

COMBINATORIAL VOTING*

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Abstract

We study elections with multiple issues, where voters have independent private values over bundles of issues. The innovation is considering nonseparable preferences, where issues may be complements or substitutes. Equilibrium exists when distributions over values have full support or when issues are complements. Suppose that, conditional on a voter being pivotal for a fixed issue, the outcomes of the residual issues are asymptotically certain. Then equilibrium strategies reduce to issue-by-issue voting in large elections. We characterize when large elections exhibit this reduction. With two issues, issue-by-issue voting is characterized with an inequality on the distribution over type realizations. Finally, there exists a nonempty open set of distributions where all equilibria induce aggregate uncertainty even in the limit. Thus, contrary to the single-issue case, predictability of large elections is not a generic feature of independent private values.

1 Introduction

Propositions 1A and 1B of the 2006 California General Election both aimed to increase funding for transportation improvements.¹ Consider a voter who prefers some increased funding and supports either proposition by itself, but given the state's fiscal situation, prefers to see both measures fail together than to see both pass together. This voter views the propositions as substitutes. However, the ballot does not ask for her preferences over bundles of transportation measures, but only elicits a separate up-down vote on each proposition. If she votes up on Proposition 1A while Proposition 1B passes, she contributes to the undesired passage of both measures. On the other hand, if Proposition 1B were to fail, she would like to see Proposition 1A pass to fund some transportation improvements.

How should she vote? Some subtle considerations complicate the answer to this question. What is the likelihood she is pivotal on either proposition or both? If she is pivotal on some

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¹Proposition 1A dedicated gasoline taxes for transportation improvements, at the exclusion of other uses, while Proposition 1B issued \$20 billion in bonds to fund improvements. Both measures passed by large margins.

proposition, what is the conditional likelihood that the other will pass or fail? The natural model for these questions is a game of incomplete information. The model begs other questions. Does equilibrium exist? What does it look like? Does it exhibit special properties in large elections? Is it efficient? Do other mechanisms generate better equilibrium outcomes? For elections with nonseparable issues, these basic questions are still undecided. To our knowledge, this paper is the first to follow the strategic implications of complementarity or substitution across issues to their equilibrium conclusions, and makes some initial progress in addressing these concerns.

The following example illustrates the strategic subtlety of elections with multiple issues. There are two issues, say Propositions 1 and 2. Each voter's private values for the four possible bundles $\emptyset, \{1\}, \{2\}, \{1, 2\}$ can be represented as a four-dimensional type $\theta = (\theta_\emptyset, \theta_1, \theta_2, \theta_{12})$, where θ_A denotes the value for bundle A . Voters' types are independent and identically distributed with the following discrete distribution:

$$\theta = \begin{cases} (\delta, 0, 0, 1) & \text{with probability } 1 - 2\varepsilon \\ (1, 0, 0, 0) & \text{with probability } \varepsilon \\ (0, 1, 0, 0) & \text{with probability } \varepsilon \end{cases}$$

where $\delta, \varepsilon > 0$ are arbitrarily small. With high probability $1 - 2\varepsilon$, a voter wants both issues to pass, but slightly prefers both issues failing than either issue passing alone. With small probability ε , a voter is type $(1, 0, 0, 0)$ and wants both issues to fail or is type $(0, 1, 0, 0)$ and wants issue 1 to pass alone. But in either case, she is indifferent between her less preferred alternatives. It is dominant for type $(1, 0, 0, 0)$ to vote down on both issues and for type $(0, 1, 0, 0)$ to support issue 1 and vote against issue 2. The question is how type $(\delta, 0, 0, 1)$ should vote.

A natural conjecture is that type $(\delta, 0, 0, 1)$ should vote up on both issues. Then the conjectured equilibrium strategy s^* is

$$\begin{aligned} s^*(\delta, 0, 0, 1) &= \{1, 2\} \\ s^*(1, 0, 0, 0) &= \emptyset \\ s^*(0, 1, 0, 0) &= \{1\}, \end{aligned}$$

where $s^*(\cdot)$ refers to the issues which that type supports. When voters play this strategy, then both issues will have majority support in large elections, which is ex ante efficient. One might then argue that the suggested strategy is incentive compatible, since $(\delta, 0, 0, 1)$ should vote up for either issue when she is confident that the other will pass.

However, the proposed strategy is not an equilibrium in large elections. This is because the conditional probability that the residual issue passes is starkly different from the unconditional probability. To see this, consider a voter deciding whether to support issue 1. She correctly reasons that her support only matters when she is pivotal for issue 1. When the other votes on issue 1 are split, she is in an unlikely state of the world where half of the other voters were of type $(1, 0, 0, 0)$,

since this is the only type who votes against issue 1. Moreover, in large elections, there will be some voters of type $(0, 1, 0, 0)$. Then voters of type $(\delta, 0, 0, 1)$ comprise a strict minority of the other voters. Since these are the only types who support issue 2, one should conclude that issue 2 will surely fail when she is pivotal for issue 1 in large elections. Therefore, if the pivotal voter is of type $(\delta, 0, 0, 1)$, she should vote down on issue 1 because she prefers the bundle \emptyset yielding utility δ to the bundle $\{1\}$ yielding utility 0. In fact, the only equilibrium in weakly undominated strategies is for type $(\delta, 0, 0, 1)$ to vote down on both issues, inducing the ex ante inefficient social outcome of the empty bundle in large elections.

The basic complication for elections with nonseparable issues is the wedge between the unconditional probability that an issue will pass and the conditional probability when a voter is pivotal on another issue. This resonates with existing analyses of strategic voting on a single issue with interdependent values; see for example, Austen-Smith and Banks (1996) or Feddersen and Pesendorfer (1997). In these models, being pivotal provides additional information regarding other voters' signals about an unknown state of the world. The intuition there is analogous to the importance of strategic conditioning in common value auctions for a single item, where it leads to the winner's curse and strategic underbidding. In both single-issue elections and single-object auctions with common values, strategic conditioning complicates information aggregation and efficiency. This is because the expected value of the object or the proposal is different when the player conditions on being the winner of the auction or the pivotal voter of the election.

The intuition here is different. We assume purely private values. Here, the wedge is related to the exposure problem in combinatorial auctions for multiple items, which exists even with private values. Suppose two items are sold in separate auctions. Consider a bidder with complementary valuations who desires only the bundle of both items. She must bid in both auctions to have any chance of obtaining this package. But, she should recognize that doing so exposes her to the risk of losing the second auction while winning the first, forcing her to pay for an undesired single item bundle. Moreover, she should condition the risk of losing the second auction on the event that she wins the first auction. So, the unconditional probability of winning the second auction is not appropriate in computing her exposure, but rather the conditional probability of winning the second auction assuming that she wins the first auction. Likewise, a voter in an election who desires a bundle of two issues to pass, but does not want either issue to pass alone, faces an exposure problem. In deciding her vote for issue 1, she should consider the probability that issue 2 will pass or fail, but also condition this probability on the assumption that she is pivotal on issue 1.

This exposure disappears when values are separable across issues, in which case each issue can be treated like a separate election. Consequently, the following intuitions for voting on a single issue are delicate. First, voting sincerely for the preferred outcome (pass or fail) is a weakly dominant strategy for every voter. In contrast, with nonseparable preferences, voting sincerely is never an equilibrium. Second, for a generic class of distributions over values, those distributions where the probability of preferring an issue's passage is not exactly one half, the outcome of each issue is asymptotically predictable with large electorates. In our model, even maintaining the assumption of

independent private values, there nevertheless exists a nontrivial set of type distributions where the outcomes of large elections are unpredictable. This aggregate *endogenous* uncertainty is despite the fact that values are purely private and independent, i.e. there is no aggregate *primitive* uncertainty in large elections. So, this unpredictability is orthogonal to considerations about interdependent values or a failure of information aggregation.

The paper proceeds as follows. Section 2 introduces the Bayesian game of voting over multiple issues. Section 3 shows the existence of equilibrium using two arguments. One is topological and converts the infinite-dimensional fixed point problem over strategies to a finite-dimensional problem over probabilities regarding which issues a voter is pivotal for and which issues will pass irrespective of her vote. This conversion yields later dividends in characterizing equilibria as the size of the electorate grows large. It also implies that sincere voting is not an equilibrium for every density over voters' types. The second argument assumes complementarity between issues and shows the existence of a monotone equilibrium, where types who have a stronger preference for passing more issues also vote for more issues in equilibrium. This proof relies on recent general monotone existence results due to Reny (2008).

Section 4 moves to characterizing the asymptotic behavior of equilibria for large electorates. In particular, we are interested in studying when the probability of an issue y passing, conditional on a voter being pivotal on issue x , goes to zero or one. When all issues other than x are either conditionally certain to pass or fail in large elections when the vote on x is split, we say that the equilibrium exhibits conditional certainty. Conditional certainty simplifies the computation of best response. In particular, there exists some set D_x of issues which are certain to pass when a voter is pivotal on x . To decide her ballot, the voter only needs to compare whether passing x is better given the resolution D_x of the other issues. In particular, if the bundle $\{x\} \cup D_x$ is preferred to the bundle D_x , then she should support issue x . To characterize conditional certainty, we highlight an intuitive inequality on the limit strategy: the conditional probability that another voter supports issue $y \in D_x$, when the vote on x is split, is greater than a half. The strict version of the inequality is sufficient for conditional certainty, while the weak version is necessary. While the intuition for the importance of the expression is clear, the limit results do not follow directly from laws of large numbers. This is because when a voter is pivotal for the issue x , others' votes on issue x are no longer independent. This then introduces statistical dependence across their votes on every issue.

These conditions are phrased on equilibrium strategies. Section 5 turns to the simpler case with two issues. In this environment, we relate these strategic conditions to conditions on the primitives. In particular, we identify an inequality on the distribution over type realizations which is sufficient (in its strict form) and necessary (in its weak form) for conditional certainty. Using these results, we construct a nonempty open set of densities where the conditional probability of either issue passing when the other issue is split is uniformly bounded away from zero and one in large elections. Moreover, we show that this implies there is also aggregate uncertainty regarding the outcome of the election, even for very large elections. This establishes that predictability of outcomes in is not a generic feature of large elections with multiple issues.

2 Model

There is a finite and odd set of I voters. They vote over a finite set of binary issues X , whose power set is denoted \mathcal{X} . Each voter i submits a ballot $A_i \in \mathcal{X}$, with $x \in A_i$ meaning that i votes “up” on issue x , and $y \notin A_i$ that she votes “down” on y . Issues can be interpreted as policy referenda which will pass or fail, or as elected offices decided between two political parties where one is labelled “up.” The social outcome $F(A_1, \dots, A_I) \in \mathcal{X}$ is decided with issue-by-issue majority rule:

$$F(A_1, \dots, A_I) = \{x \in X : \#\{i \in I : x \in A_i\} > I/2\}.$$

We assume each voter knows her own private values over outcomes, but allow uncertainty about others’ values. Each voter has a (normalized) type space $\Theta_i = \mathbb{R}^{\mathcal{X}}$, with typical element θ_i .² Then $\theta_i(A)$ denotes type θ_i ’s utility for all the issues in A passing and all those in its complement failing: so θ_i ’s utility for the ballot profile (A_1, \dots, A_I) is $\theta_i(F(A_1, \dots, A_I))$. Let $\Theta = \prod_i \Theta_i$ denote the space of all type profiles, and $\Theta_{-i} = \prod_{j \neq i} \Theta_j$. Voter i ’s type realization follows the distribution $\mu_i \in \Delta\Theta_i$. We assume types are independent across voters, letting $\mu = \mu_1 \otimes \dots \otimes \mu_I \in \Delta\Theta$ refer to the product distribution across voters.

