Framing Contingencies*

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Abstract

The subjective likelihood of a contingency often depends on the manner in which it is described to the decision maker. To accommodate this dependence, we introduce a model of decision making under uncertainty which takes as primitive a family of preferences indexed by partitions of the state space. Each partition corresponds to a description of the state space. We characterize the following *partition-dependent expected utility* representation. The decision maker has a nonadditive set function ν over events. Given a partition of the state space, she computes expected utility with respect to her partition-dependent belief, which weights each cell in the partition by ν. Nonadditivity of ν allows the probability of an event to depend on the way in which the state space is described. We propose behavioral definitions for those events which are transparent to the decision maker and those which are completely overlooked, and connect these definitions to conditions on the representation.

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1 Introduction

This paper formally incorporates the framing of contingencies into decision making under uncertainty. Its primitives are descriptions of acts, which map contingencies to outcomes. For example, the following health insurance policy associates deductibles on the left with contingencies on the right:

$$\begin{pmatrix} \$500 & \text{surgery} \\ \$100 & \text{prenatal care} \\ \vdots & \vdots \end{pmatrix}.$$

Compare this to the following policy, which includes some redundancies:

$$\begin{pmatrix} \$500 & \text{laminotomy} \\ \$500 & \text{other surgeries} \\ \$100 & \text{prenatal care} \\ \vdots & \vdots \end{pmatrix}.$$

Both policies provide effectively identical levels of coverage. Nonetheless, a consumer might evaluate them differently. The second one explicitly mentions laminotomies, which she may overlook or fail to fully consider when evaluating the first contract. This oversight is behaviorally revealed if the consumer is willing to pay a higher premium for the second contract, reflecting an increased personal belief of the likelihood of surgery after laminotomies are mentioned.

The primary methodological innovation of the paper is its ability to discriminate between different presentations of the same act. Our general model expands the standard subjective model of decision making under uncertainty. We introduce a richer set of primitives which distinguishes the different expressions for an act as distinct choice objects. In particular, lists of contingencies with associated outcomes are the primitive objects of choice. Choices over lists are captured by a family of preferences, where each preference is indexed by a partition of the state space. We interpret the partition as a description of the different events. Equipped with this primitive, we present axioms which characterize the suggested partition-dependent expected utility representation. To our knowledge, this is the first axiomatic attempt to incorporate framing of contingencies as a consideration in decision making.[1]

We characterize the following utility function, which we call *partition-dependent expected utility*. Each event E carries a weight $\nu(E)$. Each outcome x delivers a utility $u(x)$. When presented a list $E_1, \ldots, E_n$ of contingencies which partition the state space, the decision

[1] A recent paper by Bourgeois-Gironde and Giraud (forthcoming) considers presentation effects in the Bolker–Jeffrey decision model.

maker judges the probability of E_i to be $\nu(E_i)/\sum_j \nu(E_j)$. Suppose $E = F \cup G$ with F, G disjoint. Since ν is not necessarily additive, the judged likelihood of event $E = F \cup G$ can depend on whether it is coarsely expressed as E or finely expressed as the union of two subevents $F \cup G$. The utility for a list

$$\begin{pmatrix} x_1 & E_1 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix}$$

is obtained by aggregating her utilities $u(x_i)$ over the consequences x_i by the normalized weights $\nu(E_i)/\sum_j \nu(E_j)$ on their corresponding events E_i. This particular functional form departs modestly and parsimoniously from standard expected utility by relaxing the additivity of ν. Indeed, given a fixed list $E_1, \ldots, E_n$ of events, it maintains the affine aggregation and probabilistic sophistication of standard expected utility.[2]

Savage (1954) and Anscombe and Aumann (1963) do not distinguish different presentations of the same act. They implicitly assume the psychological principle of extensionality, that the framing of an event is inconsequential to its judged likelihood. Despite its normative appeal, extensionality is violated in experiments where unpacking a contingency into finer subcontigencies affects its perceived likelihood. In a classic experiment, Fischoff, Slovic, and Lichtenstein (1978) told car mechanics that a car fails to start and asked for the likelihood that different parts could cause the failure. The mechanics' likelihood assessments depended on whether a part's subcomponents were explicitly listed.

Tversky and Koehler (1994) proposed a nonextensional model of judgment, which they called support theory. Support theory begins with a function $P(A, B)$, which reflects the likelihood of a hypothesis A given that A or the mutually exclusive hypothesis B holds. It connects these likelihoods by asserting $P(A, B) = \frac{s(A)}{s(A)+s(B)}$ where $s(\cdot)$ is a nonadditive "support function" over different hypotheses. Support theory enjoys considerable success among psychologists for its ability to "accommodate many mechanisms ... that influence subjective probability, but integrate them via the construct of the support" (Brenner, Koehler, and Rottenstreich 2002). This paper contributes to the development of support theory by: first, providing a decision theoretic model and foundation for support theory; second, studying the uniqueness of the support function under different assumptions on the behavioral data; third, identifying new classes of events which have special properties in terms of their support.

One interpretation of nonextensionality is through unforeseen contingencies. The general idea of a decision maker with a coarse understanding of the state space appears in

[2] In fact, we will shortly directly impose the Anscombe–Aumann representation on preferences given a fixed list of contingencies. As will be clear in the sequel, the belief will change between lists.

papers by Dekel, Lipman, and Rustichini (2001), by Epstein, Marinacci, and Seo (2007), by Ghirardato (2001), and by Mukerji (1997). Our contribution is to compare preferences across descriptions to identify which contingencies had been unforeseen. This basic insight of using the explicit expression of unforeseen contingencies as a foundation for their identification was anticipated in psychology and in economics. Tversky and Koehler (1994, p. 565) connect nonextensional judgment and unforeseen contingencies: "The failures of extensionality ... highlight what is perhaps the fundamental problem of probability assessment, namely the need to consider unavailable possibilities ... People ... cannot be expected ... to generate all relevant future scenarios." Dekel, Lipman, and Rustichini (1998a, p. 524) distinguish unforeseen contingencies from null events, because "an 'uninformative' statement – such as 'event x might or might not happen' – can change the agent's decision." Our model formally executes their suggested test.

Beyond unforeseen contingencies are other psychological sources for nonextensional judgment. A first source is limited memory or recall. For example, the car mechanics surveyed by Fischoff, Slovic, and Lichtenstein (1978) had surely heard of the mechanical failures before. To explain nonextensionality, Tversky and Koehler (1994, p. 549) appeal to "memory and attention ... Unpacking a category ... into its components ... might remind people of possibilities that would not have been considered otherwise."

A second source of nonextensionality is that different descriptions alter the salience of events. For example, Fox and Rottenstreich (2003) ask subjects to report the probability that Sunday will be the hottest day of the coming week. Their reports depend significantly on whether the rest of the week is described as a single event or separated as Monday, Tuesday, and so on, with a mean of $\frac{1}{3}$ in the first case and of $\frac{1}{7}$ in the latter. In such cases, descriptions effect probability judgments without suggesting unforeseen or unrecalled cases.

The next section introduces the primitives of our theory. Section 3 defines the suggested partition-dependent expected utility representation. Section 4 axiomatizes the representation and discusses the uniqueness of its components. Finally, Section 5 defines two families of events, those which are completely understood and those which are completely overlooked, and examines the structure of these families when the representation holds.

2 A nonextensional model of decision making

This section introduces the primitives of the model. The closest formalism of which we are aware is the model of decision making under ignorance by Cohen and Jaffray (1980), which also considers different descriptions of the state space.[3] However, they impose as a normative condition that preference is invariant to the manner in which the states are

[3]We thank Raphaël Giraud for bringing this work to our attention.

expressed, while this dependence is exactly our focus.

Let S denote a state space. A finite partition of S is a nonempty and pairwise disjoint collection of subsets $\pi = \{E_1, \dots, E_n\}$ such that $S = \bigcup_{i=1}^n E_n$. The events $E_1, \dots, E_n$ are called the cells of partition π.[4] Let Π^* denote the collection of all finite partitions of S. We interpret each partition $\pi \in \Pi^*$ as a description of the state space S: it explicitly mentions categories of possible states, where each cell of the partition is a category, and these categories are comprehensive. For any $\pi \in \Pi^*$, let $\sigma(\pi)$ denote the algebra induced by π.[5] Define the binary relation $\geq$ on Π^* by $\pi' \geq \pi$ if $\sigma(\pi') \supset \sigma(\pi)$, i.e. if π' is finer than π. If $\pi' \geq \pi$, then π' is a richer description of the state space than π. The meet $\pi \wedge \pi'$ and join $\pi \vee \pi'$ respectively denote the finest common coarsening and the coarsest common refinement of π and π'. For any event $E \subset S$, let Π^*_E denote the set of finite partitions of E. If $E \in \pi \in \Pi^*$ and $\pi_E \in \Pi^*_E$, we slightly abuse notation and let $\pi \vee \pi_E$ denote $\pi \vee [\pi_E \cup \{E^{\complement}\}]$. The model considers a set of descriptions $\Pi \subset \Pi^*$. We assume that Π includes the vacuous description $\{S\}$ and is closed under $\wedge$ and $\vee$. Some definitions in the sequel will reference two collections of events. First, let $\mathcal{C} = \cup_{\pi \in \Pi} \pi$ denote the collection of cells of partitions in Π. Second, let $\mathcal{E} = \bigcup_{\pi \in \Pi} \sigma(\pi)$ denote the collection of all unions of cells of some partition in Π. Clearly, $\mathcal{E}$ is the algebra generated by $\mathcal{C}$. Most results will focus on two cases of Π. In the first case, descriptions can be indexed so they become progressively finer, in which case Π is a filtration. In the second, all possible descriptions are included, in which case $\Pi = \Pi^*$. We will discuss the distinction shortly.

Let X denote a finite set of consequences or prizes. Invoking the Anscombe–Aumann structure, let ΔX denote the set of lotteries on X. An act $f : S \to \Delta X$ maps states to lotteries. Slightly abusing notation, let $p \in \Delta X$ also denote the corresponding constant act. Let $\mathcal{F}_\pi$ denote the family of acts which respect the partition π, i.e. $f^{-1}(p) \in \sigma(\pi)$ for all $p \in \Delta X$. In words, the act f is $\sigma(\pi)$-measurable if it assigns a constant lottery to all states in a particular cell of the partition: if $s, s' \in E \in \pi$, then $f(s) = f(s')$. Informally, $\mathcal{F}_\pi$ is the set of acts of contracts which can be described using the descriptive power of π; an act $g \notin \mathcal{F}_\pi$ requires a finer categorization than is available in π. Let $\mathcal{F} = \bigcup_{\pi \in \Pi} \mathcal{F}_\pi$ denote the universe of acts under consideration. For any act $f \in \mathcal{F}$, let $\pi(f)$ denote the coarsest available partition $\pi \in \Pi$ such that $f \in \mathcal{F}_\pi$.[6] Note that when $\Pi \neq \Pi^*$, because $\pi(f)$ is the coarsest partition within Π, it could be strictly finer than the partition induced by f, i.e. the coarsest partition (among all partitions) which makes f measurable. Similarly for any pair of acts $f, g \in \mathcal{F}$, let $\pi(f, g)$ be the coarsest available partition $\pi \in \Pi$ such that

[4] For any partition $\pi \in \Pi^*$, we adopt the convention where $\pi \cup \{\emptyset\}$ is identified with π.

[5] Since π is finite, $\sigma(\pi)$ is the family of unions of cells in π and the empty set.

[6] The existence of $\pi(f)$ is guaranteed by our assumption that Π is closed under the operation $\wedge$. To see this, let $\pi \in \Pi$ be any partition according to which f is measurable. Since π is a finite partition, there are finitely many partitions that are (weakly) coarser than π. Hence the set $\Pi' = \{\pi \in \Pi \mid \pi' \in \Pi, \pi' \leq \pi, \& f \in \mathcal{F}_{\pi'}\}$ is finite and nonempty, and $\pi(f) = \wedge_{\pi' \in \Pi'} \pi'$.

$f, g \in \mathcal{F}_\pi$.

Our primitive is a *family* of preferences $\{\succsim_\pi\}_{\pi \in \Pi}$ indexed by partitions π, where each $\succsim_\pi$ is defined over the family $\mathcal{F}_\pi$ of π-measurable acts. Our interpretation of $f \succsim_\pi g$ is that f is weakly preferred to g when the state space is described as the partition π. If $f \notin \mathcal{F}_\pi$, then the description π is too coarse to express the structure of f. If either f or g is not π-measurable, then the statement $f \succsim_\pi g$ is nonsensical. The strict and symmetric components $\succ_\pi$ and $\sim_\pi$ carry their standard meanings.

The restriction to π-measurable acts is not innocuous, particularly when framing effects are interpreted as reflecting unawareness. Consider a health insurance contract which covers eighty percent of the cost of surgery. The exact benefit of the insurance depends on which surgery is required, about which the consumer might have only a vague understanding. Nonetheless, its terms are described without explicitly mentioning every possible surgery. The measurability assumption precludes such contracts, a limitation of our model.

Our original motivation was to study preferences over lists. The family of preferences $\{\succsim_\pi\}$ provides a parsimonious primitive which loses little descriptive power relative to a model which begins with preferences over lists. Suppose we started with a list

$$\begin{pmatrix} x_1 & E_1 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix}$$

which is a particular expression of the act f. This list is more compactly represented as a pair (f, π), where the partition $\pi = \{E_1, \dots, E_n\}$ denotes the list of explicit contingencies on the right. This description π is necessarily richer than the coarsest expression of f, so $f \in \mathcal{F}_\pi$. Now suppose the decision maker is deciding between two lists, which are represented as (f, π_1) and (g, π_2). Then the events in both π_1 and π_2 are explicitly mentioned. So the family of described events is the coarsest common refinement of π_1 and π_2, their join $\pi = \pi_1 \vee \pi_2$. Then (f, π_1) is preferred to (g, π_2) if and only if (f, π) is preferred to (g, π). We can therefore restrict attention to the preferences over pairs (f, π) and (g, π) where $f, g \in \mathcal{F}_\pi$. Moving the partition from being carried by the acts to being carried as an index of the preference relation arrives at exactly the model studied here.

The lists are expressed through indexed preference relations for the resulting economy of notation. The partition π which indexes $f \succsim_\pi g$ is the coarsest refinement of the observable descriptions in the lists π_1 and π_2 which accompanied f and g. The partition π is not meant to be interpreted as anything more. It is exogenous information which is an observable component of the decision problem, and should not be taken as a direct measure of the decision maker's subjective understanding of the state space. In fact, Section 5 suggests a method for inferring her subjective understanding of the state space from her preferences over lists.

