q_0^2}\left\{\frac{q_{0,l}q_{0,l}'}{\rho_l} - \frac{q_{1,l}q_{0,l}'}{\rho_l} - \frac{q_{0,l}q_{1,l}'}{\rho_l} + \frac{q_{1,l}q_{1,l}'}{\rho_l}\right. \\
&\quad \left. +\frac{\Sigma_{1,l}}{\rho_l} + \frac{\Sigma_{0,l}}{1-\rho_l}\right\} \\
&= \sum\nolimits_{l=1}^{L}\tau_l\frac{\rho_l^2}{Q_0^2}\left\{\frac{\Sigma_{1,l}}{\rho_l} + \frac{\Sigma_{0,l}}{1-\rho_l} + \frac{1}{\rho_l}\left(q_{1,l}-q_{0,l}\right)\left(q_{1,l}-q_{0,l}\right)'\right\} \\
&= \mathbb{E}\left[\frac{p(X)^2}{Q^2}\left\{\frac{\Sigma_1(X)}{p(X)} + \frac{\Sigma_0(X)}{1-p_0(X)} + \frac{1}{p(X)}\left(q_1(X)-q_0(X)\right)\left(q_1(X)-q_0(X)\right)'\right\}\right] \\
&= \mathbb{E}\left[\Phi(X)\right],
\end{aligned}
$$

for $\Phi(x)$ as defined by (10) of the paper.

For the case where $p(X)$ is known we need to evaluate

$$
\begin{aligned}
V_{12}' V_{11}^{-1} V_{12} &= \frac{1}{Q^2} \begin{pmatrix} \tau_1 (1-\rho_1) q_{1,1} + \tau_1 \rho_1 q_{0,1} & \cdots & \tau_L (1-\rho_L) q_{1,L} + \tau_L \rho_L q_{0,L} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \tau_1 \frac{1-\rho_1}{\rho_1} & & 0 \\ & \ddots & \\ 0 & & \tau_L \frac{1-\rho_L}{\rho_L} \end{pmatrix}^{-1} \times \begin{pmatrix} \tau_1 (1-\rho_1) q_{1,1}' + \tau_1 \rho_1 q_{0,1}' \\ \vdots \\ \tau_L (1-\rho_L) q_{1,L}' + \tau_L \rho_L q_{0,L}' \end{pmatrix} \\
&= \frac{1}{Q^2} \begin{pmatrix} \rho_1 q_{1,1} + \frac{\rho_1^2}{1-\rho_1} q_{0,1} & \cdots & \rho_L q_{1,L} + \frac{\rho_L^2}{1-\rho_L} q_{0,L} \end{pmatrix} \times \begin{pmatrix} \tau_1 (1-\rho_1) q_{1,1}' + \tau_1 \rho_1 q_{0,1}' \\ \vdots \\ \tau_L (1-\rho_L) q_{1,L}' + \tau_L \rho_L q_{0,L}' \end{pmatrix} \\
&= \frac{1}{Q^2} \sum_{l=1}^{L} \tau_l \rho_l (1-\rho_l) q_{1,l} q_{1,l}' + \tau_l \rho_l^2 q_{1,l} q_{0,l}' + \tau_l \rho_l^2 q_{0,l} q_{1,l}' + \tau_l \frac{\rho_L^3}{1-\rho_L} q_{0,l} q_{0,l}' \\
&= \sum_{l=1}^{L} \tau_l \frac{\rho_l^2}{Q^2} \left\{ \frac{1-\rho_l}{\rho_l} q_{1,l} q_{1,l}' + q_{1,l} q_{0,l}' + q_{0,l} q_{1,l}' + \frac{\rho_L}{1-\rho_L} q_{0,l} q_{0,l}' \right\} \\
&= \mathbb{E}\left[\frac{p(X)^2}{Q^2} \left\{ \frac{1-p(X)}{p(X)} q_1(X) q_1(X)' + q_1(X) q_0(X)' \right.\right. \\
&\quad \left.\left. + q_0(X) q_1(X)' + \frac{p(X)}{1-p(X)} q_0(X) q_0(X)' \right\} \right],
\end{aligned}
$$

which then gives

$$
V_{22} - V_{12}' V_{11}^{-1} V_{12} = \mathbb{E}[\Phi(X)],
$$

with $\Phi(x)$ as defined by (8) of the paper.