A (pure) strategy s_i for each voter i is a measurable function $s_i : \Theta_i \rightarrow \mathcal{X}$ assigning a ballot to each of her types. The space of strategies for each voter is S_i . The space of strategy profiles is $S = \prod_i S_i$, and let $S_{-i} = \prod_{j \neq i} S_j$.

Definition 1. A strategy profile s^* is a **voting equilibrium** if it is a Bayesian-Nash equilibrium in weakly undominated strategies.

A voter’s values might exhibit certain structural characteristics. For example, she might view the issues as complements, as substitutes, or as having no interaction. These are captured by the following definitions.

Definition 2. θ_i is **supermodular** if, for all $A, B \in \mathcal{X}$:

$$\theta_i(A \cup B) + \theta_i(A \cap B) \geq \theta_i(A) + \theta_i(B).$$

θ_i is **submodular** if, for all $A, B \in \mathcal{X}$:

$$\theta_i(A \cup B) + \theta_i(A \cap B) \leq \theta_i(A) + \theta_i(B).$$

θ_i is **additively separable** if it is both supermodular and submodular.

The model and its subsequent analysis stand nicely aside existing literatures in a diversity of disciplines and methodologies. Existing theoretical work on aggregating preferences over multiple issues largely focuses attention to separable preferences. On the normative side, Barberà, Sonnenschein, and Zhou (1991) use social choice axioms to characterize generalizations of issue-by-issue

²The results are unaffected by normalizing the type space as any subset of $\mathbb{R}^{\mathcal{X}}$ with full dimension.

majority rule for the domain of separable preference profiles. They also prove that no mechanism is strategy proof, efficient, and nondictatorial over separable (and, a fortiori, nonseparable) preference profiles. Given their general impossibility results, understanding the features of specific institutions seems important. We complement their normative insights with a positive study of the strategic implications of majority rule.

A separate strand of work considers alternative institutions to improve efficiency in deciding multiple issues, but under the assumption of separable preferences. Casella (2005) suggests a storable votes mechanism which decides issues sequentially and voters can abstain and save their current vote to cast multiple votes for a later issue. Casella and Gelman (forthcoming) propose giving voters bonus votes to use for an issue of their discretion in an election simultaneously deciding multiple referenda. Hortala-Vallve (2007) introduces a qualitative voting mechanism where voters can distribute a fixed stock of votes arbitrarily among the issues. To date, this line of research has not considered the nonseparable preferences we study here.

Several papers in political science recognize the potential problems introduced by nonseparable preferences for multiple referenda. Brams, Kilgour, and Zwicker (1998) point out that the final set of approved outcomes may not match any single submitted ballot, which they call the “paradox of multiple elections.”³ Lacy and Niou (2000) construct an example of preferences for three strategic voters where the final outcome is not the Condorcet winner. Our proposal enriches this literature in two directions. First, while this literature largely focuses on sincere voting, we analyze the implications of strategic voting for this setting.⁴ Second, we introduce uncertainty regarding other’s preferences. This uncertainty is crucial for a voter with nonseparable preferences, whose optimal vote for a particular issue depends on her conjecture regarding the resolution of other issues.

3 Existence of equilibrium

We begin by proving existence of voting equilibria.

3.1 General existence of equilibrium

Theorem 1. *Suppose μ_i admits a density function with full support. There exists a voting equilibrium s^* .*

Remark. The full support assumption can be replaced with the following weaker condition: for every bundle A , there is a positive measure of types whose unique weakly undominated strategy is to submit the ballot A . Alternatively, assuming a set of naive voters who submit the ballot A in all circumstances would also guarantee an equilibrium among the sophisticated voters.

The proof lifts the infinite-dimensional problem of finding a fixed point in the space of strategy profiles to a finite-dimensional space of probabilities. Specifically, when other voters submit the

³The paradox is extended by Özkal-Sanver and Sanver (2007) and reinterpreted by Saari and Sieberg (2001).

⁴The exceptions are the mentioned example by Lacy and Niou (2000) and a behavioral single-person model of sequential survey responses by Lacy (2005).

ballots A_{-i} , the strategically relevant information for voter i is summarized as the set of issues C for which voter i is pivotal and the set of issues D which will pass irrespective of the i 's ballot. The outcome of submitting the ballot A_i is that those issues which she supports and on which she is pivotal will pass, along with those issue which will pass no matter how she votes: $[A_i \cap C] \cup D$. The relevant uncertainty can therefore be summarized as a probability over the ordered disjoint pairs of subsets of X , which we write as $\mathcal{D} = \{(C, D) \in \mathcal{X} \times \mathcal{X} : C \cap D = \emptyset\}$. Each strategy profile $s \in S$ induces a probability $\pi_i(s) \in \Delta\mathcal{D}$ for each voter i over \mathcal{D} , where $\Delta\mathcal{D}$ denotes the space of probabilities on \mathcal{D} . Viewed as a function, $\pi_i : S \rightarrow \Delta\mathcal{D}$ can be verified to be continuous, essentially by its construction.⁵

In turn, each belief $P_i \in \Delta\mathcal{D}$ over these ordered pairs induces an optimal ballot $[\sigma_i(P_i)](\theta_i) \in \mathcal{X}$ for a voter with values θ_i , which is the ballot A_i that maximizes the expected utility $\sum_{\mathcal{D}} \theta_i(A_i \cap C \cup D) \cdot P_i(C, D)$. Observe that this expression for interim expected utility is a linear function with coefficients $P_i(C, D)$ on θ_i . Then the set of types for whom A_i is an optimal ballot are those where $\sum_{\mathcal{D}} \theta_i(A_i \cap C \cup D) \cdot P_i(C, D) \geq \sum_{\mathcal{D}} \theta_i(A'_i \cap C \cup D) \cdot P_i(C, D)$, which defines a finite intersection of half-spaces. Small changes in P_i induce small geometric changes in these half-spaces. The density assumption implies that these small geometric changes also have small measure, proving that $\sigma_i : \Delta\mathcal{D} \rightarrow S_i$ is continuous.

Let $\pi(s) = (\pi_1(s), \dots, \pi_I(s)) : S \rightarrow [\Delta\mathcal{D}]^I$ and $\sigma(P_1, \dots, P_I) = (\sigma_1(P_1), \dots, \sigma_I(P_I)) : [\Delta\mathcal{D}]^I \rightarrow S$. Then the composition $\pi \circ \sigma : [\Delta\mathcal{D}]^I \rightarrow S \rightarrow [\Delta\mathcal{D}]^I$ defines a continuous function between finite-dimensional spaces. However, before applying a fixed point theorem, we still need to prove that we can restrict attention to undominated strategies.

Consider a strategy profile s in weakly undominated strategies. The induced probability $\pi_i(s)$ that voter i will be pivotal for the issues in C while the issues in D pass is at least as large as the probability that half the other voter submit $C \cup D$ while the other half submits D . By the full support assumption, there is a strictly positive probability any voter submits C or $C \cup D$ in any weakly undominated strategy. The independence assumption allows us to multiply these probabilities across voters, to conclude that $[\pi_i(s)](C, D)$ is strictly positive. Then the probabilities induced by strictly undominated strategy profiles S^U lives in a compact subsimplex Δ^U in the interior of the entire $\Delta\mathcal{D}$: $\pi_i(S^U) \subseteq \Delta^U$. However, since all the strategically relevant events have positive probability in Δ^U , the induced best replies must also be weakly undominated Δ^U . Therefore, the restriction $\pi \circ \sigma : [\Delta^U]^I \rightarrow S^U \rightarrow [\Delta^U]^I$ defines a continuous function from a compact subset of a finite-dimensional space to itself. By Brouwer's Theorem, there exists a fixed point P^* with strictly positive probabilities on all pairs. Then $\sigma(P^*)$ is a Bayesian-Nash equilibrium in weakly undominated strategies.

The key step in the proof, converting the existence problem from the infinite-dimensional space of strategies to a finite-dimensional space of probabilities, is adapted from Oliveros (2007). The broad approach of reducing the dimensionality of the problem to a finite simplex is reminiscent of Radner and Rosenthal (1982) and Milgrom and Weber (1985), who show existence of purified

⁵The topology on S_i is defined by the distance $d(s_i, s'_i) = \mu_i(\{\theta_i : s_i(\theta_i) \neq s'_i(\theta_i)\})$.

equilibria in games of incomplete information with finite actions. However, these results are not immediately applicable, as we require equilibrium in weakly undominated strategies. Beside the technical benefit, there is a methodological insight in conceptualizing equilibrium as a fixed point of probabilities over pivot and passing events. Later, this approach will enable sharper characterization of voting behavior in large elections.

A similar proof can be used to demonstrate existence without a common prior, as long as i 's belief for the others' values is constant across her own types θ_i . Another variant can be used to show that if μ_i is identically distributed across voters, there exists a symmetric equilibrium.

Corollary 2. *Suppose μ admits a density function with full support and $\mu_i = \mu_j$ for all $i, j \in I$. There exists a symmetric voting equilibrium where $s_i^* = s_j^*$ for all $i, j \in I$.*

Some parts of the sequel will focus attention to symmetric settings to obtain limit characterizations.

As mentioned in the introduction, when μ_i has full support, sincere voting is never an equilibrium. The result has a simple intuition. Optimal voting is determined cardinally by utility differences across bundles, while sincere voting is determined ordinally by the best bundle.

Proposition 3. *If each μ_i admits a density with full support, then sincere voting, where*

$$s_i(\theta_i) = \arg \max_A \theta_i(A).$$

is not a voting equilibrium.

3.2 Existence of monotone equilibrium

With complementary issues, equilibrium can be qualified to be monotone in the increasing differences order: those types who have a stronger preference for more issues passing will support more issues in equilibrium. Monotonicity of the equilibrium ballot across types is useful for empirical identification. Monotonicity justifies the following inference: those who are observed to vote for more issues have a preference for larger bundles. For example, suppose X is a number of political offices and voting “up” corresponds to voting for the Republican candidate while voting “down” corresponds to voting for the Democratic candidate. If all voters prefer to have politicians of the same party in government, then we can infer that those who vote for more Republicans are more right-leaning than those who vote for fewer Republicans. However, if some voters are concerned with balancing party representation, i.e. if issues are substitutes, then this inference is no longer justified, as it confounds ideological centrism with a desire for party balance.