An important consideration is exactly which preferences are available or observable to the analyst. How rich are the preferences which can be sensibly elicited from the decision maker? This question speaks directly to the structure of the collection Π. Consider the interpretation of framing in terms of availability or recall. Once an event is explicitly mentioned to the decision maker, this pronouncement cannot be reversed. In this case, after being presented with prior partitions $\pi_1, \ldots, \pi_{t-1}$, the relevant behavior after also being told π_t is with respect to the refinement of the prior presentations $\pi_1, \ldots, \pi_{t-1}$ and the current π_t. So the appropriate assumption in this case is that Π is a filtration.

On the other hand, under different motivations for framing, it seems more reasonable to consider the family of all descriptions. For example, if framing effects are due to salience, these effects are independent of the decision maker's ability to recall events. A similar argument can be made for the representativeness heuristic.[7] Even for motivations where preferences under the full set of descriptions cannot be elicited for a single subject, the analyst could believe there is enough uniformity in the population to elicit preferences across subjects, in which case a particular description could be given to one subject while alternative descriptions are given to others. Similarly, it might be useful to consider counterfactual assessments about what a particular decision maker would have done if she had been presented alternative sequences of descriptions.

We therefore consider two canonical cases. In the first, Π is a filtration. In the second, Π is the family of all finite partitions. The appropriateness of either case depends on the application. Neither case is obviously more technically challenging. When Π is larger, the theory leverages more information about the decision maker, but also must rationalize more of her choices.

Given a partition $\pi = \{E_1, \ldots, E_n\} \subset \mathcal{E}$ and acts $f_1, \ldots, f_n \in \mathcal{F}$ define a new act by:

$$\begin{pmatrix} f_1 & E_1 \\ \vdots & \vdots \\ f_n & E_n \end{pmatrix}(s) = \begin{cases} f_1(s) & \text{if } s \in E_1 \\ \vdots & \vdots \\ f_n(s) & \text{if } s \in E_n \end{cases}.$$ [8]

The following defines null events for our setting with a family of preferences.

[7] Consider the famous "Linda problem," where subjects are told that "Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations." The subjects believe the event "Linda is a bank teller" is less probable than the event "Linda is a bank teller and is active in the feminist movement" (Tversky and Kahneman 1983, p. 297).

[8] Note that the partition π does not necessarily belong to Π. However the assumption that $\pi \subset \mathcal{E}$ guarantees that π is coarser than some partition $\pi' \in \Pi$. To see this, let $\pi_i \in \Pi$ be such that $E_i \in \sigma(\pi_i)$ for each $i = 1, \ldots, n$ and let $\pi' = \pi_1 \vee \pi_2 \vee \ldots \vee \pi_n \in \Pi$. Then $\pi \leq \pi'$ and the new act defined above belongs to $\mathcal{F}_{\pi' \vee \pi(f_1) \vee \ldots \vee \pi(f_n)} \subset \mathcal{F}$.

Definition 1. Given $\pi \in \Pi$, an event $E \in \sigma(\pi)$ is **π-null** if

$$\begin{pmatrix} p & E \\ f & E^{\complement} \end{pmatrix} \sim_\pi \begin{pmatrix} q & E \\ f & E^{\complement} \end{pmatrix},$$

for all $f \in \mathcal{F}_\pi$ and $p, q \in \Delta X$. $E \in \sigma(\pi)$ is **π-nonnull** if it is not π-null. The event E is **null** if E is π-null for any π such that $E \in \pi$. E is **nonnull** if it is not null.[9]

3 Partition-dependent expected utility

We study the following utility representation. The decision maker has a nonnegative set function $\nu : \mathcal{C} \to \mathbb{R}_+$ over relevant contingencies. Presented with a description $\pi = \{E_1, E_2, \ldots, E_n\}$ of the state space, she places a weight $\nu(E_k)$ on each described event. Following Tversky and Koehler (1994), we refer to $\nu(E)$ as the support of E. Normalizing by the sum, $\mu_\pi(E_k) = \nu(E_k)/\sum_i \nu(E_i)$ defines a probability measure μ_π over $\sigma(\pi)$. Then, her utility for the act f expressed as:

$$f = \begin{pmatrix} p_1 & E_1 \\ p_2 & E_2 \\ \vdots & \vdots \\ p_n & E_n \end{pmatrix},$$

is $\sum_{i=1}^n u(p_i)\mu_\pi(E_i)$, where $u : \Delta X \to \mathbb{R}$ is an affine utility function over objective lotteries.

The following avoids division by zero during the normalization

Definition 2. A **support function** is a weakly positive set function $\nu : \mathcal{C} \to \mathbb{R}_+$ such that $\sum_{E \in \pi} \nu(E) > 0$ for all $\pi \in \Pi$.

Although $\emptyset$ is not in $\mathcal{C}$ since it is not an element of any partition, we will adhere to the convention that $\nu(\emptyset) = 0$. We can now formally define the utility representation.

Definition 3. $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a **partition-dependent expected utility (PDEU)** representation if there exist a nonconstant affine vNM utility function $u : \Delta X \to \mathbb{R}$ and a support function $\nu : \mathcal{C} \to \mathbb{R}_+$ such that for all $\pi \in \Pi$ and $f, g \in \mathcal{F}_\pi$:

$$f \succsim_\pi g \iff \int_S u \circ f \, d\mu_\pi \geq \int_S u \circ g \, d\mu_\pi,$$

[9]Note that for an event E to be nonnull, E only needs to be nonnull for some partition π including E, but not necessarily for all partitions whose algebras include E.

where μ_π is the unique probability measure on $(S, \sigma(\pi))$ such that, for all $E \in \pi$:

$$\mu_\pi(E) = \frac{\nu(E)}{\sum_{F \in \pi} \nu(F)}. \tag{1}$$

When such a pair (u, ν) exists, we call it a PDEU representation.

The support $\nu(E)$ corresponds to the relative weight of E in lists where E, but not its subevents, are explicitly mentioned. The nonadditivity of ν allows for framing effects: E and F can be disjoint yet $\nu(E) + \nu(F) \neq \nu(E \cup F)$. The normalization of dividing by $\sum_{E \in \pi} \nu(E)$ is also significant. If the complement of E is unpacked into finer subsets, then the assessed likelihood of E will be indirectly affected in the denominator. So, the probability of E depends directly on its description π_E and indirectly on the description $\pi_{E^\complement}$ of its complement.

PDEU is closely related to support theory, introduced by Tversky and Koehler (1994) and extended by Rottenstreich and Tversky (1997). Support theory begins with descriptions of events, called hypotheses. Tversky and Koehler (1994) analyze comparisons of likelihood between pairs (A, B) of mutually exclusive hypotheses that they call evaluation frames, which consist of a focal hypothesis A and an alternative hypothesis B. The probability judgment of A relative to B is $P(A, B) = s(A)/[s(A) + s(B)]$, where $s(A)$ is the support assigned to hypothesis A based on the strength of its evidence. They focus on the case of nonadditive support for the same motivations as we do. They also characterize the formula for $P(A, B)$. However, they directly treat P, rather than preference, as primitive (Tversky and Koehler 1994, Theorem 1). Our theory translates support theory from judgment to decision making and extends its scope beyond binary evaluation frames. Our results provide behavioral axiomatic foundations for the model and precise requirements for identifying a unique support function from behavioral data.

Alongside its psychological pedigree, there are sound methodological arguments for PDEU. These points will develop in the sequel, but we summarize a few here. First, while the beliefs μ_π could be left unconnected across partitions, the consequent lack of basic structure would not be amenable to applications or comparative statics. Second, PDEU has an attractively compact form. As in the standard case, preference is summarized by two mathematical objects, one function for utility and another for likelihood. Third, an inherited virtue of the standard model is that a large number of implied preferences can be determined from a small number of choice observations. Under PDEU, once the weights of specific events are fixed, the weights of many others can be computed by comparing likelihood ratios. This tractably generates counterfactual predictions about behavior under alternative descriptions of the state space, an exercise that would be difficult without any structure across partitions.

Finally, PDEU associates interesting classes of behavior with features of ν. For example, specific kinds of framing effects are characterized by subadditivity. The availability heuristic associates the probability of events with the number of cases that the decision maker can recall; if more precise description aid recall, then the support function is subadditive. These sorts of characterizations are provided in the online supplement. PDEU also guarantees natural structure on special collections of events, in particular those which are immune to framing and those which are completely overlooked without explicit mention. These results are presented in Section 5.

We will sometimes refer to the following prominent example of PDEU.

Example 1 (Principle of insufficient reason)**.** Suppose ν is a constant function, for example $\nu(E) = 1$, for every nonempty E. Then the decision maker puts equal probability on all described contingencies. Such a criterion for cases of extreme ignorance or unawareness was advocated by Laplace and Leibnitz as the principle of insufficient reason, but is sensitive to the framing of the states.

When the set function ν is additive, the probabilities of events do not depend on their expressions, and the model reduces to standard subjective expected utility.

Definition 4. $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a **partition-independent expected utility** representation if it admits a PDEU representation (u, ν) with finitely additive ν.

4 Axioms and representation theorems

This section provides axiomatic characterizations of PDEU in two settings: when Π is a filtration, and when Π includes all finite partitions.

4.1 Basic axioms

We first present axioms which will be required in both settings. The first five are standard, and are collectively denoted as the Anscombe–Aumann axioms.

Axiom 1 (Weak Order)**.** $\succsim_\pi$ is complete and transitive for all $\pi \in \Pi$.

Axiom 2 (Independence)**.** For all $\pi \in \Pi$, $f, g, h \in \mathcal{F}_\pi$ and $\alpha \in (0, 1)$: if $f \succ_\pi g$, then $\alpha f + (1 - \alpha)h \succ_\pi \alpha g + (1 - \alpha)h$.

Axiom 3 (Archimedean Continuity)**.** For all $\pi \in \Pi$ and $f, g, h \in \mathcal{F}_\pi$: if $f \succ_\pi g \succ_\pi h$, then there exist $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1 - \alpha)h \succ_\pi g \succ_\pi \beta f + (1 - \beta)h$.

Axiom 4 (Nondegeneracy)**.** For all $\pi \in \Pi$, there exist $f, g \in \mathcal{F}_\pi$ such that $f \succ_\pi g$.

Axiom 5 (State Independence)**.** For all $\pi \in \Pi$, π-nonnull $E \in \sigma(\pi)$, $p, q \in \Delta X$, and $f \in \mathcal{F}_\pi$:

$$p \succsim_{\{S\}} q \Longleftrightarrow \begin{pmatrix} p & E \\ f & E^{\complement} \end{pmatrix} \succsim_\pi \begin{pmatrix} q & E \\ f & E^{\complement} \end{pmatrix}.$$

State independence has some additional content here: not only is the utility for a consequence invariant to the event in which it obtains, and also invariant to the description of the state space.

These axioms guarantee a collection of probability measures $\mu_\pi : \sigma(\pi) \to [0, 1]$ and an affine function $u : \Delta X \to \mathbb{R}$ such that $\int_S u \circ f \, d\mu_\pi$ represents $\succsim_\pi$. That is, fixing a partition π, the preference $\succsim_\pi$ is standard expected utility given the subjective belief μ_π. The model's interest derives from the relationship between preferences across descriptions.

Any expression of a contract f must mention at least the different events in which it delivers the various payments. At a minimum, the events in $\pi(f)$ must be explicitly mentioned, recalling that $\pi(f)$ is the coarsest available partition $\pi \in \Pi$ such that $f \in \mathcal{F}_\pi$. Similarly, when comparing two acts f and g, the coarsest description available to express both f and g is $\pi(f, g) = \pi(f) \vee \pi(g)$, where none of the payoff-relevant contingencies are unpacked into finer subevents. This motivates the following binary relation $\succsim$ on $\mathcal{F}$.

Definition 5. For all $f, g \in \mathcal{F}$ define $f \succsim g$ if $f \succsim_{\pi(f,g)} g$, where $\pi(f, g) = \pi(f) \vee \pi(g)$.

Under the Anscombe–Aumann axioms, the single relation $\succsim$ compactly summarizes the entire family of relations $\{\succsim_\pi\}_{\pi \in \Pi}$. For example, consider the preference between two acts f and g given a description π which is strictly finer than $\pi(f, g)$. Does the preference $f \succsim_\pi g$ hold? To answer this question equipped only with $\succsim$, take any act h such that $\pi = \pi(h)$. Then $f \succ_\pi g$ if and only if $\alpha f + (1 - \alpha)h \succ_\pi g$ for $\alpha \in (0, 1)$ close to 1, since the mixture act $\alpha f + (1 - \alpha)h$ is close to f in terms of payoffs but requires the minimal description π.

We will use $\succsim$ in the sequel for its notational convenience. However, where $\succsim$ is invoked, much of the force is implicit in its construction. These assumptions should therefore be delicately interpreted.

The following is a verbatim application of the classic axiom of Savage (1954) to the defined relation $\succsim$.

Axiom 6 (Sure-Thing Principle)**.** For all events $E \in \mathcal{E}$ and acts $f, g, h, h' \in \mathcal{F}$,

$$\begin{pmatrix} f & E \\ h & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} g & E \\ h & E^{\complement} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} f & E \\ h' & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} g & E \\ h' & E^{\complement} \end{pmatrix}.$$

The sure-thing principle is usually invoked to establish coherent conditional preferences: the relative likelihood of subevents of E is independent of the prizes associated with $E^{\complement}$.

But in our context, this coherence is already guaranteed by the Anscombe–Aumann axioms. Here, the marginal power of the axiom is to require that the preference conditional on E is independent of the description of $E^{\complement}$ induced by h or h'. To see this, assume for simplicity that the images of h and h' are disjoint from the images of f and g. Then, the implied descriptions to make the comparison in the left hand side can be divided into two parts: the description of E implied by f and g, and description of $E^{\complement}$ implied by h. The descriptions in the right hand side can be similarly divided: the *same* description of E generated by f and g, and the possibly different description of $E^{\complement}$ generated by h'. The sure-thing principle requires that the relative likelihoods of subevents of E are independent of how the complement $E^{\complement}$ is expressed.[10]

There are situations where such separability might be restrictive. For example, the judged relative likelihood of a failure of an automobile's alarm system to a failure of its transmission might depend on how finely its audio system is described. This is because alarm and audio systems are electronic components, while the transmission is mechanical. Nonetheless, such separability is required in classic support theory, where the relative likelihood in an evaluation frame (A, B) of hypothesis A to hypothesis B is independent of how any third hypothesis is described. This separability is a consequence of summarizing likelihood with a single function ν, and therefore necessary for PDEU representation.