Consider the partial order \geq on types defined as follows: $\theta'_i \geq \theta_i$ if the inequality $\theta'_i(A) - \theta'_i(B) \geq \theta_i(A) - \theta_i(B)$ holds for all $A \supseteq B$. This order captures the notion that a larger type θ' has a uniformly stronger preference for more issues to pass, as the difference in her utility between a larger bundle A and a smaller bundle B always dominates that difference for a smaller type θ . Going back to the ideology example, if “up” is coded as a Republican candidate for that office, the difference in utility

between a more Republican (A) and a less Republican (B) legislature is greater for a right-leaning type θ' than it is for a left-leaning type θ . The following theorem demonstrates that assuming issues are complementary, i.e. that a more unified legislature is more desirable, suffices for the desired inference that more right-leaning types will vote for more Republican candidates.

Theorem 4. *Suppose μ admits a density whose support is the set of all supermodular type profiles. Define the increasing differences order \geq on Θ_i by $\theta'_i \geq \theta_i$ if*

$$\theta'_i(A) - \theta'_i(B) \geq \theta_i(A) - \theta_i(B), \quad \forall A \supseteq B.$$

Then there exists a monotone voting equilibrium s^ , i.e. for all $i \in I$, $s_i^*(\theta'_i) \supseteq s_i^*(\theta_i)$ whenever $\theta'_i \geq \theta_i$.*

Note that the election is not a supermodular game. Assuming sufficiently large strategies by the other voters guarantees that no voter is ever pivotal on any issue. Precluding a voter from being pivotal eliminates the difference in interim utility between any of her two strategies. Moreover, the restriction to weakly undominated strategies is important, since the trivial equilibrium where all voters play the same constant strategy is monotone. The proof relies on recent monotone existence results by Reny (2008), which improve earlier theorems by Athey (2001) and McAdams (2003) by allowing for general orders on types, such as the increasing differences order, and for restrictions on strategies, such as the exclusion of weakly dominated strategies.

4 Asymptotic behavior in large elections

This section examines asymptotic voting behavior as the number of voters tends large. To this end, we assume that voters are identically distributed, $\mu_i = \mu_j$ for all voters i, j , and focus attention to symmetric equilibria where $s_i^* = s_j^*$. For notational ease, we henceforth drop the subscript as a reference to a particular player's strategy and let s_j^* denote the equilibrium strategy for an anonymous player in the game with I voters.

As the number of voters gets large, the probability of being pivotal for any single issue x becomes small. But, the probability of being pivotal on two issues simultaneously vanishes at a much faster rate. So, when a voter conditions on the unlikely event of being pivotal for issue x , she can ignore the doubly unlikely event of also being pivotal for another issue. Then, the only relevant uncertainty is how these other issues besides x are resolved by the rest of the electorate, after appropriately conditioning on being pivotal for issue x .

Suppose that as the electorate grows, conditioning on a voter being pivotal for issue x , it becomes certain that the issues in $D_x \subseteq X \setminus \{x\}$ will pass while the other issues outside of this set fail. When deciding how to vote on issue x , a voter's strategic calculus reduces to whether the expected conditional outcome D_x on the residual issues is better for her with or without the addition of issue x , i.e. whether $\{x\} \cup D_x$ is better than D_x alone. So, the equilibrium strategy is determined ordinally issue-by-issue, with the proviso that the status quo D_x is appropriately conditioned on

x . This substantially reduces the complexity of deciding a voter's equilibrium ballot, which is then determined by only $\#X$ ordinal rankings. Even though she has generally nonseparable preferences, equilibrium voting behavior in large elections is as simply determined as that of someone with separable preferences, if this asymptotic certainty holds. The following definition and proposition formalize this.

Definition 3. Fix a sequence of strategies $s_I \rightarrow s$ and an issue x . An issue $y \neq x$ is **conditionally certain to pass (fail) at x** if:

$$\mathbf{P} \left(\#\{j \neq i : y \in s_I(\theta_j)\} > (<) \frac{I-1}{2} \mid \#\{j \neq i : x \in s_I(\theta_j)\} = \frac{I-1}{2} \right) \rightarrow 1 \quad (1)$$

If each issue $y \neq x$ is conditionally certain to either pass or fail at x , then we say the sequence exhibits **conditional certainty at x** . In this case, we let D_x denote those issues which are conditionally certain to pass and write that **D_x is conditionally certain at x** .

As claimed, conditional certainty at x simplifies the decision to support or vote against issue x . Suppose the issues in D_x are conditionally certain to pass when a voter is pivotal on issue x , while the issues outside of D_x are conditionally certain to fail. Then for large elections, the voter should support issue x if and only if adding x to D_x makes her better off: $\theta_i(D_x \cup \{x\}) \geq \theta_i(D_x)$.

Proposition 5. Consider a sequence of equilibrium strategies $s_I^* \rightarrow s^*$. If D_x is conditionally certain at x , then

$$x \in s^*(\theta_i) \iff \theta_i(D_x \cup \{x\}) \geq \theta_i(D_x).$$

The proof of Proposition 5 exploits the probabilistic structure used to prove Theorem 1, the general existence result. There, the strategically information was summarized by an induced probability $P(C, D)$ over ordered disjoint pairs (C, D) of subsets of issues, interpreting C as the issues for which voter is pivotal and D as the issues which will pass irrespective of her ballot. In deciding whether to support issue x , the relevant probability is the conditional probability $P(C, D \mid x \in C)$. The incentive condition for whether $A \cup \{x\}$ is better than A is summarized as:

$$\sum_{C, D \in \mathcal{D}} P(\{x\}, D \mid x \in C) \cdot \theta([A \cap C] \cup D) \geq \sum_{C, D \in \mathcal{D}} P(C, D \mid x \in C) \cdot \theta([A \cup \{x\} \cap C] \cup D).$$

Conditional certainty of D_x implies that $P(\{x\}, D_x \mid x \in C) \rightarrow 1$. For any fixed population I , there exist some set of types θ_i whose weighted differences in values across bundles beside D_x and $D_x \cup \{x\}$ more than offset the difference $\theta(D_x \cup \{x\}) - \theta(D_x)$ in deciding whether to support x . But, these differences across other bundles need to get arbitrarily large since they are being weighted by a vanishing probability. By the density assumption, types with these large differences across values, or dimensions, become very unlikely for large elections. So, for nearly all types, the decision as to whether to submit the ballot A or the ballot $A \cup \{x\}$ is mediated by the difference $\theta(D_x \cup \{x\}) - \theta(D_x)$. Since this defines the incentive condition across a finite number of comparisons, the incentive

condition for x to be included in the optimal ballot is also determined by $\theta(D_x \cup \{x\}) - \theta(D_x)$ at the limit.

A recent strand of theoretical computer science studies the computational complexity of expressing the large number of rankings over combinatorial domains and of aggregating those preferences.⁶ For example, Lang (2004) studies the completeness of different logical languages for expressing preferences and characterizes the class of computational complexity for different aggregation rules, while Grabos (2005) suggests using recent techniques from artificial intelligence to design constructive algorithms which find the winning combination for different voting rules. Proposition 5 suggests that large electorates might mitigate the complexity of strategic voting. In particular, if conditional certainty holds for every issue x , each voter’s equilibrium ballot is determined by only her preference over only a few of the possible bundles. This is because most of bundles become extremely unlikely to be the outcome of the election, irrespective of how she votes. However, this argument assumes conditional certainty holds at all issues. The rest of the paper examines when this assumption is valid, and the implications of its failure.

The remainder of this section characterizes when a sequence of symmetric strategies exhibits conditional certainty, hence admitting issue-by-issue voting in large elections.

Proposition 6. *Let $s_I \rightarrow s$. If*

$$\mu(y \in s(\theta) \mid x \in s(\theta)) > (<) \mu(y \notin s(\theta) \mid x \notin s(\theta)). \quad (2)$$

Then y is conditionally certain to pass (fail) at x .

A heuristic intuition for the inequality is straightforward: inequality (2) is equivalent to

$$\frac{1}{2}\mu(y \in s(\theta) \mid x \in s(\theta)) + \frac{1}{2}\mu(y \in s(\theta) \mid x \notin s(\theta)) > \frac{1}{2}$$

When a voter is pivotal on issue x , the conditional probability that someone else supports x or votes against x is a half, so the left hand side is the conditional probability that someone else supports issue y . The suggested inequality guarantees that the conditional probability another voter supports issue y is strictly larger than one half.

However, the result is not a simple application of the law of large numbers, which requires independence across observations. Conditioning on being pivotal for x breaks the unconditional statistical independence across the other voters’ ballots. Suppose voter i is pivotal on issue x . Then knowing that voter j supported issue x makes it more likely that another voter j' voted against it, since an equal number voted each for and against. From the pivotal voter’s perspective, the votes on x are like an urn with an equal number of black or white balls, and without replacement. This indirectly introduces statistical dependence for any issue which is strategically correlated with

⁶Lang (2005) and Chevaleyre, Endriss, Lang, and Maudet (2007) review the state of the art in computational social choice, including combinatorial domains.

x . Such dependence precludes a straightforward application of convergence results for independent sequences.

The proof handles this dependence by making an artificial conditioning assumption. This additional conditioning restores independence of votes across players, but has no effect on the conditional distribution of the sum.⁷ In particular, suppose that the highest-indexed voter I is pivotal on issue x . Assume that the lowest-indexed half of the other voters $1, \dots, \frac{I-1}{2}$ supported issue x , while the higher-indexed voters $\frac{I+1}{2}, \dots, I-1$ voted against it. When I is pivotal on x , the others' votes are no longer independent, but are still exchangeable. Moreover, we are interested only in the total distribution of the sum, and not in the identity of the supporters. This artificial assumption regarding the identities of the supporters has no effect on the distribution of the vote count on y . However, with this assumption, knowing player j 's vote on issue x is no longer informative regarding player k 's vote on x , since k 's vote is now assumed to be known. The statistical independence between votes on y is now recovered. We can then apply the law of large numbers separately to each artificial subsample, the sample of those assumed to support issue x or those assumed to vote against it.

As any issue which is conditionally certain to pass cannot be conditionally certain to fail, the contrapositive of Proposition 6 provides a necessary condition for conditional certainty.

Corollary 7. *Let $s_I \rightarrow s$. If y is conditionally certain to pass (*fail*) at x , then*

$$\mu(y \in s(\theta) | x \in s(\theta)) \geq (\leq) \mu(y \notin s(\theta) | x \notin s(\theta)).$$

5 The two issue case

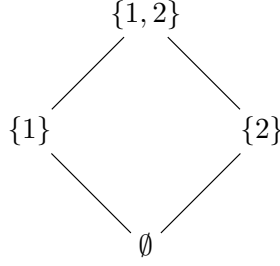
5.1 Characterization of limit equilibria

The previous section characterized conditional certainty through inequalities on strategies. Turning to the simpler case with only two issues, the first part of this section expresses these inequalities on the primitive details of the model, namely the distribution of values over bundles. For either issue $x = 1, 2$, let $x' \neq x$ denote the complementary issue. Restricting attention to two issues affords a notational simplification, since each issue can be conditionally certain only at its complement.

Definition 4. For the two issue case, we write $A \subseteq \{1, 2\}$ is **conditionally certain** if x is conditionally certain to pass at x' for every $x \in A$ and x is conditionally certain to fail at x' for every $x \notin A$.

⁷Recall that de Finetti's Theorem ensures that the probability distribution of any exchangeable sequence X^i can be expressed as a mixture of probability distributions of conditionally independent sequences X_γ^i where γ is an (unobserved) piece of additional information. In our case, this mixture can be expressed in a way where all conditionally independent distributions share the same induced distribution on the sum $\sum_i X_\gamma^i$. Insofar as we are interested only in the sum, we can proceed taking any of the components of the mixture as the true distribution of the sequence.

Proposition 8. Consider the following graph:



Let A', A'' denote the two nodes connected to A . Suppose

$$\frac{\mu[\theta_A \geq \max\{\theta_{A'}, \theta_{A''}\}]}{\mu[\theta_A \leq \min\{\theta_{A'}, \theta_{A''}\}]} > \max \left\{ \frac{\mu[\theta_{A'} \geq \theta_A \geq \theta_{A''}]}{\mu[\theta_{A''} \geq \theta_A \geq \theta_{A'}]}, \frac{\mu[\theta_{A''} \geq \theta_A \geq \theta_{A'}]}{\mu[\theta_{A'} \geq \theta_A \geq \theta_{A''}]} \right\}.$$

Then there exists a sequence of equilibria $s_I^* \rightarrow s^*$ such that A is conditionally certain.

When the sufficient inequality in Proposition 8 holds, computing an equilibrium for large elections is simple. By virtue of Proposition 5, the asymptotic equilibrium decision to support issue x is determined by

$$x \in s^*(\theta) \iff \theta(A \cup \{x\}) \geq \theta(A \setminus \{x\}).$$

The sufficiency of the inequality, which ignores the value for the bundle $\{1, 2\} \setminus A$ that is unconnected to A , is suggestive of the scope for inefficiency or miscoordination in elections with nonseparable preferences.

Example 1. Let the values for bundles θ_A be independently drawn from normal distributions μ_A with common variance but different means. Suppose the expectations are ordered

$$\mathbf{E}(\theta_{12}) > \mathbf{E}(\theta_\emptyset) > \mathbf{E}(\theta_1) = \mathbf{E}(\theta_2).$$

Then for large elections, equilibrium approximates one of the following strategies. In the first, each voter assumes both issues are conditionally certain to pass: $x \in s^*(\theta)$ if and only if $\theta_{12} \geq \theta_{x'}$. In the second, each voter assumes both issues are conditionally certain to fail: $x \in s^*(\theta)$ if and only if $\theta_x \geq \theta_\emptyset$. While passing both issues can be arbitrarily more efficient ex ante than having both fail, i.e. $\mathbf{E}(\theta_{12}) - \mathbf{E}(\theta_\emptyset)$ can be arbitrarily large, there still exists a miscoordinated equilibrium where the pivotal voter on x assumes that the other issue has failed. These are both limit equilibria because the right hand side of the inequality in Proposition 8 is exactly one while the left hand side is strictly greater than one when A is set to either bundle $\{1, 2\}$ or \emptyset .

On the other hand, suppose

$$\mathbf{E}(\theta_{12}) > \mathbf{E}(\theta_1) = \mathbf{E}(\theta_2) > \mathbf{E}(\theta_\emptyset).$$

This eliminates the second limit, and both issues are conditionally certain to pass in all sequences

of equilibria.⁸

The following provides a partial converse to Proposition 8.⁹

Proposition 9. *Consider the graph in Proposition 8. Let A', A'' denote the two nodes connected to A . If A is conditionally certain, then*

$$\frac{\mu[\theta_A \geq \max\{\theta_{A'}, \theta_{A''}\}]}{\mu[\theta_A \leq \min\{\theta_{A'}, \theta_{A''}\}]} \geq \max \left\{ \frac{\mu[\theta_{A'} \geq \theta_A \geq \theta_{A''}]}{\mu[\theta_{A''} \geq \theta_A \geq \theta_{A'}]}, \frac{\mu[\theta_{A''} \geq \theta_A \geq \theta_{A'}]}{\mu[\theta_{A'} \geq \theta_A \geq \theta_{A''}]} \right\}.$$

The contrapositive statement of Proposition 9 provides a sufficient condition for conditional uncertainty: if the inequality in Proposition 9 is in the opposite direction for all A , then every sequence of equilibria must exhibit conditional uncertainty. We will use this fact in the next subsection to construct a set of densities which preclude conditional certainty.

5.2 Conditional certainty is not generic

In this section, we focus on the following set of densities.

Example 2. Pick some small $\varepsilon > 0$.¹⁰ Consider the class of densities \mathcal{C} which satisfy the following restrictions:

$$\begin{aligned} \frac{1-\varepsilon}{4} &< \mu(\theta_{12} \geq \theta_1 \geq \theta_\emptyset \geq \theta_2) < \frac{1}{4} \\ \frac{1-\varepsilon}{4} &< \mu(\theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12}) < \frac{1}{4} \\ \frac{1-\varepsilon}{4} &< \mu(\theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1) < \frac{1}{4} \\ \frac{1-\varepsilon}{4} &< \mu(\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset) < \frac{1}{4} \end{aligned}$$

This class is open and nonempty.¹¹

We will shortly claim that every convergent sequence of equilibria for any distribution in \mathcal{C} must exhibit conditional uncertainty. A key step in establishing that claim is the following generalization of 5, which is of independent interest. The equilibrium strategy s^* is summarized by two parameters $\alpha_1, \alpha_2 \in [0, 1]$ for large elections, where α_x corresponds to the asymptotic conditional probability that issue x' passes when a voter is pivotal for issue x . Proposition 5 is the special case where α_1 and α_2 are degenerately 0 or 1.

⁸In fact, these characterizations hold as long as the distributions of μ_A are independent and analogously ranked in terms of first-order stochastic dominance.

⁹It is not a corollary of the contrapositive of Proposition 8, because the fact A is conditionally certain for one sequence of equilibria does not preclude another set B from also being conditionally certain for another sequence of equilibria.

¹⁰In fact, ε can be as large as $\frac{1}{16}$.

¹¹It is open in both the sup and weak convergence topologies.

Proposition 10. *There exists $\alpha_x \in [0, 1]$ such that:*

$$x \in s^*(\theta) \iff \alpha_x \theta_{12} + (1 - \alpha_x) \theta_x \geq \alpha_x \theta_{x'} + (1 - \alpha_x) \theta_\emptyset.$$

The next result is that if $\mu \in \mathcal{C}$, then no limit equilibrium can exhibit conditional certainty on either issue.

Proposition 11. *There exists a nonempty open set of densities, namely \mathcal{C} , where every convergent sequence of equilibria exhibits conditional uncertainty on both issues.*

The claim is not just that there exists one sequence of equilibria with this property, but rather that *all* convergent sequences of equilibria must maintain conditional uncertainty in the limit. Therefore, the conditional uncertainty is not a consequence of selection, but endemic to all limit equilibria of the game.

To obtain some intuition for Proposition 11, notice that every distribution in \mathcal{C} induces a connected Condorcet cycle for large electorates. More precisely, pairwise majority rule across bundles would rank $\{1, 2\} \succ \{1\} \succ \emptyset \succ \{2\} \succ \{1, 2\}$. The connectedness of this cycle is important, because it implies that for any fixed outcome A , there is a dominating bundle which differs from A on only one issue. For example, if both issues were conditionally certain to pass, the relevant comparisons are to the bundle $\{1, 2\}$. Then most types would vote for issue 2 (since $\{1, 2\} \succ \{1\}$) and against issue 1 (since $\{2\} \succ \{1, 2\}$). This provides some tension against the initial assumption that both issues are conditionally certain to pass, since nearly everyone is voting against issue 1. However, this tension needs to be maintained *conditionally* when a voter is pivotal. The proof fleshes out this tension conditionally, showing that that it indeed leads to a contradiction. In particular, the necessary inequality for conditional certainty in Proposition 9 is violated for every bundle A by the construction of \mathcal{C} , because the ratio of the probability $\mu(\theta_1 \geq \theta_{12} \geq \theta_2)$ to the probability $\mu(\theta_2 \geq \theta_{12} \geq \theta_1)$ is very large, as the denominator $\mu(\theta_2 \geq \theta_{12} \geq \theta_1)$ is bounded above by ε .

This demonstrates that there must be conditional uncertainty on at least one issue. The proof proceeds by arguing that conditional uncertainty on one issue implies conditional uncertainty on the other. Suppose issue 1 is conditionally uncertain. We invoke Proposition 10 to parametrize the equilibrium decision to support issue 2 by some probability $\alpha_2 \in (0, 1)$. In particular, in large elections a type θ will support issue 2 if and only if $\alpha_2 \theta_{12} + (1 - \alpha_2) \theta_2 \geq \alpha_2 \theta_1 + (1 - \alpha_2) \theta_\emptyset$. This parameterization implies that certain subsets of types must support or vote down on issue 2 in large elections, for example any type θ where $\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset$ must vote down on issue 2. The proof shows that this partial information precludes issue 2 from being conditionally certain to pass or fail.

While \mathcal{C} is a particular open set which induces conditional uncertainty, there are other type distributions which also do so. For example, let \mathcal{I} be the set of densities where the values across bundles are independent and identically distributed. Any strict convex combination of such a density and a cycling density, $\mu \in \alpha \mathcal{I} + (1 - \alpha) \mathcal{C}$ with $\alpha \in (0, 1)$ also exhibits conditional uncertainty on both issues.

5.3 Predictable outcomes for large elections are not generic

Finally, we show that all limit equilibria for the densities in Example 2 display not only conditional uncertainty from the perspective of a pivotal voter, but also display unconditional or aggregate uncertainty. For every sequence of equilibria s_I^* , the probability that an issue will pass or fail remains uniformly bounded away from 0 and 1 for large elections. In this case, the outcome of the election remains unpredictable, no matter how large the number of voters is.

This contrasts with the case of separable preferences. Assuming separability, excepting knife-edge distributions where voters are equally likely to prefer an issue's passage or its failure, the outcome of each issue is certain in large elections. Unpredictability in election outcomes is therefore difficult to reconcile with a model of voting with private separable values. While modifications to a model with private separable values can explain this unpredictability, these modifications are unnecessary if preferences are allowed to be nonseparable. With nonseparable preferences, the predictability of large elections is a consequence of the shape of the type distribution.

The following formally defines what it means for a sequence of equilibria to exhibit aggregate uncertainty regarding the outcomes of an election.

Definition 5. Consider a sequence of strategies $s_I \rightarrow s$. Issue x is *unconditionally certain to pass (fail)* if

$$\mathbf{P} \left(\#\{i : x \in s_I^*(\theta_i)\} > (<) \frac{I}{2} \right) \rightarrow 1.$$

If issue x is neither unconditionally certain to pass or fail, we say issue x is *unconditionally uncertain*.

So far, all of our results have implicated the *conditional* probabilities given that a voter is pivotal for the residual issues. On the other hand, the unpredictability of election outcomes is related to the *unconditional* probabilities of passage. The following proposition connects the uncertainty of conditional outcomes with the uncertainty of unconditional outcomes in large elections.