We will occasionally reference the following standard condition, which excludes any nonempty null events:

Axiom 7 (Strict Admissibility)**.** If $f(s) \succsim g(s)$ for all $s \in S$ and $f(s') \succ g(s')$ for some $s' \in S$, then $f \succ g$.

4.2 Π is a filtration

We write Π is a filtration if the refinement relation $\geq$ is complete on Π. Given the restriction to finite partitions, Π can then be indexed by a finite or countably infinite sequence as $\Pi = \{\pi_t\}_{t=0}^T$ with $\pi_0 = \{S\}$ and $\pi_{t+1} > \pi_t$ for $0 \leq t < T$. When T is finite, π_T is the finest partition in Π, therefore $\mathcal{F} = \bigcup_{\pi \in \Pi} \mathcal{F}_\pi = \mathcal{F}_{\pi_T}$ and $\mathcal{E} = \bigcup_{\pi \in \Pi} \sigma(\pi) = \sigma(\pi_T)$. For any expressible act $f \in \mathcal{F}$, here $\pi(f)$ refers to the first partition in $\{\pi_t\}_{t=0}^T$ for which f is measurable, but $\pi(f)$ could be strictly finer than the algebra induced by f. Similarly, $\pi(f, g)$ refers to the first partition in the filtration where f and g become describable.

Theorem 1. *Given a filtration $\{\pi_t\}_{t=0}^T$, $\{\succsim_{\pi_t}\}_{t=0}^T$ admits a PDEU representation if and only if it satisfies the Anscombe–Aumann axioms and the sure-thing principle.*

[10] Given the Anscombe–Aumann axioms, this feature of the sure-thing principle is perhaps more transparently expressed by the following equivalent condition: Fix an event $E \in \mathcal{E}$. Let $\pi|_E = \{A \cap E : A \in \pi\}$. For any $\pi, \pi' \in \Pi$ such that $E \in \sigma(\pi), \sigma(\pi')$, if $\pi|_E = \pi'|_E$ and $f, g \in \mathcal{F}_\pi \cap \mathcal{F}_{\pi'}$ with $f|_{E^{\complement}} = g|_{E^{\complement}}$, then: $f \succsim_\pi g \Longleftrightarrow f \succsim_{\pi'} g$.

Proof. See Appendix B.1.

Some intuition for Theorem 1 is provided after presenting the uniqueness result. A precise statement regarding the uniqueness of u and ν requires an additional definition.

Definition 6. A filtration $\Pi = \{\pi_t\}_{t=0}^T$ is **gradual with respect to** $\{\succsim_{\pi_t}\}_{t=0}^T$ if there exists a π_t-nonnull event $E \in \pi_t \cap \pi_{t+1}$ for all $t < T$.

In words, Π is gradual if it never splits all of the π_t-nonnull events into finer descriptions. For example, suppose $\pi_1 = \{\{a,b\},\{c,d\}\}$ and $\pi_2 = \{\{a\},\{b\},\{c\},\{d\}\}$. This filtration is not gradual because π_2 splits every event in π_1. An alternative elicitation could describe the state space as $\pi_2' = \{\{a\},\{b\},\{c,d\}\}$ and then as $\pi_3' = \{\{a\},\{b\},\{c\},\{d\}\}$. This filtration collects a strictly richer set of preferences. In the alternative elicitation, ν is uniquely identified, up to a constant scalar.[11]

Theorem 2. *Suppose $\{\pi_t\}_{t=0}^T$ is a filtration and $\{\succsim_{\pi_t}\}_{t=0}^T$ admits a PDEU representation (u,ν). Then the following are equivalent:*

(i) *$\{\pi_t\}_{t=0}^T$ is gradual with respect to $\{\succsim_{\pi_t}\}_{t=0}^T$.*

(ii) *If (u',ν') also represents $\{\succsim_{\pi_t}\}_{t=0}^T$, then there exist numbers $a,c > 0$ and $b \in \mathbb{R}$ such that $u'(p) = au(p) + b$ for all $p \in \Delta X$ and $\nu'(E) = c\nu(E)$ for all $E \in \mathcal{C} \setminus \{S\}$.*

Proof. See Appendix B.2.

The identification of the support function ν is surprisingly delicate. This delicacy provides some intuition for how the support function is elicited. When two cells are in the same partition, identifying ν is simple. For example, if $E, F \in \pi$, then the likelihood ratio $\nu(E)/\nu(F)$ is identified by $\mu_\pi(E)/\mu_\pi(F)$, where μ_π is the probability measure on π implied by the Anscombe–Aumann axioms on $\succsim_\pi$. When E and F are not part of the same partition, an appropriate chain of available partitions and betting preferences calibrates the likelihood ratio $\nu(E)/\nu(F)$. For example, suppose $S = \{a,b,c,d\}$, $T = 2$, $\pi_1 = \{\{a,b\},\{c,d\}\}$, and $\pi_2 = \{\{a\},\{b\},\{c,d\}\}$. Consider the ratio $\nu(\{a,b\})/\nu(\{a\})$. First, consider preferences when the states are described as the partition π_1 to identify the likelihood ratio $\frac{\nu(\{a,b\})}{\nu(\{c,d\})}$ of $\{a,b\}$ to $\{c,d\}$. Next, considering the preferences when the states are described as π_2 reveals the ratio $\frac{\nu(\{c,d\})}{\nu(\{a\})}$ of $\{c,d\}$ to $\{a\}$. Then, we can identify $\frac{\nu(\{a,b\})}{\nu(\{a\})} = \frac{\nu(\{a,b\})}{\nu(\{c,d\})} \times \frac{\nu(\{c,d\})}{\nu(\{a\})}$, i.e. "the $\{c,d\}$'s cancel" when the revealed likelihood ratios multiply out.

This approach of indirectly linking the cells with intermediate connections might encounter two obstacles. First, if $\{c,d\}$ is π_1-null, then the ratio is undefined. Second, if the

[11] The exception is the value $\nu(S)$ at the vacuous description $\{S\}$, which is unidentified because this quantity always divides itself to unity.

filtration specifies $\pi_2 = \{\{a\}, \{b\}, \{c\}, \{d\}\}$, then it is not gradual and there is no cell common to π_1 and π_2 with which to execute the indirect comparison of $\{a, b\}$ to $\{b\}$. Instead, the ratios would reflect $\frac{\nu(\{a,b\})}{\nu(\{c,d\})}$ in the first and $\frac{\nu(\{c\})+\nu(\{d\})}{\nu(\{a\})}$ in the second. However, since generally $\nu(\{c, d\}) \neq \nu(\{c\}) + \nu(\{d\})$, these ratios are not useful in identifying $\frac{\nu(\{a,b\})}{\nu(\{a\})}$.

If two events E and F can be connected through such a chain of disjoint nonnull cells across partitions, then the ratio of $\nu(E)$ to $\nu(F)$ is pinned down. Otherwise, the ratio cannot be identified. Assuming that the filtration is gradual ensures that all cells can be connected, hence provides unique identification of ν up to a scalar multiple.

4.3 Π is the collection of all finite partitions

We now consider the case where $\Pi = \Pi^*$, the collection of all finite partitions of S. Then $\mathcal{E} = 2^S$ and $\mathcal{C} = 2^S \backslash \{\emptyset\}$. Unlike when Π is a filtration, the sure-thing principle is insufficient for PDEU. The problem is that the calibrated likelihood ratio of E to F can depend on the particular chain of comparisons used to link them. When Π is a filtration, there is only one such sequence available. The next example illustrates the potential dependence.

Example 2. Let $S = \{a, b, c, d\}$ and $\Delta X = [0, 1]$. Let $\pi^* = \{\{a, b\}, \{c, d\}\}$ with $\mu_{\pi^*}(\{a, b\}) = \frac{2}{3}$ and $\mu_{\pi^*}(\{c, d\}) = \frac{1}{3}$. For any $\pi \neq \pi^*$, let $\mu_\pi(C) = \frac{1}{|\pi|}$ for all cells $C \in \pi$. Suppose $u(p) = p$, so $\succsim_\pi$ is represented by $\int_S f \, d\mu_\pi$. These preferences satisfy the Anscombe–Aumann axioms and the sure-thing principle, but admit no PDEU representation. To the contrary, suppose (u, ν) was such a representation. Let $\pi_1 = \{\{a, b\}, \{c\}, \{d\}\}$, $\pi_2 = \{\{a, d\}, \{b\}, \{c\}\}$, and $\pi_3 = \{\{a\}, \{b\}, \{c, d\}\}$. Then, multiplying relevant likelihood ratios:

$$\frac{\nu(\{a,b\})}{\nu(\{c,d\})} = \frac{\nu(\{a,b\})}{\nu(\{c\})} \times \frac{\nu(\{c\})}{\nu(\{b\})} \times \frac{\nu(\{b\})}{\nu(\{c,d\})} = \frac{\mu_{\pi_1}(\{a,b\})}{\mu_{\pi_1}(\{c\})} \times \frac{\mu_{\pi_2}(\{c\})}{\mu_{\pi_2}(\{b\})} \times \frac{\mu_{\pi_3}(\{b\})}{\mu_{\pi_3}(\{c,d\})} = 1.$$

We can directly obtain a contradictory conclusion:

$$\frac{\nu(\{a,b\})}{\nu(\{c,d\})} = \frac{\mu_{\pi^*}(\{a,b\})}{\mu_{\pi^*}(\{c,d\})} = 2.$$

The example suggests that an additional assumption on implied likelihood ratios across different sequences of comparisons is required. Preferences across partitions are summarized by the defined relation $\succsim$, which compares acts assuming their coarsest available description. This relation is intransitive: the implied partitions $\pi(f, g)$, $\pi(g, h)$, and $\pi(f, h)$ are generally distinct. The following is a common generalization of transitivity.

Axiom 8 (Acyclicity)**.** For all acts $f_1, \ldots, f_n \in \mathcal{F}$,

$$f_1 \succ f_2, \ldots, f_{n-1} \succ f_n \Longrightarrow f_1 \succsim f_n.$$

This generalization is still too strong. Given the Anscombe–Aumann axioms, acyclicity guarantees additivity of ν.

Proposition 1. $\{\succsim_\pi\}_{\pi\in\Pi^*}$ *admits a partition-independent expected utility representation if and only if it satisfies the Anscombe–Aumann axioms and acyclicity.*[12]

Proof. See Appendix C.1.

Acyclicity therefore precludes nonadditive support functions. It is behaviorally restrictive because some cycles seem intuitive in the presence of framing effects, such as the following.

Example 3. This example is inspired by Tversky and Kahneman (1983), who report that the predicted frequency across subjects of seven-letter words ending with `ing` is higher than those with `n` as the sixth letter. Consider the following events regarding a random seven-letter word:

E_1 `_____t_`
E_2 `_____n_`

The decision maker might consider E_1 more likely than E_2 because the letter `t` is more common than `n`.

Now consider the following pair of events:

E_2 `_____n_`
E_3 `____ing`

The decision maker considers E_2 more likely than E_3, because it is a strict superset.

But, when presented with E_1 and E_3:

E_1 `_____t_`
E_3 `____ing`

she thinks E_3 is more likely, since she is now reminded of the large number of present participles which end with `ing`. Letting $p \succ q$, we have a strict cycle:

$$\begin{pmatrix} p & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succ \begin{pmatrix} p & E_2 \\ q & E_2^{\complement} \end{pmatrix}, \begin{pmatrix} p & E_2 \\ q & E_2^{\complement} \end{pmatrix} \succ \begin{pmatrix} p & E_3 \\ q & E_3^{\complement} \end{pmatrix}, \begin{pmatrix} p & E_3 \\ q & E_3^{\complement} \end{pmatrix} \succ \begin{pmatrix} p & E_1 \\ q & E_1^{\complement} \end{pmatrix}.$$ [13]

[12] It is clear that Proposition 1 remains true if acyclicity of $\succsim$ is replaced with transitivity of $\succsim$. Define the certainty equivalence relation $\succsim^*$ on $\mathcal{F}$ by: $f \succsim^* g$ if there exists $p, q \in \Delta X$ such that $f \sim p \succsim q \sim g$. The relation $\succsim^*$ is monotone (or weakly admissible) if $f \succsim^* g$ whenever $f(s) \succsim^* g(s)$ for all $s \in S$. Then, Proposition 1 also remains true if acyclicity of $\succsim$ is replaced with monotonicity of $\succsim^*$. Details are available from the authors upon request.

[13] We are very grateful to an anonymous referee for suggesting this example.

The heart of Example 3 is the nonempty intersection shared by E_2 and E_3. When E_2 and E_3 are mentioned together, this intersection primes the consideration of subevents of E_3, namely words ending with "`ing`." The decision maker is not comparing the likelihood of "seven-letter words ending with `ing`" against "seven-letter words with `n` in the sixth place," but against "seven-letter words with `n` in the sixth place which may or may not end with `ing`." On the other hand, when comparing E_1 to E_2, she is directly comparing the two events, without explicit mention of $E_2 \cap E_3$.

Finally, suppose E_3 had been `_____d_`. This event is disjoint from E_1 and E_2, and a cycle now seems less plausible. This suggests that cycles where subsequent events are disjoint should be excluded, since these have meaningful likelihood interpretations even in the presence of framing. This motivates the following.

Definition 7. A cycle of events $E_1, E_2, \ldots, E_n, E_1$ is **sequentially disjoint** if $E_1 \cap E_2 = E_2 \cap E_3 = \cdots = E_{n-1} \cap E_n = E_n \cap E_1 = \emptyset$.

Axiom 9 (Binary Bet Acyclicity)**.** For any sequentially disjoint cycle of sets $E_1, \ldots, E_n, E_1$ and lotteries $p_1, \ldots, p_n; q \in \Delta X$,

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_2 & E_2 \\ q & E_2^{\complement} \end{pmatrix}, \ldots, \begin{pmatrix} p_{n-1} & E_{n-1} \\ q & E_{n-1}^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix} \Longrightarrow \begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

This consistency on likelihoods is only applicable across comparisons of disjoint events, a sensible restriction given our model of framing. If A and B intersect, then eliciting whether A is judged more likely than B is delicate. The delicacy is that we cannot directly measure the likelihood of the coarsest expression of "A" versus the coarsest expression of "B," because no partition allows a comparison of A to B. The best we can do is assess the subjective likelihood of "$A \setminus B$ or $A \cap B$" versus "$B \setminus A$ or $A \cap B$." But once framing effects are allowed, this is a conceptually distinct question.

We can now characterize PDEU preferences when Π is rich.

Theorem 3. $\{\succsim_\pi\}_{\pi \in \Pi^*}$ *admits a PDEU representation if and only if it satisfies the Anscombe–Aumann axioms, the sure-thing principle, and binary bet acyclicity.*

Proof. See Appendix C.2.

Turning to uniqueness, the following translates Definition 6 of a gradual filtration to the current setting with all partitions.

Definition 8. A sequence of events $E_1, E_2, \ldots, E_n$ is **sequentially disjoint** if $E_1 \cap E_2 = E_2 \cap E_3 = \cdots = E_{n-1} \cap E_n = \emptyset$.

Axiom 10 (Event Reachability)**.** For any distinct nonnull events $E, F \subsetneq S$, there exists a sequentially disjoint sequence of nonnull events $E_1, \ldots, E_n$ such that $E_1 = E$ and $E_n = F$.

Theorem 4. *Assume that* $\{\succsim_\pi\}_{\pi\in\Pi^*}$ *admits a PDEU representation* (u, ν)*. The following are equivalent:*

(i) $\{\succsim_\pi\}_{\pi\in\Pi^*}$ *satisfies event reachability.*

(ii) *If* (u', ν') *also represents* $\{\succsim_\pi\}_{\pi\in\Pi^*}$*, then there exist numbers* $a, c > 0$ *and* $b \in \mathbb{R}$ *such that* $u'(p) = au(p) + b$ *for all* $p \in \Delta X$ *and* $\nu'(E) = c\nu(E)$ *for all* $E \subsetneq S$*.*

Proof. Follows from Lemma 4 in Appendix A.

Strict admissibility implies event reachability, but the converse is false: event reachability is strictly weaker than strict admissibility.

4.4 Binary bet acyclicity and the product rule

Binary bet acyclicity is reminiscent of an implication of support theory called the product rule, which is well-known in the psychological literature. Roughly speaking, if $R(A, B)$ denotes the relative likelihood of hypothesis A to a mutually exclusive hypothesis B, the product rule requires $R(A, C)R(C, B) = R(A, D)R(D, B)$. Rewritten as $R(A, C)R(C, B)R(B, D) = R(A, D)$, this is a special case of the consistency across likelihood ratios implied by binary bet acyclicity. The product rule and binary bet acyclicity have similar intuition: the particular comparison event, C or D, used to calibrate the quantitative likelihood ratio of A to B is irrelevant. One way to think of the product rule is as a limited version of binary bet acyclicity which only precludes cycles of size four, but allows for larger cycles. Given strict admissibility, if there are no cycles of size four, then there are no cycles of any size. Therefore, binary bet acylicity is equivalent to the product rule. The product rule also enjoys some empirical support.[14]

The next result formally states this equivalence. As Appendix D argues in more detail, Theorem 1 of Tversky and Koehler (1994) can be restated as a representation result for the relative likelihoods ratios $R(A, B)$. Theorem 5.ii directly follows from Tversky and Koehler (1994) and from Nehring (2008), who independently provided a proof of the same result.[15]

[14] In an experiment involving judging the likelihoods that professional basketball teams would defeat others, Fox (1999) elicits ratios of support values and finds an "excellent fit of the product rule for these data" at both the aggregate and individual subject level.

[15] To clarify the relationship of the result to Tversky and Koehler (1994), we provide a proof of Theorem 5.ii based on the proof of Theorem 1 in Tversky and Koehler (1994) (see Lemma 7 in Appendix D). As suggested above, Theorem 5.ii can also be proven by showing that, under the hypotheses of the Theorem, if there are no binary bet cycles of size four, then there are no binary bet cycles of any size. Details are available from the authors upon request.

Theorem 5 (Tversky and Koehler 1994, Nehring 2008). *Suppose that* $\{\succsim_\pi\}_{\pi\in\Pi^*}$ *satisfies the Anscombe–Aumann axioms, the sure-thing principle, and strict admissibility. Then,*

(i) *There exists an affine vNM utility function* $u : \Delta X \to \mathbb{R}$ *and a unique family of probabilities* $\{\mu_\pi\}_{\pi\in\Pi^*}$ *with* $\mu_\pi : \sigma(\pi) \to [0, 1]$*, such that:*

   a. For any $\pi \in \Pi^*$ *and* $f, g \in \mathcal{F}_\pi$*,* $f \succsim_\pi g \iff \int_S u \circ f \, d\mu_\pi \geq \int_S u \circ g \, d\mu_\pi$*, and for any* $E \in \pi$*,* $\mu_\pi(E) > 0$*.*

   b. For any nonempty disjoint events A, B*, the ratio defined by*

$$R(A, B) := \frac{\mu_\pi(A)}{\mu_\pi(B)}$$

   is independent of $\pi \in \Pi^*$ *such that* $A, B \in \pi$*.*[16]

(ii) $\succsim$ *satisfies binary bet acylicity if and only if* R *satisfies the* **product rule** *(Tversky and Koehler 1994):*

$$R(A, B)R(B, C) = R(A, D)R(D, C),$$

for all nonempty events A, B, C, D *such that* $[A \cup C] \cap [B \cup D] = \emptyset$*.*

Proof. See Appendix D. □

Theorem 5 can be leveraged to connect the cases where Π is a filtration with the case where Π includes all partitions. Specifically, binary bet acyclicity is equivalent to assuming that the likelihood ratio of E to F would not have changed if another filtration had been used for elicitation. Details can be found in the online supplement.

5 Transparent events and completely overlooked events

Throughout this section, let $\Pi = \Pi^*$. We now define two interesting families of events. The first family consists of those events which are completely transparent to the decision maker, prior to any further description of the state space. The second family is the opposite:

[16] Under the assumptions of the theorem, these ratios can also be directly defined through preference. Fix any $p \succ q$. For all disjoint nonempty A, B, define $R(A, B)$ as follows. Without loss of generality, suppose

$$\begin{pmatrix} p & A \\ q & A^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p & B \\ q & B^{\complement} \end{pmatrix}.$$

Then there exists a unique $\alpha \in (0, 1]$ such that

$$\begin{pmatrix} \alpha p + (1-\alpha) q & A \\ q & A^{\complement} \end{pmatrix} \sim \begin{pmatrix} p & B \\ q & B^{\complement} \end{pmatrix}.$$

Define $R(A, B) = 1/\alpha$ and $R(B, A) = \alpha$.

those events which are completely overlooked until they are explicitly described to her. These definitions are imposed on the preferences directly. When preferences admit a PDEU representation, the families of transparent and of overlooked events are closed under union and intersection, which is potentially useful for applications. Moreover, these events can be readily identified from the support function ν, another useful consequence of PDEU for applications.

5.1 Transparent events

We now consider those events whose explicit descriptions have no effect on choice. If event A was already in mind when deciding between acts f and g, then mentioning it explicitly should have no bearing on preference. Conversely, if its explicit description reverses preference, then A must not have been completely considered.

Definition 9. Fix $\{\succsim_\pi\}_{\pi\in\Pi^*}$. An event A is **transparent** if for any $\pi \in \Pi^*$ and for any $f, g \in \mathcal{F}_\pi$:

$$f \succsim_\pi g \Longleftrightarrow f \succsim_{\pi\vee\{A,A^\complement\}} g.$$

Let $\mathcal{A}$ denote the family of all transparent events.

The events in $\mathcal{A}$ are those which are immune to manipulation by framing or description. Someone designing a contract, and deciding which contingencies to explicitly mention, cannot change the decision maker's willingness to pay for the contract by mentioning an event in $\mathcal{A}$. The family $\mathcal{A}$ has some nice features when preferences admit a strictly admissible PDEU representation.

Proposition 2. *Suppose $\{\succsim_\pi\}_{\pi\in\Pi^*}$ admits a PDEU representation (u, ν) and satisfies strict admissibility. Then*

(i) *$A \in \mathcal{A}$ if and only if $\nu(E) = \nu(E \cap A) + \nu(E \cap A^\complement)$ for all events $E \neq S$.*

(ii) *$\mathcal{A}$ is an algebra.*[17]

(iii) *ν is additive on $\mathcal{A} \setminus \{S\}$, i.e. for all disjoint $A, B \in \mathcal{A}$ such that $A \cup B \neq S$:*

$$\nu(A \cup B) = \nu(A) + \nu(B).$$

Moreover, $\nu(A) + \nu(A^\complement) = \nu(B) + \nu(B^\complement)$ for any $A, B \in \mathcal{A} \setminus \{\emptyset, S\}$.

[17] The algebraic structure of $\mathcal{A}$ is similar to the structure of unambiguous events under some definitions (Nehring 1999). This structure is arguably restrictive for unambiguous events, but does not carry these shortcomings for our interpretation. Moreover, the behavioral definitions which induce algebras in that literature are logically independent of our definition of transparent events.

(iv) $\mathcal{A} = 2^S$ *if and only if* ν *is additive on* $2^S \setminus \{S\}$.

Proof. See Appendix E.1.

Given a strictly admissible PDEU representation, an event A is transparent if every other event is additive with respect to its intersection and relative complement with A. This is natural, since if A was already understood, mentioning it should have no effect on the judged likelihood of any other event. Moreover, the family $\mathcal{A}$ is an algebra. Thus $(S, \mathcal{A})$ can be sensibly interpreted as the prior understanding of the state space before any descriptions. This understanding might vary across agents, i.e. one decision maker might understand more events than another, but $\mathcal{A}$ can be elicited from preferences. Finally, the support function is additive over the transparent events. Since complementary weights sum to a constant number, if we redefine the value $\nu(S) \equiv \nu(A) + \nu(A^{\complement})$ for an arbitrary $A \in \mathcal{A} \setminus \{\emptyset, S\}$, then $(S, \mathcal{A}, \nu|_{\mathcal{A}})$ defines a probability space after appropriate normalization.

Example 4. Let $\pi^* = \{A_1, \dots, A_n\}$ be a partition of the state space. Interpret π^* as the decision maker's *a priori* understanding of the state space before any additional details are provided in the description. Suppose that, when the state space is described as the partition π, the decision maker understands both the explicitly described events in π and those events in π^* which she understood *a priori*. She then adapts the principle of insufficient reason over the refinement $\pi \vee \pi^*$. In terms of the representation, this is captured by setting $\nu(E) = |\{i : E \cap A_i \neq \emptyset\}|$. For example, a consumer might understand that chemotherapy, surgery, drugs, and behavioral counseling are possible treatments when purchasing health insurance, even if they are not specifically mentioned. But, when a specific disease is mentioned, she applies the principle of insufficient reason over its relevant treatments.

In this case, $\mathcal{A}$ is the algebra generated by π^*. Even if the prior understanding π^* of the decision maker is unknown to the analyst, the example confirms that Definition 9 recovers π^* from preferences.

The notion of transparency can also be defined relative to a partition. In other words, one can define the events $\mathbf{A}(\pi)$ which are understood once the partition π is announced to the decision maker. The operator $\mathbf{A}(\pi)$ has appealing properties across partitions. Under strictly admissible PDEU representations, $\mathbf{A}(\pi)$ has the properties of $\mathcal{A}$ described in Proposition 2 for every π. Details are in the online supplement.

5.2 Completely overlooked events

As a counterpoint to the events which are understood perfectly, we now discuss the events which are completely overlooked. In the unforeseen contingencies interpretation of our model these will correspond to the completely unforeseen events.

Definition 10. Fix $\{\succsim_\pi\}_{\pi\in\Pi^*}$. An event $E \subset S$ is **completely overlooked** if $E = \emptyset$ or if, for all three cell partitions $\{E, F, G\}$ of S and $p, q, r \in \Delta X$:

$$\begin{pmatrix} p & E\cup F \\ q & G \end{pmatrix} \sim r \Longleftrightarrow \begin{pmatrix} p & F \\ q & E\cup G \end{pmatrix} \sim r.$$

In words, E is completely overlooked if the decision maker never puts any weight on E unless it is explicitly described to her. In the first comparison of the definition, she attributes all the likelihood of receiving p to F, because E carries no weight when it is not separately mentioned; in the second comparison, all the likelihood of q is similarly attributed to G. Due to the framing of both acts, E remains occluded and the certainty equivalents are equal because both appear to be bets on F or G.

It is important to notice that an event does not have to be either transparent in the sense of Definition 9 or completely overlooked. The two definitions represent extreme cases admitting many intermediate possibilities.

A completely overlooked event is distinct from a null event. Whenever $E \cup F \neq S$, the following preference is consistent with E being completely overlooked :

$$\begin{pmatrix} p & E\cup F \\ q & G \end{pmatrix} \succ \begin{pmatrix} p' & E \\ p & F \\ q & G \end{pmatrix}.$$

Here, the presentation of the second act explicitly mentions E, at which point she assigns it some positive likelihood. In contrast, this strict preference is precluded whenever E is null, because then the decision maker would be indifferent to whether p' or p was assigned to the impossible event E. On the other hand, all null events are completely overlooked.

The event E might contribute no additional likelihood to $E \cup F$ for two reasons. First, the decision maker may have completely overlooked the event E when it was grouped as $E \cup F$. Second, she may have actually considered its possibility, but concluded that E was impossible. These cases are behaviorally indistinguishable.

Proposition 3. *Suppose $\{\succsim_\pi\}_{\pi\in\Pi^*}$ satisfies strict admissibility and admits the PDEU representation (u, ν) where ν is monotone. Then*

(i) *E is completely overlooked if and only if $\nu(E \cup F) = \nu(F)$ for any nonempty event F disjoint from E such that $E \cup F \neq S$.*

(ii) *If E and F are completely overlooked and $E \cup F \neq S$, then $E \cap F$ and $E \cup F$ are also completely overlooked.*

(iii) *If* $|S| \geq 3$ *and all nonempty events are completely overlooked, then* $\nu(E) = \nu(F)$ *for all nonempty* $E, F \neq S$.

The first part of the proposition relates completely overlooked events with their marginal contribution to the weighting function ν. The second part shows that the family of completely overlooked events has some desirable properties: closure under set operations is guaranteed when the sets do not cover all of S. The third part characterizes the principle of insufficient reason. This extreme case where all nonempty events are completely overlooked is represented by a constant support function where $\nu(E) = 1$ for every nonempty E. The decision maker places a uniform distribution over the events which are explicitly mentioned in a description π.[18]

Appendix

A Preliminary observations

In this section we state and prove a set of preliminary lemmas and a uniqueness result for general Π. We note that the results in this section apply to both the case where Π is a filtration and the case where Π is the set of all finite partitions. We first state, without proof, the straightforward observation that the first five axioms provide a simple analog of the Anscombe–Aumann expected utility theorem.