Proposition 12. *There is unconditional uncertainty for both issues if and only if there is conditional uncertainty for both issues*

To provide some intuition for Proposition 12, recall that assuming conditional uncertainty on both issues imposes restrictions on the limits of conditional probabilities, such as the probability issue 1 will pass when a voter is pivotal for issue 2. Reusing the artificial conditioning argument used in the proof of Proposition 6, the conditional distribution of the vote count on issue 1, when the vote on issue 2 is split, can be expressed as the sum of two arrays of independent random variables. The first array consists of binary variables whose success probabilities are $\mu[1 \in s_I^*(\theta) | 2 \in s(\theta)]$, the conditional probability of supporting issue 1 given a positive vote for issue 2. The other array similarly consist of binary variables with success probability $\mu[1 \in s(\theta) | 2 \notin s(\theta)]$. We can apply the central limit theorem to each array separately. The distribution of the (normalized) sum is a convolution of these two distributions, therefore itself asymptotically normal. Therefore, the conditional probability of the issue passing can be approximated by a normal cumulative distribution

function evaluated at one half. For there for to be conditional uncertainty, the expectation of the conditional distribution must converge to one half at rate faster than \sqrt{I} , otherwise the distribution function will collapse too quickly and will be degenerately 0 or 1 at one half. In other words, we need

$$\lim_{I \rightarrow \infty} \left| \sqrt{I} \left[\frac{1}{2} \mu[1 \in s_I^*(\theta) | 2 \in s(\theta)] + \frac{1}{2} \mu[1 \in s(\theta) | 2 \notin s(\theta)] - \frac{1}{2} \right] \right| < \infty.$$

A similar rate of convergence must hold for the conditional probability of supporting issue 1 given a split on issue 2.

To move from the conditional probabilities to the unconditional probabilities, observe that, for a fixed I , the unconditional probability of a voter supporting issue 1 can be written as a linear equation of the conditional probabilities given her vote on issue 2:

$$\mu[1 \in s_I^*(\theta)] = \mu[2 \in s_I^*(\theta)] \mu[1 \in s_I^*(\theta) | 2 \in s_I^*(\theta)] + \mu[2 \notin s_I^*(\theta)] \mu[1 \in s_I^*(\theta) | 2 \notin s_I^*(\theta)].$$

A similar equation can be written for the unconditional probability of a voter supporting issue 2. Jointly, the pair defines a system of two linear equations with two unknowns, namely the unconditional probabilities $\mu[1 \in s(\theta)]$ and $\mu[2 \in s(\theta)]$. The coefficients of the system are given by the conditional probabilities. The resulting solutions for the unconditional probabilities imply that root convergence for both conditional probabilities is equivalent to root convergence for both unconditional probabilities.

A consequence of Propositions 11 and 12 is that for a nontrivial set of primitive distributions over type realizations, election outcomes will remain unpredictable for arbitrarily large electorates.

Corollary 13. *There exists a nonempty open set of densities where every convergent sequence of equilibria exhibits unconditional uncertainty on both issues.*

As with Proposition 11, we stress that every convergent sequence of equilibria must exhibit unconditional uncertainty. So unpredictability is not a special feature of a particular sequence of equilibria, but will hold no matter which equilibria are selected.

A Appendix

Throughout the appendix, we normalize types to live in $[0, 1]^{\mathcal{X}}$ for expositional convenience.¹²

A.1 Proof of Theorem 1

We first verify that every undominated strategy assigns an open set of types to each ballot. Additive separability is too strong for this purpose because the set of additively separable types is Lebesgue null. This motivates the following weaker notion of separability:

Definition 6. θ_i is **quasi-separable** if

$$\theta_i(A) \geq [>] \theta_i(A \cap B) \iff \theta_i(A \cup B) \geq [>] \theta_i(B).$$

¹²Take any increasing homeomorphism from \mathbb{R} to $(0, 1)$, e.g. $\theta(A) \mapsto \frac{\tanh(\theta(A))+1}{2}$.

When θ_i is quasi-separable, the voter's preference for whether any issue x is voted up or down is invariant to which of the other issues in $A \setminus \{x\}$ pass or fail. The following observation is also made by Lacy and Niou (2000, Result 4).

Lemma 14. *Suppose θ_i is quasi-separable and $A_i^*(\theta_i) = \arg \max_{A \in \mathcal{X}} \theta_i(A)$ is unique. Then $s_i(\theta_i) = A_i^*(\theta_i)$ whenever s_i is weakly undominated.*

Proof. Suppose θ_i is quasi-separable and $A_i^* = A_i^*(\theta_i)$ is unique. We first prove that if $x \in A_i^*$, then s_i is weakly dominated whenever $x \notin s_i(\theta_i)$. Since $\theta_i(A_i^*) > \theta_i(A_i^* \setminus \{x\})$, we have $\theta_i(\{x\}) > \theta_i(\emptyset)$. Consider any strategy s_i where there exists some quasi-separable $x \notin s_i(\theta_i)$. Compare this to the strategy $s'_i(\theta_i) = s_i(\theta_i) \cup \{x\}$ for θ_i and equal for all other types. Now, for any fixed ballot profile A_{-i} for the other voters, either i is pivotal for issue x or is not. If not, then the same set of issues passes under both strategies, so there is no loss of utility to θ_i . If she is pivotal on x , then the set of issues $F(s_i(\theta_i), A_i) \cup \{x\}$ passes, which leaves her strictly better off by quasi-separability. So s_i is weakly dominated.

Similarly, if $x \notin A_i^*$, then s_i is weakly dominated whenever $x \in s_i(\theta_i)$. Therefore $A_i^* = s_i(\theta_i)$ for every weakly undominated strategy s_i . \square

We now begin the proof of existence. We endow each voter's strategy space S_i with the topology induced by the following distance: $d(s_i, s'_i) = \mu_i(\{\theta_i : s_i(\theta_i) \neq s'_i(\theta_i)\})$ and endow the space of strategy profiles S with the product topology.¹³ For a fixed strategy profile s , let the function $G^{s-i} = (G_0^{s-i}, G_+^{s-i}) : \Theta_{-i} \rightarrow \mathcal{X} \times \mathcal{X}$ be defined by

$$G_0^{s-i}(\theta_{-i}) = \left\{ x \in X : \#\{j \neq i : x \in s_j(\theta_j)\} = \frac{I-1}{2} \right\},$$

i.e. the set of issues where voter i is pivotal, and

$$G_+^{s-i}(\theta_{-i}) = \left\{ x \in X : \#\{j \neq i : x \in s_j(\theta_j)\} > \frac{I-1}{2} \right\},$$

i.e. the set of issues which pass irrespective of voter i 's ballot. Then, for type θ_i , her utility for a fixed ballot profile (A_1, \dots, A_I) is $\theta_i([A_i \cap G_0^{A-i}] \cup G_+^{A-i})$, i.e. the union of two sets: first, the set of issues where she is pivotal and she votes up; second, the set of issues which are passed irrespective of her ballot. Let $\mathcal{D} \subseteq \mathcal{X} \times \mathcal{X}$ denote the space of ordered disjoint pairs of sets of issues, $\mathcal{D} = \{(C, D) \in \mathcal{X} \times \mathcal{X} : C \cap D = \emptyset\}$. For a type θ_i , her expected utility for a strategy profile s is

$$\sum_{(C,D) \in \mathcal{D}} \theta_i([s_i \cap C] \cup D) \times \mu_{-i}([G^{s-i}]^{-1}(C, D)).$$

An important observation is that the type's expected utility for a ballot depends only on her belief about for which issues she will be pivotal and which issues will pass irrespective of her ballot. Let $\Delta\mathcal{D}$ denote the probability distributions over \mathcal{D} . For convenience, we work with the sup metric, $\|P - P'\| = \max_{(C,D) \in \mathcal{D}} |P(C, D) - P'(C, D)|$.

Define the probability $\pi_i(s) \in \Delta\mathcal{D}$ by

$$[\pi_i(s)](C, D) = \mu_{-i}([G^{s-i}]^{-1}(C, D)). \quad (3)$$

In words, $[\pi_i(s)](C, D)$ is the probability, from voter i 's perspective, that she will be pivotal on the issues in C and that the issues in D will pass no matter how she votes, given that the strategy s is being played by the

¹³To be precise, this is defined over equivalence classes of strategies whose differences are μ_i -null.

other voters. Fix $(C, D) \in \mathcal{D}$. If s_j is weakly undominated, by Lemma 14 there exists some quasi-separable type θ_j for whom $s_j(\theta_j) = D$. Moreover, these conditions are satisfied in an open neighborhood U^D of θ_i . By the full support assumption, there is some strictly positive probability $\mu_j^D(U^D) > 0$ of a type for j with $s_j(U^D) = D$, and similarly $\mu_j^{C \cup D}$ of a set of types $U^{C \cup D}$ for which $s_j(U^{C \cup D}) = C \cup D$. Enumerate $I \setminus \{i\} = \{j_1, \dots, j_{I-1}\}$. By independence of μ across voters, for any weakly undominated strategy profile the joint probability that D is submitted for the first $\frac{I-1}{2}$ other voters and $C \cup D$ is submitted by the second $\frac{I-1}{2}$ other voters is at least

$$L_i(C, D) = \prod_{k=1}^{\frac{I-1}{2}} \mu_{j_k}^D \times \prod_{k=\frac{I+1}{2}}^{I-1} \mu_{j_k}^{C \cup D} > 0.$$

Thus $[\pi_i(s)](C, D) \geq L_i(C, D)$ for all $(C, D) \in \mathcal{D}$, whenever s is weakly undominated. Let

$$L = \min\{L_i(C, D) : i \in I, (C, D) \in \mathcal{D}\}$$

and define the following compact convex subset of probabilities:

$$\Delta^U = \left\{ P \in \Delta\mathcal{D} : \min_{(C, D) \in \mathcal{D}} P(C, D) \geq L \right\}.$$

So, letting S^U denote the space of weakly undominated strategy profiles, we can consider the function $\pi_i : S^U \rightarrow \Delta^U$. By independence of μ ,

$$\begin{aligned} [\pi_i(s)](C, D) &= \mu_{-i}([G^{s_{-i}}]^{-1}(C, D)) \\ &= \sum_{\{A_{-i} \in \mathcal{X}^{I-1} : G_i^{A_{-i}} = (C, D)\}} \mu_{-i}(\{\theta_{-i} : s_{-i}(\theta_{-i}) = A_{-i}\}) \\ &= \sum_{\{A_{-i} \in \mathcal{X}^{I-1} : G_i^{A_{-i}} = (C, D)\}} \left[\prod_{j \neq i} \mu_j(\{\theta_j : s_j(\theta_j) = [A_{-i}]_j\}) \right]. \end{aligned}$$

The last expression is a sum of products of probabilities $\mu_j(\{\theta_j : s_j(\theta_j) = [A_{-i}]_j\})$ which, considered as functions dependent on S^U , are immediately continuous in the defined topology on S^U . Hence π_i is continuous. Then the function $\pi : S^U \rightarrow [\Delta^U]^I$ defined by $\pi(s) = (\pi_1(s), \dots, \pi_I(s))$ is continuous.

Fix a belief $P_i \in \Delta^U$. Then the set of types for voter i for which it is optimal to submit the ballot A_i is defined by

$$A_i(P_i) = \bigcap_{A'_i \in \mathcal{X}} \left\{ \theta_i : \sum_{\mathcal{D}} \theta_i([A_i \cap C] \cup D) \times P_i(C, D) \geq \sum_{\mathcal{D}} \theta_i([A'_i \cap C] \cup D) \times P_i(C, D) \right\}.$$

Fix an enumeration $\mathcal{X} = \{A^1, \dots, A^{|\mathcal{X}|}\}$ and define the function $\sigma_i : \Delta\mathcal{D} \rightarrow S$ as follows. Let A^0 denote the set of types which are not quasi-separable or do not have a unique $\arg \max_{A \in \mathcal{X}} \theta_i(A)$. For all $\theta_i \in A^k(P_i) \setminus [A^0 \cup \dots \cup A^{k-1}]$, let $[\sigma_i(P_i)](\theta_i) = A^k$.¹⁴ Since P_i is in the interior of $\Delta\mathcal{D}$, $\sigma_i(P_i)$ is not weakly dominated: $\sigma_i(P_i) \in S_i^U$. Observe that the set of types θ_i which play A_i is a convex polytope (with open and closed faces).

¹⁴This construction is to avoid ambiguous assignments on the μ_i -null set of types with multiple optimal ballots given P_i . Alternatively, one can consider the space S modulo differences of μ -measure zero, in which case the ambiguous assignment is irrelevant.

We now prove that $\sigma_i : \Delta^U \rightarrow S_i^U$ is continuous. Since $P_i \in \Delta^U$ is strictly bounded away from zero, the set of types which have multiple optimal ballots given belief P_i is of strictly lower dimension than Θ_i , hence μ_i -null since μ_i admits a density. Then $\mu_i(A_i(P_i) \setminus [\sigma(P_i)]^{-1}(A_i)) = 0$, so it suffices to show that $\mu_i(A_i(P_i))$ is continuous in P_i . Fix $\varepsilon > 0$. The set $A_i(P_i)$ is nonempty, since there exists a nonempty neighborhood of quasi-separable types which submit A_i in any undominated strategy. By outer regularity of μ_i , the probability of the closed set $A_i(P_i)$ is arbitrarily approximated by its neighborhoods (cf. Parthasarathy 1967, Theorem 1.2), i.e. there exists some δ -neighborhood of $A_i(P_i)$, denoted $U_\delta[A_i(P_i)]$, such that $\mu_i(U_\delta[A_i(P_i)]) < \mu_i(A_i(P_i)) + \varepsilon$. Moreover, there exists a sufficiently small $\gamma > 0$ such that if, for all $A'_i \in \mathcal{X}$,

$$\sum_{\mathcal{D}} \theta_i([A_i \cap C] \cup D) \times P_i(C, D) \geq \sum_{\mathcal{D}} \theta_i([A'_i \cap C] \cup D) \times P_i(C, D) - \gamma,$$

then $\theta_i \in U_\delta[A_i(P_i)]$; this is because of the continuity of the quantities on both sides of the inequality in θ_i . Suppose $\|P_i - P'_i\| < \gamma/2$. The difference in expected utility for any action across the two probabilities is bounded by $\gamma/2$, since values were normalized to live in the unit interval. Then, fixing $\theta_i \in A_i(P'_i)$, i.e. a type θ_i for whom A_i is optimal given conjecture P'_i , for all $A'_i \in \mathcal{X}$:

$$\begin{aligned} \sum_{\mathcal{D}} \theta_i([A_i \cap C] \cup D) \times P_i(C, D) &\geq \sum_{\mathcal{D}} \rho_i([A_i \cap C] \cup D) \times P'_i(C, D) - \gamma/2 \\ &\geq \sum_{\mathcal{D}} \theta_i([A'_i \cap C] \cup D) \times P'_i(C, D) - \gamma/2 \\ &\geq \sum_{\mathcal{D}} \theta_i([A'_i \cap C] \cup D) \times P_i(C, D) - \gamma. \end{aligned}$$

So, $A_i(P'_i)$ is contained in $U_\delta[A_i(P_i)]$. Then $\mu_i(A_i(P'_i) \setminus A_i(P_i)) \leq \mu_i(U_\delta[A_i(P_i)] \setminus A_i(P_i)) < \varepsilon$. Similarly, there exists a sufficiently small distance $\gamma' > 0$ such that if $\|P_i - P'_i\| < \gamma'$, then $\mu_i(A_i(P_i) \setminus A_i(P'_i)) < \varepsilon$. But

$$\begin{aligned} \mu_i(\{\theta_i : [\sigma_i(P_i)](\theta_i) \neq [\sigma_i(P'_i)](\theta_i)\}) &\leq \sum_{A_i \in \mathcal{X}} \mu_i(A_i(P_i) \Delta A_i(P'_i)) \\ &= \sum_{A_i \in \mathcal{X}} (\mu_i(A_i(P'_i) \setminus A_i(P_i)) + \mu_i(A_i(P_i) \setminus A_i(P'_i))) \\ &< 2|\mathcal{X}|\varepsilon \end{aligned}$$

whenever $\|P_i - P'_i\| < \min\{\gamma/2, \gamma'\}$. So the function $\sigma : [\Delta^U]^I \rightarrow S^U$ defined by $\sigma(P_1, \dots, P_I) = (\sigma_1(P_1), \dots, \sigma_I(P_I))$ is continuous.

Then the composition $\pi \circ \sigma : [\Delta^U]^I \rightarrow S^U \rightarrow [\Delta^U]^I$ is continuous, hence yields a fixed point (P_1^*, \dots, P_I^*) by Brouwer's Theorem. Then $s^* = \sigma(P_1^*, \dots, P_I^*)$ is, by construction, a best response to itself, hence the desired equilibrium in weakly undominated strategies.

A.2 Proof of Proposition 3

We maintain the notation from the proof of Theorem 1. Let s^0 denote the sincere voting profile, where $s_i^0(\theta_i) \in \arg \max_A \theta_i(A)$. By full support assumption, for all A and i , there is a strictly positive measure of types θ_i where the sincere ballot is A . Hence the induced probability $P_i^0 = \pi_i(s^0)$ is in the interior of $\Delta\mathcal{D}$. Therefore $P_i(C, D) > 0$. Consider θ_i with $\theta_i(\emptyset) = 1$ and $\theta_i(\{1\}) = 0$, and $\theta_i(A) = 1 - \delta$ whenever $A \neq \emptyset, \{1\}$. When $2 \in C$ and $D = \{1\}$

$$\theta_i([\{2\} \cap C] \cup \{1\}) - \theta_i([\emptyset \cap C] \cup \{1\}) = \theta_i(\{1, 2\}) - \theta_i(\{1\}) = 1 - \delta.$$

For all other (C, D) , the difference $\theta_i(\{\{2\} \cap C\} \cup D) - \theta_i(\{\emptyset \cap C\} \cup D)$ is either 0 or $-\delta$. Since P_i^0 has full support, we have

$$\sum_{\mathcal{D}} \theta_i(\{\{2\} \cap C\} \cup D) \times P_i^0(C, D) > \sum_{\mathcal{D}} \theta_i(\{\emptyset \cap C\} \cup D) \times P_i^0(C, D)$$

for sufficiently small $\delta > 0$. So submitting the ballot $\{2\}$ is a stricter better reply than the sincere ballot \emptyset for this θ_i . This property is maintained in a neighborhood of θ_i , so sincere voting is not a best reply for a strictly positive measure of types.

A.3 Proof of Theorem 4

We use recent results due to Reny (2008). Namely, we will verify the assumptions of Theorem 4.1 and 4.2, which we summarize in the following statement.

Theorem 15 (Reny 2008). *Suppose that, for every player i :*

- G.1 Θ_i is a complete separable metric space endowed with a measurable partial order
- G.2 μ_i assigns probability zero to any Borel subset of T_i having no strictly ordered points
- G.3 A_i is a compact locally-complete metric semilattice
- G.4 $u_i(\cdot, \theta) : A \rightarrow \mathbb{R}$ is continuous for every $\theta \in \Theta$.

For each i , let C_i be a join-closed, piecewise-closed, and pointwise-limit-closed subset of pure strategies containing at least one monotone pure strategy, such that the intersection of C_i with i 's set of monotone best replies is nonempty whenever every other player j employs a monotone pure strategy in C_j . Then G possesses a monotone pure-strategy equilibrium in which each player i 's pure strategy is in C_i .

We first show that the election is weakly quasi-supermodular and obeys single-crossing in \geq , which will be useful later.

Lemma 16. *The voting game is weakly quasi-supermodular in actions, i.e.*

$$\begin{aligned} \int_{\Theta_{-i}} \theta_i(A_i, s_{-i}(\theta_{-i})) d\mu_{-i} &\geq \int_{\Theta_{-i}} \theta_i(A_i \cap B_i, s_{-i}(\theta_{-i})) d\mu_{-i} \\ \implies \int_{\Theta_{-i}} \theta_i(A_i \cup B_i, s_{-i}(\theta_{-i})) d\mu_{-i} &\geq \int_{\Theta_{-i}} \theta_i(A_i, s_{-i}(\theta_{-i})) d\mu_{-i} \end{aligned}$$

Proof. We show that supermodularity *in outcomes* of the ex post utilities implies weak quasi-supermodularity *in actions* of the interim utilities. So, suppose the hypothesis inequality holds. Carrying the notation from the proof of Theorem 1, this can be rewritten as

$$\sum_{C, D \in \mathcal{D}} \theta_i([A_i \cap C] \cup D) \times [\pi_i(s)](C, D) \geq \sum_{C, D \in \mathcal{D}} \theta_i([A_i \cap B_i \cap C] \cup D) \times [\pi_i(s)](C, D).$$

Applying supermodularity of θ_i to the sets $[A_i \cap C] \cup D$ and $[B_i \cap C] \cup D$:

$$\sum_{C, D \in \mathcal{D}} [\theta_i([A_i \cap C] \cup D) - \theta_i([A_i \cap B_i \cap C] \cup D)] \times [\pi_i(s)](C, D) \geq 0$$

implies

$$\sum_{C,D \in \mathcal{D}} [\theta_i([(A_i \cup B_i) \cap C] \cup D) - \theta_i([A_i \cap C] \cup D)] \times [\pi_i(s)](C, D) \geq 0,$$

which can be rewritten as the desired conclusion. \square

Lemma 17. *The voting game satisfies weak single-crossing in \geq , i.e. if $\theta'_i \geq \theta_i$ and $A'_i \supseteq A_i$, then*

$$\theta_i(F(A'_i, s_{-i}(\theta_{-i}))) \geq \theta_i(F(A_i, s_{-i}(\theta_{-i}))) \implies \theta'_i(F(A'_i, s_{-i}(\theta_{-i}))) \geq \theta'_i(F(A_i, s_{-i}(\theta_{-i})))$$

for any profile s_{-i} of monotone strategies by the other voters.