Lemma 1. *The collection* $\{\succsim_\pi\}_{\pi \in \Pi}$ *satisfies the Anscombe–Aumann axioms if and only if there exist an affine utility function* $u : \Delta X \to \mathbb{R}$ *with* $[-1, 1] \subset u(\Delta X)$ *and a unique family of probability measures* $\{\mu_\pi\}_{\pi \in \Pi}$ *with* $\mu_\pi : \sigma(\pi) \to [0, 1]$ *such that*

$$f \succsim_\pi g \Longleftrightarrow \int_S u \circ f \, d\mu_\pi \geq \int_S u \circ g \, d\mu_\pi$$

for any $f, g \in \mathcal{F}_\pi$.

The next lemma states that the sure-thing principle is necessary for a PDEU representation.

Lemma 2. *If* $\{\succsim_\pi\}_{\pi \in \Pi}$ *admits a partition-dependent expected utility representation, then* $\succsim$ *satisfies the sure-thing principle.*

[18] In fact, part (iii) of Proposition 3 can be strengthened to the following. If two disjoint sets E and F, with $E \cup F \neq S$, are completely overlooked, then the principle of insufficient reason is applied to subevents of their union: $\nu(D) = \nu(D')$ for all $D, D' \subset E \cup F$. Then $E \cup F$ can be considered an area of the state space of which the decision maker has no understanding.

Proof. For any $f, g \in \mathcal{F}$, note that $D(f,g) \equiv \{s \in S : f(s) \neq g(s)\} \in \sigma(\pi(f,g))$, hence:

$$\begin{aligned} f \succsim g \quad &\iff \quad f \succsim_{\pi(f,g)} g \\ &\iff \quad \int_{D(f,g)} u \circ f \, d\mu_{\pi(f,g)} \geq \int_{D(f,g)} u \circ g \, d\mu_{\pi(f,g)} \\ &\iff \quad \sum_{\substack{F \in \pi(f,g): \\ F \subset D(f,g)}} u(f(F))\nu(F) \geq \sum_{\substack{F \in \pi(f,g): \\ F \subset D(f,g)}} u(g(F))\nu(F), \end{aligned}$$

where the second equivalence follows from multiplying both sides by $\sum_{F' \in \pi(f,g)} \nu(F')$.

Now to demonstrate the sure-thing principle, let $E \in \mathcal{E}$ and $f, g, h, h' \in \mathcal{F}$. Let

$$\hat{f} = \begin{pmatrix} f & E \\ h & E^{\complement} \end{pmatrix}; \qquad \hat{g} = \begin{pmatrix} g & E \\ h & E^{\complement} \end{pmatrix};$$

$$\hat{f}' = \begin{pmatrix} f & E \\ h' & E^{\complement} \end{pmatrix}; \qquad \hat{g}' = \begin{pmatrix} g & E \\ h' & E^{\complement} \end{pmatrix}.$$

Note that $D \equiv D(\hat{f}, \hat{g}) = D(\hat{f}', \hat{g}') \subset E$ and $\pi_D \equiv \{F \in \pi(\hat{f}, \hat{g}) : F \subset D(\hat{f}, \hat{g})\} = \{F \in \pi(\hat{f}', \hat{g}') : F \subset D(\hat{f}', \hat{g}')\}$. Hence by the observation made in the first paragraph:

$$\begin{aligned} \hat{f} \succsim \hat{g} \quad &\iff \quad \sum_{F \in \pi_D} u(\hat{f}(F))\nu(F) \geq \sum_{F \in \pi_D} u(\hat{g}(F))\nu(F) \\ &\iff \quad \sum_{F \in \pi_D} u(f(F))\nu(F) \geq \sum_{F \in \pi_D} u(g(F))\nu(F) \\ &\iff \quad \sum_{F \in \pi_D} u(\hat{f}'(F))\nu(F) \geq \sum_{F \in \pi_D} u(\hat{g}'(F))\nu(F) \\ &\iff \quad \hat{f}' \succsim \hat{g}'. \end{aligned}$$

□

The next lemma summarizes the general implications of the Anscombe-Aumann axioms and the sure-thing principle.

Lemma 3. *Assume that $\{\succsim_\pi\}_{\pi \in \Pi}$ satisfies the Anscombe–Aumann axioms and the sure-thing principle. Then $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a representation $(u, \{\mu_\pi\}_{\pi \in \Pi})$ as in Lemma 1. For any events $E, F \in \mathcal{C}$ and partitions $\pi, \pi' \in \Pi$:*

(i) *If $E \in \pi, \pi'$, then $\mu_\pi(E) = 0 \Leftrightarrow \mu_{\pi'}(E) = 0$.*

(ii) *If $E, F \in \pi, \pi'$ and $E \cap F = \emptyset$, then $\mu_\pi(E)\mu_{\pi'}(F) = \mu_\pi(F)\mu_{\pi'}(E)$*

Proof. To prove part (i), it suffices to show that if $E \in \pi, \pi'$, then $\mu_\pi(E) = 0 \Rightarrow \mu_{\pi'}(E) = 0$. Suppose that $\mu_\pi(E) = 0$. Select any two lotteries $p, q \in \Delta X$ satisfying $u(p) > u(q)$ and any two acts $h, h' \in \mathcal{F}$ such that $\pi(h) = \pi$, and $\pi(h') = \pi'$. Then

$$\begin{pmatrix} p & E \\ h & E^{\complement} \end{pmatrix} \sim \begin{pmatrix} q & E \\ h & E^{\complement} \end{pmatrix}$$

by Lemma 1. Hence

$$\begin{pmatrix} p & E \\ h' & E^{\complement} \end{pmatrix} \sim \begin{pmatrix} q & E \\ h' & E^{\complement} \end{pmatrix}$$

by the sure-thing principle. Since $u(p) > u(q)$, the last indifference can hold only if $\mu_{\pi'}(E) = 0$ by Lemma 1.

To prove part (ii), observe that if either side of the desired equality is zero, then part (ii) is immediately implied by part (i). So now assume that both sides are strictly positive. Then all of the terms $\mu_\pi(E)$, $\mu_{\pi'}(F)$, $\mu_\pi(F)$, and $\mu_{\pi'}(E)$ are strictly positive. As before, select any two lotteries $p, q \in \Delta X$ such that $u(p) > u(q)$, and define a new lottery r by

$$r = \frac{\mu_\pi(E)}{\mu_\pi(E) + \mu_\pi(F)} p + \frac{\mu_\pi(F)}{\mu_\pi(E) + \mu_\pi(F)} q.$$

Select any two acts $h, h' \in \mathcal{F}$ such that $p, q, r \notin h(S) \cup h'(S)$, $\pi(h) = \pi$, and $\pi(h') = \pi'$. By the choice of r and the expected utility representation of $\succsim_\pi$, we have:

$$\begin{pmatrix} p & E \\ q & F \\ h & (E \cup F)^{\complement} \end{pmatrix} \sim \begin{pmatrix} r & E \cup F \\ h & (E \cup F)^{\complement} \end{pmatrix}$$

Hence by the sure-thing principle,

$$\begin{pmatrix} p & E \\ q & F \\ h' & (E \cup F)^{\complement} \end{pmatrix} \sim \begin{pmatrix} r & E \cup F \\ h' & (E \cup F)^{\complement} \end{pmatrix}.$$

This indifference, in conjunction with the expected utility representation of $\succsim_{\pi'}$, implies that

$$u(r) = \frac{\mu_{\pi'}(E)}{\mu_{\pi'}(E) + \mu_{\pi'}(F)} u(p) + \frac{\mu_{\pi'}(F)}{\mu_{\pi'}(E) + \mu_{\pi'}(F)} u(q).$$

We also have

$$u(r) = \frac{\mu_\pi(E)}{\mu_\pi(E) + \mu_\pi(F)} u(p) + \frac{\mu_\pi(F)}{\mu_\pi(E) + \mu_\pi(F)} u(q),$$

by the definition of r. Subtracting $u(q)$ from each side of the prior two expressions for $u(r)$ above, we obtain

$$\frac{\mu_{\pi'}(E)}{\mu_{\pi'}(E) + \mu_{\pi'}(F)} [u(p) - u(q)] = \frac{\mu_\pi(E)}{\mu_\pi(E) + \mu_\pi(F)} [u(p) - u(q)],$$

which further simplifies to

$$\frac{\mu_{\pi'}(F)}{\mu_{\pi'}(E)} = \frac{\mu_\pi(F)}{\mu_\pi(E)}$$

since both sides of the previous equality are strictly positive. □

By part (i) of Lemma 3, for all $\pi, \pi' \in \Pi$, an event $E \in \pi, \pi'$ is π-null if and only if it is π'-null. Hence under the Anscombe–Aumann axioms and the sure-thing principle, we can change quantifiers in the definitions of null and nonnull events in $\mathcal{C}$. An event $E \in \mathcal{C}$ is null if and only if E is π-null for *some* partition $\pi \in \Pi$ with $E \in \pi$. Similarly, an event $E \in \mathcal{C}$ is nonnull if and only if E is π-nonnull for *every* partition $\pi \in \Pi$ with $E \in \pi$.[19]

We will next state and prove a general uniqueness result which will imply the uniqueness Theorems 2 and 4. To do so, we first need to generalize event reachability so that it applies to our general model.

Axiom 11 (Generalized Event Reachability)**.** For any distinct nonnull events $E, F \in \mathcal{C} \setminus \{S\}$, there exists a sequence of nonnull events $E_1, \ldots, E_n \in \mathcal{C}$ such that $E = E_1$, $F = E_n$, and for each $i = 1, \ldots, n-1$ there is $\pi \in \Pi$ such that $E_i, E_{i+1} \in \pi$.

Note that when Π is the set of all finite partitions, generalized event reachability is equivalent to event reachability.

Lemma 4. *Assume that $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a PDEU representation (u, ν). Then, the following are equivalent:*

(i) *$\{\succsim_\pi\}_{\pi \in \Pi}$ satisfies Generalized Event Reachability.*

(ii) *If (u', ν') also represents $\{\succsim_\pi\}_{\pi \in \Pi}$, then there exist numbers $a, c > 0$ and $b \in \mathbb{R}$ such that $u'(p) = au(p) + b$ for all $p \in \Delta X$ and $\nu'(E) = c\nu(E)$ for all $E \in \mathcal{C} \setminus \{S\}$.*

Proof. Assume that $\{\succsim_\pi\}_{\pi \in \Pi}$ admits the PDEU representation (u, ν). Let $\mathcal{C}^*$ denote the set of nonnull events in $\mathcal{C}$. The collection $\mathcal{C}^*$ is nonempty since nondegeneracy ensures that $S \in \mathcal{C}^*$. Define the binary relation $\approx$ on $\mathcal{C}^*$ by $E \approx F$ if there exist a sequence of events $E_1, \ldots, E_n \in \mathcal{C}^*$ with $E = E_1$, $F = E_n$, and for each $i = 1, \ldots, n-1$ there is $\pi \in \Pi$ such that $E_i, E_{i+1} \in \pi$. The relation $\approx$ is reflexive, symmetric, and transitive, defining an equivalence relation on $\mathcal{C}^*$. For any $E \in \mathcal{C}^*$, let $[E] = \{F \in \mathcal{C}^* : E \approx F\}$ denote the equivalence class of E with respect to $\approx$. Let $\mathcal{C}^*/\approx\, = \{[E] : E \in \mathcal{C}^*\}$ denote the quotient set of all equivalence classes of $\mathcal{C}^*$ modulo $\approx$, with a generic class $R \in \mathcal{C}^*/\approx$. Note that, given the above definitions, event reachability is equivalent to $\mathcal{C}^*/\approx$ consisting of two indifference classes $\{S\}$ and $\mathcal{C}^* \setminus \{S\}$.

We first show the "(i) $\Rightarrow$ (ii)" part. Suppose that (u', ν') is a PDEU representation of $\{\succsim_\pi\}_{\pi \in \Pi}$ and that Generalized Event Reachability is satisfied. For each $\pi \in \Pi$, let μ_π and μ'_π respectively denote the probability distributions derived from ν and ν' by Equation (1). Applying the uniqueness component of the Anscombe–Aumann expected utility theorem to $\succsim_\pi$, we have $\mu_\pi = \mu'_\pi$ and $u' = au + b$ for some $a > 0$ and $b \in \mathbb{R}$.

If $E \in \mathcal{C}$ is null, then $\nu(E) = \mu_\pi(E) = 0 = \mu'_\pi(E) = \nu'(E)$ for any $\pi \in \Pi$ with $E \in \pi$. Also note that if $E, F \in \mathcal{C}^*$ are such that there exists $\pi \in \Pi$ with $E, F \in \pi$, then

$$\frac{\nu(E)}{\nu(F)} = \frac{\mu_\pi(E)}{\mu_\pi(F)} = \frac{\mu'_\pi(E)}{\mu'_\pi(F)} = \frac{\nu'(E)}{\nu'(F)}.$$

[19] Note that $\emptyset$ is null and S is nonnull by nondegeneracy. Also, there may exist a nonnull event $E \in \mathcal{C}$, which is π-null for some $\pi \in \Pi$ such that $E \in \sigma(\pi)$. From the above observation concerning the quantifiers, this can only be possible if E is not a cell in π but a union of its cells. This would correspond to a representation where, for example, E is a union of two disjoint subevents $E = E_1 \cup E_2$ and $\nu(E) > 0$, yet $\nu(E_1) = \nu(E_2) = 0$.

We will next extend the equality $\frac{\nu(E)}{\nu(F)} = \frac{\nu'(E)}{\nu'(F)}$ to any pair of events $E, F \in \mathcal{C}^* \setminus \{S\}$, in order to conclude that there exists $c > 0$ such that $\nu'(E) = c\nu(E)$ for all $E \in \mathcal{C} \setminus \{S\}$. Let $E, F \in \mathcal{C}^* \setminus \{S\}$. By Generalized Event Reachability, there exist $E_1, \ldots, E_n \in \mathcal{C}^*$ such that $E = E_1$, $F = E_n$, and for each $i = 1, \ldots, n-1$ there is $\pi \in \Pi$ such that $E_i, E_{i+1} \in \pi$. Then:

$$\frac{\nu(E)}{\nu(F)} = \frac{\nu(E_1)}{\nu(E_2)} \times \ldots \times \frac{\nu(E_{n-1})}{\nu(E_n)} = \frac{\nu'(E_1)}{\nu'(E_2)} \times \ldots \times \frac{\nu'(E_{n-1})}{\nu'(E_n)} = \frac{\nu'(E)}{\nu'(F)}$$

where the middle equality follows from the existence of $\pi \in \Pi$ such that $E_i, E_{i+1} \in \pi$, for each $i = 1, \ldots, n-1$. Thus ν' is a scalar multiple of ν on $\mathcal{C}^* \setminus \{S\}$, determined by the constant $c = \nu'(E)/\nu(E)$ for any $E \in \mathcal{C}^* \setminus \{S\}$.