Proof. Suppose $\theta'_i \geq \theta_i$ and fix a monotone strategy profile s_{-i} for the other voters. Suppose $A'_i \supseteq A_i$. Then $F(A'_i, s_{-i}(\theta_{-i})) \supseteq F(A_i, s_{-i}(\theta_{-i}))$ for any $\theta_{-i} \in \Theta_{-i}$. By construction of the partial order \geq ,

$$\theta'_i(F(A'_i, s_{-i}(\theta_{-i}))) - \theta'_i(F(A_i, s_{-i}(\theta_{-i}))) \geq \theta_i(F(A'_i, s_{-i}(\theta_{-i}))) - \theta_i(F(A_i, s_{-i}(\theta_{-i}))).$$

This inequality is preserved by integration:

$$\begin{aligned} & \int_{\Theta_{-i}} \theta'_i(F(A'_i, s_{-i}(\theta_{-i}))) d\mu_{-i} - \int_{\Theta_{-i}} \theta'_i(F(A_i, s_{-i}(\theta_{-i}))) d\mu_{-i} \\ & \geq \int_{\Theta_{-i}} \theta_i(F(A'_i, s_{-i}(\theta_{-i}))) d\mu_{-i} - \int_{\Theta_{-i}} \theta_i(F(A_i, s_{-i}(\theta_{-i}))) d\mu_{-i}. \end{aligned}$$

Then if

$$\theta_i(F(A'_i, s_{-i}(\theta_{-i}))) \geq \theta_i(F(A_i, s_{-i}(\theta_{-i}))),$$

the inequality implies

$$\theta'_i(F(A'_i, s_{-i}(\theta_{-i}))) \geq \theta'_i(F(A_i, s_{-i}(\theta_{-i}))). \quad \square$$

We can now check the assumptions in Reny's theorem. The technical conditions G.1 to G.4 are straightforward to verify. We will restrict attention to a space of strategies which will induce weakly undominated best responses. Let R_i be the subset of strategies for player i such that (μ_i almost surely): if θ_i is quasi-separable and $A_i^*(\theta_i) = \arg \max_{A \in \mathcal{X}} \theta_i(A)$ is unique, then $s_i(\theta_i) = A_i^*(\theta_i)$. This space is join-closed, pointwise-limit-closed, and piecewise-closed because it is the intersection of two measurable order inequalities (c.f. Reny 2008, Remark 4). To see this, let

$$f_i(\theta) = \begin{cases} A_i^*(\theta_i) & \text{if } \theta_i \text{ is quasi-separable and } A_i^*(\theta_i) = \arg \max_{A \in \mathcal{X}} \theta_i(A) \text{ is unique} \\ \emptyset & \text{otherwise} \end{cases},$$

and similarly

$$g_i(\theta) = \begin{cases} A_i^*(\theta_i) & \text{if } \theta_i \text{ is quasi-separable and } A_i^*(\theta_i) = \arg \max_{A \in \mathcal{X}} \theta_i(A) \text{ is unique} \\ X & \text{otherwise} \end{cases}.$$

And $R_i = \{s_i \in S_i : f_i(\theta_i) \subseteq s_i(\theta_i) \subseteq g_i(\theta_i), \mu_i\text{-a.s.}\}$.

We next show that there exists a monotone strategy in R_i . Define the following strategy:

$$s_i(\theta_i) = \bigcup_{\theta'_i \leq \theta_i} f_i(\theta'_i).$$

By construction, s_i is monotone. Now suppose $\theta'_i \geq \theta_i$ are quasi-separable with respective unique maximizers $A_i^*(\theta'_i), A_i^*(\theta_i)$. By repeated application of quasi-separability, we have

$$\theta(A_i^*(\theta_i) \cup A_i^*(\theta'_i)) - \theta(A_i^*(\theta'_i)) \geq 0.$$

Considering the definition of \geq ,

$$\theta'(A_i^*(\theta'_i) \cup A_i^*(\theta_i)) - \theta'(A_i^*(\theta'_i)) \geq 0.$$

Since $A_i^*(\theta'_i)$ is the unique maximizer of θ' , this forces $A_i^*(\theta'_i) \cup A_i^*(\theta_i) = A_i^*(\theta'_i)$, i.e. $A_i^*(\theta'_i) \supseteq A_i^*(\theta_i)$. So if θ'_i is separable and has a unique maximizer $A_i^*(\theta'_i)$, then $f_i(\theta'_i) \supseteq f_i(\theta_i)$ for all $\theta \leq \theta'_i$. Hence $s_i(\theta'_i) = A_i^*(\theta_i)$. So $s_i \in R_i$.

Finally, we prove that any monotone strategy in R_{-i} will induce a monotone best reply in R_i . Since R_i is a superset of the weakly undominated strategies for i , clearly for any strategy profile s_{-i} of the other voters, there is some element of R_i which is a best response. Moreover, any best reply to a strategy profile from R must be weakly undominated. This is because the quasi-separable types with unique maximizer A constitute a relatively open subset of the supermodular types, so every ballot has positive probability for each voter by the full support assumption. Standard lattice arguments show that weak quasi-supermodularity and weak single-crossing imply that the point-wise join of each type's best replies in weakly undominated strategies constitutes a monotone best reply itself; for example, see the proof of Corollary 4.3 in Reny (2008). Since R_i is join-closed, this monotone best reply lives in R_i . So, there exists an equilibrium in strategies in R_i , and by construction this equilibrium must be in weakly undominated strategies.

A.4 Proof of Proposition 5

Fix $x \in X$ and take any set $B \subseteq X \setminus \{x\}$ and consider the strategies $A = B \cup \{x\}$ and B . Recall, from the proof of Theorem 1, that for a fixed electorate size I , the expression (3) for $[\pi_i(s)](C, D)$ reflects the probability induced by strategy profile s that voter i will be pivotal on the issues in C and the issues in D will pass irrespective of her vote. For ease of notation, let $P_I^* = \pi_i(s_I^*)$ in the election with I voters.

Then the incentive condition for A being a better reply than A' to s_I^* is

$$\sum_{C, D \in \mathcal{D}} P_I^*(C, D) \cdot \theta([A \cap C] \cup D) \geq \sum_{C, D \in \mathcal{D}} P_I^*(C, D) \cdot \theta([A' \cap C] \cup D) \quad (4)$$

However, since if $x \notin C$, we have $C \cap A \cup D = C \cap A' \cup D$, the only relevant components of the sums on both sides of this inequality are:

$$\sum_{C, D \in \mathcal{D}: x \in C} P_I^*(C, D) \cdot \theta_i([A \cap C] \cup D) \geq \sum_{C, D \in \mathcal{D}: x \in C} P_I^*(C, D) \cdot \theta_i([A' \cap C] \cup D)$$

Dividing both sides by $\sum_{(C, D): x \in C} P_I^*(C, D) > 0$, we can replace the unconditional probabilities P_I^* with the conditional probabilities $P_I^*(C, D | x \in C)$:

$$\sum_{C, D \in \mathcal{D}: x \in C} P_I^*(C, D | x \in C) \cdot \theta_i(C \cap A \cup D) \geq \sum_{C, D \in \mathcal{D}: x \in C} P_I^*(C, D | x \in C) \cdot \theta_i(C \cap A' \cup D). \quad (5)$$

We can rewrite the left hand side as:

$$\begin{aligned} & \sum_{C,D \in \mathcal{D}: x \in C} P_I^*(C, D | x \in C) \cdot \theta_i(C \cap A \cup D) \\ &= P_I^*({x}, D_x | x \in C) \theta_i({x} \cup D_x) + \sum_{C,D: x \in C, D \neq D_x} P_I^*(C, D | x \in C) \cdot \theta_i(C \cap A \cup D) \end{aligned}$$

Similarly rewriting the right hand side, the incentive inequality (5) can be rewritten:

$$\begin{aligned} & P_I^*({x}, D_x | x \in C) \cdot \theta_i({x} \cup D_x) + \sum_{C,D: x \in C, D \neq D_x} P_I^*(C, D | x \in C) \cdot \theta_i(C \cap A \cup D) \\ & \geq P_I^*({x}, D_x | x \in C) \cdot \theta_i(D_x) + \sum_{C,D: x \in C, D \neq D_x} P_I^*(C, D | x \in C) \cdot \theta_i(C \cap A' \cup D) \end{aligned}$$

This is rearranged as

$$\theta({x} \cup D_x) \geq \theta_i(D_x) + \Delta(x, I) \tag{6}$$

where

$$\Delta_I = \frac{\sum_{C,D: x \in C, D \neq D_x} P_I^*(C, D | x \in C) [\theta_i(C \cap A' \cup D) - \theta_i(C \cap A \cup D)]}{P_I^*(x, D_x | x \in C)}.$$

If $|\theta_i({x} \cup D_x) - \theta_i(D_x)| > \Delta(x, I)$ for all $x \in X$, her best response A to $s_I(\theta)$ is determined by

$$x \in A \iff \theta({x} \cup D_x) \geq \theta(D_x).$$

To see this, suppose not. Then there exists some alternate ballot A' such that either $x \in A$ but $\theta(D_x) > \theta({x} \cup D_x)$, or $x \notin A$ but $\theta({x} \cup D_x) > \theta(D_x)$. Assume the former, the second case being entirely similar. Since $|\theta({x} \cup D_x) - \theta(D_x)| > \Delta_I$, the inequality is stronger: $\theta(D_x) > \theta({x} \cup D_x) + \Delta_I$. Now consider the alternative $A = {x} \cup A'$. Then, since inequality (6) was shown to be equivalent to inequality (4), the inequality $\theta({x} \cup D_x) > \theta(D_x) + \Delta_I$ implies A is a strictly better response than A' , a contradiction.

We finally show that this equivalence holds for an arbitrarily large measure of types at the limie. Whenever $D \neq D_x$, then

$$\begin{aligned} P_I^*(C, D | x \in C) &= \mathbf{P} \left(\begin{array}{l} \#\{j \neq i : y \in s_I^*(\theta_i)\} = \frac{I-1}{2} \quad \forall y \in C \\ \#\{j \neq i : y \in s_I^*(\theta_i)\} > \frac{I-1}{2} \quad \forall y \in D \\ \#\{j \neq i : y \in s_I^*(\theta_i)\} < \frac{I-1}{2} \quad \forall y \notin C \cup D \end{array} \middle| \#\{j \neq i : x \in s_I^*(\theta_i)\} = \frac{I-1}{2} \right) \\ &\leq 1 - \mathbf{P} \left(\#\{j \neq i : y \in s_I^*(\theta_i)\} > \frac{I-1}{2} \quad \forall y \in D_x \middle| \#\{j \neq i : x \in s_I^*(\theta_i)\} = \frac{I-1}{2} \right) \\ &\rightarrow 0 \end{aligned}$$

Noticing that $|\theta(\cdot)| < 1$, we have $\Delta_I \rightarrow 0$ as $I \rightarrow \infty$.

Observe that the set of types for which $|\theta({x} \cup D_x) - \theta(D_x)| > \Delta_I$ for all $x \in X$ is of full Lebesgue measure at the limit, since $\Delta_I \rightarrow 0$. Invoking the density assumption, this set also has full μ measure at the limit.