To see the "(i) $\Leftarrow$ (ii)" part, suppose that generalized event reachability is not satisfied. Then the relation $\approx$ defined above has at least two distinct equivalence classes R and R' different from $\{S\}$. Define $\nu' : \mathcal{C} \to \mathbb{R}_+$ by:

$$\nu'(E) = \begin{cases} \nu(E) & \text{if } E \in R, \\ 2\nu(E) & \text{otherwise.} \end{cases}$$

for $E \in \mathcal{C}$. Take any $\pi \in \Pi$. If $E \in \pi \cap R \neq \emptyset$, then $\nu'(E) = \nu(E)$ for all $E \in \pi$. If $\pi \cap R = \emptyset$, then $\nu'(E) = 2\nu(E)$ for all $E \in \pi$. Hence (u, ν) and (u, ν') are two partition-dependent expected utility representations of $\{\succsim_\pi\}_{\pi \in \Pi}$ such that there does not exist a $c > 0$ with $\nu'(E) = c\nu(E)$ for all $E \in \mathcal{C} \setminus \{S\}$. $\square$

B Proofs of Section 4.2: Π is a filtration

B.1 Proof of Theorem 1

Necessity is implied by Lemmas 1 and 2. We now prove sufficiency. Let u and $\{\mu_\pi\}_{\pi \in \Pi}$ be as guaranteed by Lemma 1. We will define ν on $\cup_{t=0}^{k} \pi_t$ recursively on $k \geq 0$, which will define ν on the whole $\mathcal{C} = \cup_{t=0}^{T} \pi_t$.[20]

Step 0: Let $\nu(S) := c_0$ for an arbitrary constant $c_0 > 0$.

Step 1: For all $E \in \pi_1$, set $\nu(E) := c_1 \mu_{\pi_1}(E)$, for an arbitrary constant $c_1 > 0$.

Step $\mathbf{k+1}$ $(\mathbf{k \geq 0})$: Assume the following inductive assumptions:

(i) the nonnegative set function ν has already been defined on $\cup_{t=0}^{k} \pi_t$;

(ii) for all $t = 0, 1, \ldots, k$: $\sum_{E' \in \pi_t} \nu(E') > 0$ (i.e. nondegeneracy is satisfied);

(iii) for all $t = 0, 1, \ldots, k$ and for all $E \in \pi_t$: $\mu_{\pi_t}(E) = \nu(E) / \sum_{E' \in \pi_t} \nu(E')$.

Case 1. Assume that there exists $E^* \in \pi_k \cap \pi_{k+1}$ such that $\mu_{\pi_k}(E^*) > 0$. Then by Lemma 3 $\mu_{\pi_{k+1}}(E^*) > 0$ and by the inductive assumption $\nu(E^*) > 0$. For all $E \in \pi_{k+1} \setminus \pi_k = \pi_{k+1} \setminus (\cup_{t=1}^{k} \pi_t)$

[20] The c_k constants in the iterative definition show just how flexible we are in defining ν, which also hints to the role of gradualness in guaranteeing uniqueness. In the iterative definition, step 1 is a subcase of the subsequent step, however we prefer to write it down explicitly because it is substantially simpler.

(the equality is because we have a filtration) define $\nu(E)$ by

$$\nu(E) = \frac{\nu(E^*)}{\mu_{\pi_{k+1}}(E^*)}\mu_{\pi_{k+1}}(E) \tag{2}$$

Equation (2) also holds (as an equation rather than a definition) for $E \in \pi_{k+1} \cap \pi_k$, since

$$\frac{\nu(E)}{\nu(E^*)} = \frac{\mu_{\pi_k}(E)}{\mu_{\pi_k}(E^*)} = \frac{\mu_{\pi_{k+1}}(E)}{\mu_{\pi_{k+1}}(E^*)}.$$

where the first equality is by the inductive assumption and the second by Lemma 3. It is now easy to verify that ν satisfies (i), (ii), and (iii) on $\cup_{t=1}^{k+1}\pi_t$.

Case 2. Assume that for all $E \in \pi_k \cap \pi_{k+1}$: $\mu_{\pi_k}(E) = 0$. Let $c_{k+1} > 0$ be an arbitrary constant and for all $E \in \pi_{k+1} \setminus \pi_k = \pi_{k+1} \setminus (\cup_{t=1}^{k}\pi_t)$ define $\nu(E)$ by

$$\nu(E) = c_{k+1}\mu_{\pi_{k+1}}(E) \tag{3}$$

Equation (2) actually also holds (as an equation rather than a definition) for $E \in \pi_{k+1} \cap \pi_k$, since for all such E, $\mu_{\pi_k}(E) = 0$, hence by Lemma 3 $\mu_{\pi_{k+1}}(E) = 0$ and by the inductive assumption $\nu(E) = 0$. It is now easy to verify that ν satisfies (i), (ii), and (iii) on $\cup_{t=1}^{k+1}\pi_t$. □

B.2 Proof of Theorem 2

In light of the general uniqueness result Lemma 4, we only need to prove that generalized event reachability is equivalent to gradualness for filtrations. Suppose that $\{\succsim_{\pi_t}\}_{t=0}^T$ admits a PDEU representation (u, ν).

First assume that $\{\pi_t\}_{t=0}^T$ is gradual with respect to $\{\succsim_\pi\}_{\pi\in\Pi}$. Let $E, F \in \mathcal{C} \setminus \{S\}$ be distinct nonnull events. Then there exist π_i, π_j such that $0 < i, j \le T$, $E \in \pi_i$, and $F \in \pi_j$. Without loss of generality let $i \le j$, let $E_{i-1} := E$, $E_j := F$, and for each $t \in \{i, i+1, \ldots, j-1\}$ let $E_t \in \pi_t \cap \pi_{t+1}$ be a π_t-nonnull event as guaranteed by gradualness. Then $E_{i-1}, E_i, E_{i+1} \ldots, E_j \in \mathcal{C}$ is sequence of nonnull events such that $E = E_{i-1}$, $F = E_j$, and $E_t, E_{t+1} \in \pi_{t+1} \in \Pi$ for each $t = i-1, i, \ldots, j-1$. Hence generalized event reachability is satisfied.

Now assume that generalized event reachability is satisfied. Let $0 < t^* < T$. By nondegeneracy, there exist a π_{t^*}-nonnull event $E \in \pi_{t^*}$ and a π_{t^*+1}-nonnull event $F \in \pi_{t^*+1}$. Then $E, F \in \mathcal{C} \setminus \{S\}$ are nonnull, hence by generalized event reachability, there exists a sequence of nonnull events $E_1, \ldots, E_n \in \mathcal{C}$ such that $E = E_1$, $F = E_n$, and for each $i = 1, \ldots, n-1$ there is t such that $E_i, E_{i+1} \in \pi_t$. For each $i = 1, \ldots, n$, let $\underline{t}(i) = \min\{t : E_i \in \pi_t\}$ and $\bar{t}(i) = \sup\{t : E_i \in \pi_t\}$.[21] Then $E_i \in \pi_t$ if and only if $\underline{t}(i) \le t \le \bar{t}(i)$. Note that $\underline{t}(1) \le t^* \le \bar{t}(1)$, $\underline{t}(n) \le t^* + 1 \le \bar{t}(n)$, and $\underline{t}(i+1) \le \bar{t}(i)$ for $i = 1, \ldots, n-1$. Hence $\underline{t}(i) \le t^*$ and $t^* + 1 \le \bar{t}(i)$ for some $i = 1, \ldots, n$. Then $E_i \in \pi_{t^*} \cap \pi_{t^*+1}$ and E_i is nonnull, hence E_i is π_{t^*}-nonnull by Lemma 3. We conclude that $\{\pi_t\}_{t=0}^T$ is gradual with respect to $\{\succsim_{\pi_t}\}_{t=0}^T$. □

[21] We use supremum here since this value can be $+\infty$.

C Proofs of Section 4.3: Π is the set of all finite partitions

C.1 Proof of Proposition 1

For the necessity part, assume that $\{\succsim_\pi\}_{\pi\in\Pi^*}$ admits a partition-independent expected utility representation (u,ν). Note that $f \succsim g$ if and only if $\int_S u\circ f\,d\nu \geq \int_S u\circ f\,d\nu$ for any $f,g\in\mathcal{F}$. Thus $\succsim$ is transitive, hence acyclic. The necessity of the Anscombe–Aumann axioms follows immediately from the standard Anscombe–Aumann expected utility theorem.

Now turning to sufficiency, assume that $\{\succsim_\pi\}_{\pi\in\Pi^*}$ satisfies the Anscombe-Aumann Axioms and acyclicity. Let u and $\{\mu_\pi\}_{\pi\in\Pi^*}$ be as guaranteed by Lemma 1. We will first show that

$$\forall \pi\in\Pi^*\setminus\{\{S\}\} \text{ and } E\in\pi:\;\; \mu_\pi(E)=\mu_{\{E,E^\complement\}}(E) \tag{4}$$

Suppose for a contradiction that $\mu_\pi(E)>\mu_{\{E,E^\complement\}}(E)$ in (4). Let $\mu_\pi(E)>\alpha>\mu_{\{E,E^\complement\}}(E)$. Since the range of u contains the interval $[-1,1]$, there exist $p,q\in\Delta X$ such that $u(p)=1$ and $u(q)=0$. Define the act h by

$$h=\begin{pmatrix} p & E\\ q & E^\complement\end{pmatrix}.$$

Note that $\alpha p+(1-\alpha)q\succ h$. Let $f\in\mathcal{F}$ be such that $\pi(f)=\pi$ and for all $s\in S$, $u(f(s))<0$. Then there exists $\varepsilon\in(0,1)$ such that the act $h^\varepsilon\equiv(1-\varepsilon)h+\varepsilon f$ satisfies $\pi(h^\varepsilon)=\pi$ and $h^\varepsilon\succ_\pi \alpha p+(1-\alpha)q$. Then $h\succ h^\varepsilon\succ \alpha p+(1-\alpha)q\succ h$, a contradiction to $\succsim$ being acyclic. The argument for the case where $\mu_\pi(E)<\mu_{\{E,E^\complement\}}(E)$ is entirely symmetric, hence omitted.

Define $\nu\colon 2^S\to[0,1]$ by $\nu(\emptyset)\equiv 0$, $\nu(S)\equiv 1$, and $\nu(E)\equiv\mu_{\{E,E^\complement\}}(E)$ for $E\neq\emptyset,S$. To see that μ is finitely additive, let E,F be nonempty disjoint sets. If $E\cup F=S$, then $F=E^\complement$ so

$$\nu(E)+\nu(F)=\mu_{\{E,E^\complement\}}(E)+\mu_{\{E,E^\complement\}}(E^\complement)=1=\nu(E\cup F).$$

If $E\cup F\subsetneq S$, let $\pi=\{E,F,(E\cup F)^\complement\}$ and $\pi'=\{E\cup F,(E\cup F)^\complement\}$. Then by (4),

$$\nu(E)+\nu(F)=\mu_\pi(E)+\mu_\pi(F)=1-\mu_\pi((E\cup F)^\complement)=1-\mu_{\pi'}((E\cup F)^\complement)=\mu_{\pi'}(E\cup F)=\nu(E\cup F).$$

Therefore ν is additive. To conclude, note that for any $\pi\in\Pi^*$, the definition of ν and (4) imply that $\mu_\pi(E)=\nu(E)$ for all $E\in\pi$. Hence (u,ν) is a partition-independent representation of $\{\succsim_\pi\}_{\pi\in\Pi^*}$. □

C.2 Proof of Theorem 3

The necessity of the Anscombe–Aumann axioms follow from the standard Anscombe–Aumann expected utility theorem. The necessity of the sure-thing principle was established in Lemma 2. We now establish the necessity of binary bet acyclicity.

Lemma 5. *If $\{\succsim_\pi\}_{\pi\in\Pi^*}$ admits a PDEU representation, then it satisfies binary bet acyclicity.*

Proof. First note that for any (possibly empty) disjoint events E and F, and (not necessarily distinct) lotteries $p, q, r \in \Delta X$, we have:

$$\begin{pmatrix} p & E \\ q & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} r & F \\ q & F^{\complement} \end{pmatrix} \qquad \Longleftrightarrow \qquad [u(p) - u(q)]\nu(E) \geq [u(r) - u(q)]\nu(F).$$

To see the necessity of binary bet acyclicity, let $E_1, \dots E_n, E_1$ be a sequentially disjoint cycle of events and $p_1, p_2, \dots, p_n; q \in \Delta X$ be such that

$$\forall i = 1, \dots n-1: \quad \begin{pmatrix} p_i & E_i \\ q & E_i^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_{i+1} & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix}.$$

The observation made in the first paragraph implies that

$$[u(p_1) - u(q)]\nu(E_1) > [u(p_2) - u(q)]\nu(E_2) > \cdots > [u(p_n) - u(q)]\nu(E_n).$$

Since $[u(p_1) - u(q)]\nu(E_1) > [u(p_n) - u(q)]\nu(E_n)$, we conclude that

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

□

We next prove the sufficiency part. Suppose that $\{\succsim_\pi\}_{\pi \in \Pi^*}$ satisfies the Anscombe–Aumann axioms, the sure-thing principle, and binary bet acyclicity. Let $(u, \{\mu_\pi\}_{\pi \in \Pi^*})$ be a representation of $\{\succsim_\pi\}_{\pi \in \Pi^*}$ guaranteed by Lemma 1. For any two disjoint nonnull events E, F, define the ratio:

$$\frac{E}{F} \equiv \frac{\mu_\pi(E)}{\mu_\pi(F)}$$

where π is a partition such that $E, F \in \pi$. The value of $\frac{E}{F}$ does not depend on the particular choice of π, by part (ii) of Lemma 3. Moreover, $\frac{E}{F}$ is well-defined and strictly positive since E and F are nonnull. Finally, $\frac{F}{E} \times \frac{E}{F} = 1$ by construction. The following appeals to binary bet acyclicity in generalizing this equality.

Lemma 6. *Suppose that $\{\succsim_\pi\}_{\pi \in \Pi^*}$ satisfies the Anscombe–Aumann axioms, the sure-thing principle, and binary bet acyclicity. Then, for any sequentially disjoint cycle of nonnull events $E_1, \dots, E_n, E_1 \in \mathcal{E}$:*

$$\frac{E_1}{E_2} \times \frac{E_2}{E_3} \times \cdots \times \frac{E_{n-1}}{E_n} \times \frac{E_n}{E_1} = 1. \tag{5}$$

Proof. Let $(u, \{\mu_\pi\}_{\pi \in \Pi^*})$ be a representation of $\{\succsim_\pi\}_{\pi \in \Pi^*}$ guaranteed by Lemma 1. We will first show that for any $p_1, \dots, p_n, q \in \Delta X$ such that $u(q) = 0$ and $u(p_i) \in (0, 1)$ for $i = 1, \dots, n$:

$$(\forall i = 1, \dots n-1): \begin{pmatrix} p_i & E_i \\ q & E_i^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_{i+1} & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix} \Longrightarrow \begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}. \tag{6}$$

Note that it is enough to show that the hypothesis in Equation (6) above implies

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Let $\bar{\varepsilon} \in (0, 1)$ be such that $u(p_i) + \bar{\varepsilon} < 1$ for $i = 1, \ldots, n$. Since the range of the utility function u over lotteries contains the unit interval $[-1, 1]$, for each $\varepsilon \in (0, \bar{\varepsilon})$ and $i \in \{1, \ldots, n\}$, there exists $p_i(\varepsilon) \in \Delta X$ such that $u(p_i(\varepsilon)) = u(p_i) + \varepsilon^i$, where ε^i refers to the ith power of ε. The expected utility representation of Lemma 1 and the fact that E_i is nonnull implies that for sufficiently small $\varepsilon \in (0, \bar{\varepsilon})$,

$$\begin{pmatrix} p_i(\varepsilon) & E_i \\ q & E_i^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_{i+1}(\varepsilon) & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix},$$

for $i = 1, \ldots, n-1$. By binary bet acyclicity, this implies

$$\begin{pmatrix} p_1(\varepsilon) & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_n(\varepsilon) & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Appealing to the continuity of the expected utility representation of Lemma 1 in the assigned lotteries $f(s)$ and taking $\varepsilon \to 0$ proves the desired conclusion.

We can now prove Equation (5). The case where $n = 2$ immediately follows from our definition of event ratios, so assume that $n \geq 3$. Fix $t_1 > 0$, and recursively define

$$t_i = t_1 \times \frac{E_1}{E_2} \times \frac{E_2}{E_3} \times \ldots \times \frac{E_{i-1}}{E_i}.$$

for $i = 2, \ldots, n$. By selecting a sufficiently small t_1, we may assume that $t_1, \ldots t_n \in (0, 1)$. Also note that $\frac{t_{i+1}}{t_i} = \frac{E_i}{E_{i+1}}$ for $i = 1, \ldots, n-1$. Recall the range of the utility function u over lotteries contains the unit interval $[-1, 1]$, so there exist lotteries $p_1, \ldots, p_n, q \in \Delta X$ such that $u(p_i) = t_i$ for $i = 1, \ldots, n$ and $u(q) = 0$.

Fix any $i \in \{1, \ldots, n-1\}$. Let $\pi = \{E_i, E_{i+1}, (E_i \cup E_{i+1})^{\complement}\}$. Since $\frac{t_{i+1}}{t_i} = \frac{E_i}{E_{i+1}}$, we have $\mu_\pi(E_{i+1})u(p_{i+1}) = \mu_\pi(E_i)u(p_i)$. Hence:

$$\begin{pmatrix} p_i & E_i \\ q & E_i^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_{i+1} & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix}$$

by the expected utility representation of Lemma 1. Since the above indifference holds for any $i \in \{1, \ldots, n-1\}$, by Equation (6), we have

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Hence by the expected utility representation of $\succsim_\pi$ for $\pi = \{E_1, E_n, (E_1 \cup E_n)^{\complement}\}$, we have $\mu_\pi(E_1)u(p_1) = \mu_\pi(E_n)u(p_n)$. This implies $\frac{t_n}{t_1} = \frac{E_1}{E_n}$. Recalling the construction of t_n, we then have the desired conclusion:

$$\frac{E_1}{E_2} \times \frac{E_2}{E_3} \times \ldots \times \frac{E_{n-1}}{E_n} = \frac{E_1}{E_n}. \qquad \square$$

We can now conclude the proof of sufficiency. Assume that $\{\succsim_\pi\}_{\pi \in \Pi^*}$ satisfies the Anscombe–Aumann axioms, the sure-thing principle, and binary bet acyclicity. Define $\mathcal{C}^*$ and $\approx$ as in the proof of Lemma 4. Let $\mathcal{C}^*$ denote the set of nonnull events in $\mathcal{C}$. The collection $\mathcal{C}^*$ is nonempty, since nondegeneracy ensures that $S \in \mathcal{C}^*$. Define the binary relation $\approx$ on $\mathcal{C}^*$ by $E \approx F$ if there exists a sequentially disjoint sequence of nonnull events $E_1, \ldots, E_n \in \mathcal{C}^*$ with $E = E_1$ and $F = E_n$.[22] The relation $\approx$ is reflexive, symmetric, and transitive. So $\approx$ is an equivalence relation on $\mathcal{C}^*$. For any $E \in \mathcal{C}^*$, let $[E] = \{F \in \mathcal{C}^* : E \approx F\}$ denote the equivalence class of E with respect to $\approx$. Let $\mathcal{C}^*/\approx\, = \{[E] : E \in \mathcal{C}^*\}$ denote the quotient set of all equivalence classes of $\mathcal{C}^*$ modulo $\approx$, with a generic class $R \in \mathcal{C}^*/\approx$.[23] Select a representative event $G_R \in R$ for every equivalence class $R \in \mathcal{C}^*/\approx$, invoking the Axiom of Choice if the quotient is uncountable.

We next define ν. For all null $E \in \mathcal{C}$, let $\nu(E) = 0$. For every class $R \in \mathcal{C}^*/\approx$, *arbitrarily assign* a positive value $\nu(G_R) > 0$ for its representative. We will conclude by defining $\nu(E)$, for any $E \in \mathcal{C}^* \setminus \{S\}$. If $E = G_{[E]}$, then E represents its equivalence class and $\nu(E)$ has been assigned. Otherwise, whenever $E \neq G_{[E]}$, since $E \approx G_{[E]}$, there exists a sequentially disjoint path of nonnull events $E_1, \ldots, E_n \in \mathcal{C}^*$ such that $E = E_1$, $G_{[E]} = E_n$. Then let:

$$\nu(E) = \frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \nu(G_{[E]}).$$

Note that the definition of $\nu(E)$ above is independent of the particular choice of the path $E_1, \ldots, E_n$, because for any other such sequentially disjoint path of nonnull events $E = F_1, \ldots, F_m = G_{[E]}$:

$$\frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \frac{F_m}{F_{m-1}} \times \ldots \times \frac{F_2}{F_1} = 1$$

by Lemma 6.

We will next verify that $\nu : \mathcal{C} \setminus \{S\} \to \mathbb{R}_+$ defined above is a nondegenerate set function satisfying

$$\mu_\pi(E) = \frac{\nu(E)}{\sum_{F \in \pi} \nu(F)} \tag{7}$$

for any event $E \in \pi$ of any partition $\pi \in \Pi^* \setminus \{\{S\}\}$.

Let $\pi \in \Pi^* \setminus \{\{S\}\}$. By nondegeneracy and the expected utility representation for $\succsim_\pi$, there exists a π-nonnull $F \in \pi$. Then, since Lemma 3 implies that π-nonnull events in $\mathcal{C}$ are nonnull, F is nonnull so the denominator on the right hand side of Equation (7) is strictly positive, so the fraction is well-defined. This also implies that ν is a nondegenerate set function. Observe that Equation (7) immediately holds if E is null, since then $\nu(E) = 0$ and $\mu_\pi(E) = 0$ follows from E being π-null. Let $\mathcal{C}^*_\pi \subset \pi$ denote the nonnull cells of π. To finish the proof of the Theorem, we will show that $\frac{\mu_\pi(E)}{\mu_\pi(F)} = \frac{\nu(E)}{\nu(F)}$ for any distinct $E, F \in \mathcal{C}^*_\pi$. Along with the fact that $\sum_{E \in \mathcal{C}^*_\pi} \mu_\pi(E) = 1$, this will prove Equation (7).

Let $E, F \in \mathcal{C}^*_\pi$ be distinct. Note that $[E] = [F]$ since E and F are disjoint. Suppose first that neither E nor F is $G_{[E]}$. Then there exist a sequentially disjoint path of nonnull events $E_1, \ldots, E_n \in \mathcal{C}^*$ such that $E = E_1$, $G_{[E]} = E_n$, and:

$$\nu(E) = \frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \nu(G_{[E]}).$$

[22] Note that this definition slightly differs than the one used in the general uniqueness result (Lemma 4). The two definitions can easily be verified as equivalent, since Π is the set of all finite partitions.

[23] Note that $[S] = \{S\}$ and $E \approx F$ for any disjoint nonnull E, F.

But then $F, E_1, \ldots, E_n = G_{[E]}$ forms such a path from F to $G_{[E]}$, hence we have:

$$\nu(F) = \frac{F}{E_1} \times \frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \nu(G_{[E]}).$$

Dividing the term for $\nu(E)$ by the term for $\nu(F)$, we obtain $\frac{E}{F} = \frac{\nu(E)}{\nu(F)}$.

The other possibility is that exactly one of E or F (without loss of generality E) is $G_{[E]}$. Then the nonnull events $F = E_1, E_2 = E$, make up a path from F to $E = G_{[E]}$. Then

$$\nu(F) = \frac{F}{E} \times \nu(E)$$

as desired. □

D Proof of Theorem 5

Part (i) follows from Lemma 1 and Lemma 3 in Appendix A. In part (ii), if $\{\succsim_\pi\}_{\pi\in\Pi^*}$ satisfies binary bet acyclicity, then it has a PDEU representation, implying the product rule. The next Lemma shows that the product rule is also sufficient for a PDEU representation, establishing the other direction of Theorem 5.ii.

Lemma 7 (Tversky and Koehler 1994, Nehring 2008)**.** *Suppose that $\{\succsim_\pi\}_{\pi\in\Pi^*}$ satisfies the Anscombe–Aumann axioms, the sure-thing principle, and strict admissibility. Let $u : \Delta X \to \mathbb{R}$, $\{\mu_\pi\}_{\pi\in\Pi^*}$, and R be as in Theorem 5.i. Then, the product rule implies that there exists a strictly positive support function ν such that $\mu_\pi(E) = \frac{\nu(E)}{\sum_{F\in\pi}\nu(F)}$, for any $\pi \in \Pi^*$ and $E \in \pi$.*

We will show that the above Lemma follows from the proof of Theorem 1 in Tversky and Koehler (1994). The general idea is to first establish a natural correspondence between probability judgements P (which are the primitive of their analysis) and event ratios R, and then to translate Tversky and Koehler (1994)'s axioms and arguments to event ratios. We will also argue that a key assumption of Tversky and Koehler (1994) on probability judgments, proportionality, is implied by our construction of event ratios using the Anscombe–Aumann axioms and the sure-thing principle.

Throughout the remainder of this section, we assume strict admissibility, which is also implicitly assumed in Tversky and Koehler (1994). Remember that for any two disjoint nonempty events A and B, $R(A, B) \equiv \frac{A}{B}$ and in Tversky and Koehler (1994)'s representation, $P(A, B) = \frac{\nu(A)}{\nu(A)+\nu(B)}$.[24] Therefore, the probability judgment function P is related to event ratios via:

$$\frac{A}{B} = P(A, B)/P(B, A) \quad (8)$$

$$P(A, B) = \frac{1}{1 + \frac{B}{A}} \quad (9)$$

[24]Tversky and Koehler (1994) distinguish between the collection of hypotheses H and the collection of events 2^S. They assume that every hypothesis $A \in H$ corresponds to a unique event $A' \in 2^S$, and define the functions $P(\cdot, \cdot)$ and $\nu(\cdot)$ on hypotheses rather than events. For simplicity of exposition, we directly work with events rather than hypotheses.

where A and B are nonempty disjoint events. Tversky and Koehler (1994) also use the operation $A \vee B$ for explicit disjunction of disjoint nonempty events A and B. Then, the term $P(A, B \vee C)$ is naturally related to event ratios via:

$$P(A, B \vee C) = \frac{1}{1 + \frac{B}{A} + \frac{C}{A}} \tag{10}$$

where A, B, C are nonempty disjoint events.[25] We next state Tversky and Koehler (1994)'s proportionality axiom on P (see Tversky and Koehler (1994), Equation (4), p549):

Axiom 12. (Proportionality) For all pairwise disjoint nonempty events A, B, C:

$$\frac{P(A,B)}{P(B,A)} = \frac{P(A, B \vee C)}{P(B, A \vee C)}.$$

Given Equations (9) and (10) and $\frac{A}{B} = 1/(\frac{B}{A})$ for disjoint nonempty events A and B, one can equivalently express the proportionality axiom in terms of event ratios.

Axiom 13. (Proportionality) For all pairwise disjoint nonempty events A, B, C:

$$\frac{A}{B}\frac{B}{C} = \frac{A}{C}.$$

Under the assumptions of the Lemma, event ratios satisfy proportionality since $\pi = \{A, B, C, (A \cup B \cup C)^{\complement}\}$ is a partition and

$$\frac{A}{B}\frac{B}{C}\frac{C}{A} = \frac{\mu_\pi(A)}{\mu_\pi(B)}\frac{\mu_\pi(B)}{\mu_\pi(C)}\frac{\mu_\pi(C)}{\mu_\pi(A)} = 1.$$

Therefore, the probability judgement function also satisfies proportionality. We adopt the convention that $\frac{A}{A} = 1$ for any nonempty event A.

Proof of Lemma 7. We next provide a verbatim adaptation of the proof of Theorem 1 in Tversky and Koehler (1994). To establish sufficiency, we define ν as follows. Let $\mathbf{S} = \{\{a\} : a \in S\}$ be the set of singleton events.[26] Select some $D^* \in \mathbf{S}$ and set $\nu(D^*) = 1$. For any other singleton event $C \in \mathbf{S}$, such that $C \neq D^*$, define $\nu(C) = \frac{C}{D^*}$. Given any event $A \in 2^S$ such that $A \neq S, \emptyset$, select some $C \in \mathbf{S}$ such that $A \cap C = \emptyset$ and define $\nu(A)$ through:

$$\frac{\nu(A)}{\nu(C)} = \frac{A}{C};$$

that is,

$$\nu(A) = \frac{A}{C}\frac{C}{D^*}.$$

[25]Note that the object $B \vee C$ denoting the explicit disjunction of B and C is not an event. Intuitively, $P(A, B \vee C) = \frac{1}{1+\frac{B \vee C}{A}}$ where $\frac{B \vee C}{A}$ is naturally associated with $\frac{B}{A} + \frac{C}{A}$ yielding Equation (10).

[26]Tversky and Koehler (1994) call a hypothesis A **elementary** if the associated event A' is a singleton. Therefore, the collection of singleton events $\mathbf{S}$ above take the role of the collection of elementary hypotheses $\mathbf{E}$ in their proof.

To demonstrate that $\nu(A)$ is uniquely defined, suppose $B \in \mathbf{S} \setminus \{C\}$ and $A \cap B = \emptyset$. We want to show that

$$\frac{A}{C}\frac{C}{D^*} = \frac{A}{B}\frac{B}{D^*}. \tag{11}$$

If $D^* = B$ or $D^* = C$ then Equation (11) directly follows from proportionality. If on the other hand $D^* \cap B = D^* \cap C = \emptyset$, then by repeated application of proportionality

$$\frac{A}{C} = \frac{A}{B}\frac{B}{C} = \frac{A}{B}\frac{B}{D^*}\frac{D^*}{C}$$

proving Equation (11).

To complete the definition of ν, let $\nu(\emptyset) = 0$ and fix $\nu(S) > 0$ arbitrarily.

To establish the desired representation, we first show that for any disjoint events A, B such that $A, B \neq S, \emptyset$, $\nu(A)/\nu(B) = \frac{A}{B}$. Two cases must be considered.

First suppose that $A \cup B \neq S$; hence, there exists a singleton event $C \in \mathbf{S}$ such that $A \cap C = B \cap C = \emptyset$. In this case,

$$\frac{\nu(A)}{\nu(B)} = \left(\frac{A}{C}\nu(C)\right) / \left(\frac{B}{C}\nu(C)\right) = \frac{A}{C}\frac{C}{B} = \frac{A}{B}$$

by proportionality.

Second, suppose $A \cup B = S$. In this case, there is no $C \in \mathbf{S}$ that is not included in either A or B, so the preceding argument cannot be applied. To show that $\nu(A)/\nu(B) = \frac{A}{B}$, suppose $C, D \in \mathbf{S}$ and $A \cap C = B \cap D = \emptyset$. Hence,

$$\begin{aligned} \frac{\nu(A)}{\nu(B)} &= \frac{\nu(A)\nu(C)\nu(D)}{\nu(C)\nu(D)\nu(B)} \\ &= \frac{A}{C}\frac{C}{D}\frac{D}{B} \\ &= \frac{A}{B} \quad \text{(by the product rule).} \end{aligned}$$

For any pair of disjoint events, therefore, we obtain $\frac{A}{B} = \nu(A)/\nu(B)$ and ν is unique up to a choice of unit which is determined by $\nu(D^*)$. It is easy to see that this implies that $\mu_\pi(E) = \frac{\nu(E)}{\sum_{F \in \pi} \nu(F)}$ for any $\pi \in \Pi^*$ and $E \in \pi$. □

E Proofs of Section 5

E.1 Proof of Proposition 2

Proof of (i). To see the "⇒" part of (i), assume that $A \in \mathcal{A}$ and let E be any event. Assume without loss of generality that $E \neq \emptyset$. Consider the partition $\pi = \{E, E^{\complement} \cap A, E^{\complement} \cap A^{\complement}\}$. Since $E \neq S$, the sets $E^{\complement} \cap A$ and $E^{\complement} \cap A^{\complement}$ can not both be empty. Hence by strict admissibility $\nu(E^{\complement} \cap A) + \nu(E^{\complement} \cap A^{\complement}) > 0$. Assume without loss of generality that $[0,1] \subset u(\Delta X)$ and let $p, q, r \in \Delta X$ be such that $u(p) = 1$, $u(q) = 0$, and

$$u(r) = \frac{\nu(E)}{\nu(E) + \nu(E^{\complement} \cap A) + \nu(E^{\complement} \cap A^{\complement})}. \tag{12}$$

Define the act f by

$$f = \begin{pmatrix} p & E \\ q & E^{\complement} \end{pmatrix}.$$

Then $f \in \mathcal{F}_\pi$ and $f \sim_\pi r$. Hence by $A \in \mathcal{A}$ we have that $f \sim_{\pi \vee \{A, A^{\complement}\}} r$. Since $\pi \vee \{A, A^{\complement}\} = \{E \cap A, E \cap A^{\complement}, E^{\complement} \cap A, E^{\complement} \cap A^{\complement}\}$, the last indifference implies that

$$u(r) = \frac{\nu(E \cap A) + \nu(E \cap A^{\complement})}{\nu(E \cap A) + \nu(E \cap A^{\complement}) + \nu(E^{\complement} \cap A) + \nu(E^{\complement} \cap A^{\complement})}. \tag{13}$$

By Equations (12), (13), and $\nu(E^{\complement} \cap A) + \nu(E^{\complement} \cap A^{\complement}) > 0$, we conclude that $\nu(E) = \nu(E \cap A) + \nu(E \cap A^{\complement})$.

To see the "$\Leftarrow$" part of (i), assume that $\nu(E) = \nu(E \cap A) + \nu(E \cap A^{\complement})$ for any event $E \neq S$. Take any $\pi \in \Pi^*$. If π is the trivial partition then the desired conclusion follows trivially from state independence. So assume without loss of generality that π is nontrivial and let $\pi' = \pi \vee \{A, A^{\complement}\}$. It suffices to show that $\mu_\pi(F) = \mu_{\pi'}(F)$ for all $F \in \pi$. To see this, note that

$$\mu_\pi(F) = \frac{\nu(F)}{\sum_{E \in \pi} \nu(E)} = \frac{\nu(F \cap A) + \nu(F \cap A^{\complement})}{\sum_{E \in \pi} [\nu(E \cap A) + \nu(E \cap A^{\complement})]} = \mu_{\pi'}(F),$$

where the middle equality follows from our assumption and $F \neq S, E \neq S$ since π is nontrivial. $\square$

Proof of (ii). By definition, $\mathcal{A}$ is closed under complements and $\emptyset, S \in \mathcal{A}$. It suffices to show that $\mathcal{A}$ is closed under intersections. Let $A, B \in \mathcal{A}$, and take any event $E \neq S$. We have that

$$\begin{aligned} \nu(E) &= \nu(E \cap A) + \nu(E \cap A^{\complement}) \\ &= \nu(E \cap A \cap B) + \nu(E \cap A \cap B^{\complement}) + \nu(E \cap A^{\complement}), \end{aligned}$$

by part (i), $A, B \in \mathcal{A}$, and $E, E \cap A, \neq S$. Similarly we have that

$$\begin{aligned} \nu(E \cap (A \cap B)^{\complement}) &= \nu(E \cap (A \cap B)^{\complement} \cap A) + \nu(E \cap (A \cap B)^{\complement} \cap A^{\complement}) \\ &= \nu(E \cap A \cap B^{\complement}) + \nu(E \cap A^{\complement}). \end{aligned}$$

The two equalities above imply that

$$\nu(E) = \nu(E \cap A \cap B) + \nu(E \cap (A \cap B)^{\complement}).$$

Therefore by part (i), $A \cap B \in \mathcal{A}$. $\square$

Proof of (iii). We next prove the first part of (iii). Let $A, B \in \mathcal{A}$ be disjoint events such that $A \cup B \neq S$. Since $A \in \mathcal{A}$, we have by part (i) that:

$$\nu(A \cup B) = \nu([A \cup B] \cap A) + \nu([A \cup B] \cap A^{\complement}) = \nu(A) + \nu(B).$$

Hence ν is additive on $\mathcal{A} \setminus \{S\}$.

To see the second part of (iii), let $A, B \in \mathcal{A} \setminus \{\emptyset, S\}$. Note that:

$$\nu(A) + \nu(A^{\complement}) = \nu(A \cap B) + \nu(A \cap B^{\complement}) + \nu(A^{\complement} \cap B) + \nu(A^{\complement} \cap B^{\complement}),$$

by part (i) applied twice to $B \in \mathcal{A}$ and to $A, A^{\complement} \neq S$. By the exact symmetric argument interchanging the roles of A and B we also have that

$$\nu(B) + \nu(B^{\complement}) = \nu(B \cap A) + \nu(B \cap A^{\complement}) + \nu(B^{\complement} \cap A) + \nu(B^{\complement} \cap A^{\complement})$$

Hence $\nu(A) + \nu(A^{\complement}) = \nu(B) + \nu(B^{\complement})$ as desired. □

Proof of (iv). Immediately follows from parts (i) and (iii). □

E.2 Proof of Proposition 3

Proof of (i). The "⇐" part of (i) is easily seen to hold even without monotonicity of ν. To see the "⇒" part, assume that E is completely overlooked. If $E = \emptyset$ then the conclusion is immediate, so assume without loss of generality that $E \neq \emptyset$. Take any nonempty event F disjoint from E such that $E \cup F \neq S$. Let $G = S \setminus (E \cup F) \neq \emptyset$.

We first show that

$$\frac{\nu(E \cup F)}{\nu(G)} = \frac{\nu(F)}{\nu(E \cup G)}. \tag{14}$$

The fractions above are well defined since strict admissibility guarantees that the denominators do not vanish. To see (14), let $p, q, r \in \Delta X$ be such that $u(p) > u(q)$ and

$$\frac{\nu(E \cup F)}{\nu(E \cup F) + \nu(G)} u(p) + \frac{\nu(G)}{\nu(E \cup F) + \nu(G)} u(q) = u(r) \iff \begin{pmatrix} p & E \cup F \\ q & G \end{pmatrix} \sim r. \tag{15}$$

By E being completely overlooked, we have

$$\frac{\nu(F)}{\nu(F) + \nu(E \cup G)} u(p) + \frac{\nu(E \cup G)}{\nu(F) + \nu(E \cup G)} u(q) = u(r) \iff \begin{pmatrix} p & F \\ q & E \cup G \end{pmatrix} \sim r. \tag{16}$$

Since $u(p) > u(q)$, (15) and (16) imply that

$$\frac{\nu(E \cup F)}{\nu(E \cup F) + \nu(G)} = \frac{\nu(F)}{\nu(F) + \nu(E \cup G)}$$

which is equivalent to (14).

By monotonicity of ν, we have that

$$\frac{\nu(F)}{\nu(E \cup G)} \leq \frac{\nu(F)}{\nu(G)} \leq \frac{\nu(E \cup F)}{\nu(G)}$$

By Equation (14), all the weak equalities above are indeed equalities, hence in particular $\nu(F) = \nu(E \cup F)$ as desired. □

Proof of (ii). Assume that E and F are completely overlooked and $E \cup F \neq S$. To see that $E \cup F$ is completely overlooked, let G be a nonempty event disjoint from $E \cup F$ such that $E \cup F \cup G \neq S$. Then G is disjoint from E and $E \cup G \neq S$. By part (i), we have $\nu(E \cup G) = \nu(G)$. Moreover, $E \cup G$ is disjoint from F and $E \cup F \cup G \neq S$. Again by part (i) we have, $\nu(E \cup F \cup G) = \nu(E \cup G)$. Hence $\nu(E \cup F \cup G) = \nu(G)$, as desired.

To see that $E \cap F$ is completely overlooked, suppose that G is a nonempty event disjoint from $E \cap F$ such that $[E \cap F] \cup G \neq S$. We will show that $\nu(G \cup [E \cap F]) = \nu(G)$ by consider three cases. this will imply by part (i) that $E \cap F$ is completely overlooked.

Case 1: $G \subset E$. In this case $G \setminus F \neq \emptyset$, for otherwise $G \subset E \cap F$ would not be disjoint from $E \cap F$. Moreover $(G \setminus F) \cup F = G \cup F \subset E \cup F \neq S$, hence by part (i) we have that $\nu([G \setminus F] \cup F) = \nu(G \setminus F)$. By monotonicity

$$\nu(G) \leq \nu(G \cup [E \cap F]) \leq \nu(G \cup F) = \nu([G \setminus F] \cup F) = \nu(G \setminus F) \leq \nu(G). \tag{17}$$

Hence $\nu(G \cup [E \cap F]) = \nu(G)$.

Case 2: $G \subset F$. We again have that $\nu(G \cup [E \cap F]) = \nu(G)$, by exactly the same argument as the one above, changing the roles of events E and F.

Case 3: $G \setminus E \neq \emptyset$ and $G \setminus F \neq \emptyset$. It can not be that both $G \cup E$ and $G \cup F$ are equal to S, because otherwise $[G \cup E] \cap [G \cup F] = G \cup [E \cap F] = S$ contradicting the hypothesis. Assume without loss generality that $G \cup F \neq S$. Hence by part (i) we have that $\nu([G \setminus F] \cup F) = \nu(G \setminus F)$. By Equation (17) again, we conclude that $\nu(G \cup [E \cap F]) = \nu(G)$. ☐

Proof of (iii). The "⇐" part of (iii) is easily seen to hold even without monotonicity of ν. We will only prove the "⇒" part. We first show that $\nu(G) = \nu(G^{\complement})$ if $G \neq \emptyset, S$. To see this, note that since there are at least three states G or $G^{\complement}$ is not a singleton. Without loss of generality suppose that G has at least two elements and let $\{G_1, G_2\}$ be a two element partition of G. Then by part (i),

$$\nu(G) = \nu(G_1 \cup G_2) = \nu(G_1) = \nu(G_1 \cup G^{\complement}) = \nu(G^{\complement}),$$

where the second equality follows because G_2 and $G_1 \cup G_2 \neq S$ are completely unforeseen; the third equality follows because $G^{\complement}$ and $G_1 \cup G^{\complement} \neq S$ are completely unforeseen; and the fourth equality follows because G_1 and $G_1 \cup G^{\complement} \neq S$ are completely unforeseen.

Take any distinct events $E, F \neq \emptyset, S$. If $E \setminus F \neq \emptyset$ then

$$\nu(E \setminus F) \leq \nu(E) = \nu(E^{\complement}) \leq \nu((E \setminus F)^{\complement}) = \nu(E \setminus F)$$

where the inequalities follow from monotonicity of ν, hence $\nu(E) = \nu(E \setminus F)$. Similarly

$$\nu(E \setminus F) \leq \nu(F^{\complement}) = \nu(F) \leq \nu((E \setminus F)^{\complement}) = \nu(E \setminus F),$$

hence $\nu(F) = \nu(E \setminus F) = \nu(E)$ as desired. The case where $F \setminus E \neq \emptyset$ is entirely symmetric. ☐

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