A.5 Proof of Proposition 6

Without loss of generality, suppose voter I is pivotal on issue 1 and consider whether issue 2 is conditionally certain to pass or fail. Let X_k^{Ii} ($I = 1, 3, \dots; i = 1, \dots, I-1; k = 1, 2$) denote the triangular array of indicator

functions on the events $k \in s_I^*(\theta_i)$. While X_2^{Ii} are unconditionally rowwise independent, this independence is broken once we condition on voter I being pivotal on issue 1. This precludes a straightforward application of the law of large numbers to the array and requires more delicacy.

The basic logic is to split the sample of $I - 1$ other voters into two subsamples: those $\frac{I-1}{2}$ who voted for issue 1, and those $\frac{I-1}{2}$ who did not. Within each subsample, the votes on issue k are conditionally identical and independent because the votes on issue 1 are fixed. However, by exchangeability, the particular identity of voters in each subsample is irrelevant, so we can proceed without loss of generality by assuming the first half of other voters constitute the first subsample while the remainder constitute the second.

Formally, consider the following arrays of rowwise independent binary random variables:

$$Y^{Ii} = \begin{cases} 1 & \text{with probability } \mu(2 \in s_I(\theta_i) | 1 \notin s_I(\theta_I)) \\ 0 & \text{with probability } \mu(2 \notin s_I(\theta_i) | 1 \notin s_I(\theta_I)) \end{cases}$$

and

$$Z^{Ii} = \begin{cases} 1 & \text{with probability } \mu(2 \in s_I(\theta_i) | 1 \in s_I(\theta_I)) \\ 0 & \text{with probability } \mu(2 \notin s_I(\theta_i) | 1 \in s_I(\theta_I)) \end{cases}$$

Lemma 18. *The distribution of $\sum_{i=1}^{I-1} X_2^{Ii}$ conditional on $\sum_{i=1}^{I-1} X_1^{Ii} = \frac{I-1}{2}$ is identical to the distribution of the sum*

$$\sum_{i=1}^{\frac{I-1}{2}} Y^{Ii} + \sum_{i=1}^{\frac{I-1}{2}} Z^{Ii}.$$

Proof. Suppressing the I superscript for the size of the electorate and fixing any integer n :

$$\begin{aligned} & \mathbf{P} \left(\sum_{i=1}^{I-1} X_2^i = n \mid \sum_{i=1}^{I-1} X_2^i = \frac{I-1}{2} \right) \\ &= \sum_{A \subset I-1: \#A = \frac{I-1}{2}} \left[\mathbf{P} \left(\sum_{i \in A} X_2^i = 0 \mid \sum_{i=1}^{I-1} X_1^i = \frac{I-1}{2} \right) \mathbf{P} \left(\sum_{i=1}^{I-1} X_2^i = n \mid \sum_{i \in A} X_1^i = 0, \sum_{j \notin A} X_1^j = \frac{I-1}{2} \right) \right] \end{aligned}$$

By exchangeability across voters, the particular identities of the voters in the set A that voted up on issue 1 is irrelevant. In other words, we can assume without loss that the first $\frac{I-1}{2}$ other voters included 1 in their ballots and the last $\frac{I-1}{2}$ other voters excluded 1 from their ballots. The prior expression is therefore equal to:

$$\begin{aligned} &= \sum_{A \subset I-1: \#A = \frac{I-1}{2}} \left[\mathbf{P} \left(\sum_{i=1}^{\frac{I-1}{2}} X_2^i = 0 \mid \sum_{i=1}^{I-1} X_2^i = \frac{I-1}{2} \right) \mathbf{P} \left(\sum_{i=1}^{I-1} X_2^i = n \mid \sum_{i=1}^{\frac{I-1}{2}} X_1^i = 0, \sum_{j=\frac{I+1}{2}}^{I-1} X_1^j = \frac{I-1}{2} \right) \right] \\ &= \mathbf{P} \left(\sum_{i=1}^{I-1} X_2^i = n \mid \sum_{i=1}^{\frac{I-1}{2}} X_1^i = 0, \sum_{j=\frac{I+1}{2}}^{I-1} X_1^j = \frac{I-1}{2} \right) \\ &= \sum_{m=0}^{\frac{I-1}{2}} \left[\mathbf{P} \left(\sum_{i=1}^{\frac{I-1}{2}} X_2^i = m \mid \sum_{i=1}^{\frac{I-1}{2}} X_1^i = 0 \right) \mathbf{P} \left(\sum_{j=\frac{I+1}{2}}^{I-1} X_2^j = n - m \mid \sum_{j=\frac{I+1}{2}}^{I-1} X_1^j = \frac{I-1}{2} \right) \right] \\ &= \sum_{m=0}^{\frac{I-1}{2}} \left[\mathbf{P} \left(\sum_{i=1}^{\frac{I-1}{2}} Y^{Ii} = m \right) \mathbf{P} \left(\sum_{i=1}^{\frac{I-1}{2}} Z^{Ii} = n - m \right) \right] \end{aligned}$$

$$= \mathbf{P} \left(\sum_{i=1}^{\frac{I-1}{2}} Y^{Ii} + \sum_{i=1}^{\frac{I-1}{2}} Z^{Ii} = n \right) \quad \square$$

Lemma 19. *The normalized sum $\frac{\sum_{i=1}^{I-1} X_k^{Ii}}{I-1}$ conditional on $\sum_{i=1}^{I-1} X_1^{Ii} = \frac{I-1}{2}$ converges in probability to $\frac{1}{2}\mu(y \in s_I^*(\theta) | x \in s_I^*(\theta)) + \frac{1}{2}\mu(y \in s_I^*(\theta) | x \notin s_I^*(\theta))$.*

Proof. We can apply the Strong Law of Large Numbers for triangular arrays to Y^{Ii} and Z^{Ii} to conclude that

$$\left(\frac{I-1}{2} \right)^{-1} \sum_{i=1}^{\frac{I-1}{2}} Y^{Ii} \rightarrow \mu(2 \in s_I^*(\theta_i) | 1 \notin s_I^*(\theta_i))$$

and

$$\left(\frac{I-1}{2} \right)^{-1} \sum_{i=1}^{\frac{I-1}{2}} Z^{Ii} \rightarrow \mu(2 \in s_I^*(\theta_i) | 1 \in s_I^*(\theta_i))$$

almost surely, hence in distribution. By the Continuous Mapping Theorem, the sum

$$\frac{1}{2} \left(\frac{I-1}{2} \right)^{-1} \sum_{i=1}^{\frac{I-1}{2}} Y^{Ii} + \frac{1}{2} \left(\frac{I-1}{2} \right)^{-1} \sum_{i=1}^{\frac{I-1}{2}} Z^{Ii} \quad (7)$$

converges in distribution to the constant

$$\frac{1}{2}\mu(2 \in s_I^*(\theta) | 1 \in s_I^*(\theta)) + \frac{1}{2}\mu(2 \in s_I^*(\theta) | 1 \notin s_I^*(\theta)).$$

Since, by Lemma 18, the conditional distribution of $\frac{\sum_{i=1}^{I-1} X_2^{Ii}}{I-1}$ shares the distribution of (7), it also converges in distribution to the same constant. As convergence in distribution to a constant implies convergence in probability, this delivers the desired conclusion. \square

To prove the Proposition, suppose $\mu(2 \in s(\theta) | 1 \in s(\theta)) > \mu(2 \notin s(\theta) | 1 \notin s(\theta))$. Then:

$$\begin{aligned} \mu(2 \in s(\theta) | 1 \in s(\theta)) &> \mu(2 \notin s(\theta) | 1 \notin s(\theta)) \\ \mu(2 \in s(\theta) | 1 \in s(\theta)) + 1 - \mu(2 \notin s(\theta) | 1 \notin s(\theta)) &> 1 \\ \mu(2 \in s(\theta) | 1 \in s(\theta)) + \mu(2 \in s(\theta) | 1 \notin s(\theta)) &> 1 \\ \frac{1}{2}\mu(2 \in s(\theta) | 1 \in s(\theta)) + \frac{1}{2}\mu(2 \in s(\theta) | 1 \notin s(\theta)) &> \frac{1}{2} \end{aligned}$$

Let $E = \frac{1}{2}\mu(2 \in s(\theta) | 1 \in s(\theta)) + \frac{1}{2}\mu(2 \in s(\theta) | 1 \notin s(\theta))$ and pick a strictly positive $\delta < E - \frac{1}{2}$. By Lemma 19, the probability that the normalized vote count on issue 2, conditional on voter I being pivotal on 1, is greater than $E - \delta > \frac{1}{2}$ approaches one. Thus, the conditional probability that 2 passes converges to one.

Symmetrically, assuming the opposite strict inequality, the conditional probability of y failing converges to one.

Remark. The proof only requires that there exists some $\delta > 0$ such that

$$\mu(y \in s_I(\theta) | x \in s_I(\theta)) - \mu(y \notin s_I(\theta) | x \notin s_I(\theta)) > (<) \delta$$

for sufficiently large I , irrespective of whether the sequence has a limit or not.

A.6 Proof of Proposition 8

It suffices to prove the case $A = \{1, 2\}$. Other cases then follow by appropriately permuting the direction of “pass” or “fail” on the ballot. For example, when $A = \{1\}$, consider the following permutation:

$$\{1, 2\} \mapsto \{1\}, \{1\} \mapsto \{1, 2\}, \{2\} \mapsto \emptyset, \emptyset \mapsto \{2\}.$$

Similar permutations apply for $A = \{2\}$ and $A = \emptyset$.

So let

$$\Theta_{12} = \{\theta : \theta_{12} \geq \max\{\theta_1, \theta_2\}\}$$

$$\Theta_1 = \{\theta : \theta_1 \geq \theta_{12} \geq \theta_2\}$$

$$\Theta_2 = \{\theta : \theta_2 \geq \theta_{12} \geq \theta_1\}$$

$$\Theta_\emptyset = \{\theta : \theta_{12} \leq \min\{\theta_1, \theta_2\}\}$$

These four sets of types cover Θ . Since μ has full support and admits a density, they have strictly positive probability but null pairwise intersections. By assumption,

$$\frac{\mu(\Theta_{12})}{\mu(\Theta_\emptyset)} > \max \left\{ \frac{\mu(\Theta_1)}{\mu(\Theta_2)}, \frac{\mu(\Theta_2)}{\mu(\Theta_1)} \right\}.$$

Let

$$\mathcal{P}_n = \{P \in \Delta^U : P(x, x' | x \in C) \geq 1 - \frac{1}{n}, \forall x = 1, 2\}$$

Recall that $P(C, D)$ is the probability that an anonymous voter is pivotal on the issues in C and that the issues in D will pass irrespective of her ballot. Let $A \subset \{x'\}$ and consider $A' = \{x\} \cup A$. The incentive condition for A' being a better reply than A given the belief $P \in \mathcal{P}_n$ over pivotal and passing events is:

$$\theta_{12} \geq \theta_x + \Delta_n(\theta)$$

where

$$\Delta_n(\theta) = \frac{P(x, \emptyset | x \in C) [\theta_\emptyset - \theta_x] + P(\{1, 2\}, \emptyset | x \in C) [\theta_A - \theta_{A'}]}{P(x, x' | x \in C)}.$$

Observe that $\Delta_n(\theta) \rightarrow 0$ as $n \rightarrow \infty$. Let $\Theta_n = \{\theta : |\theta_{12} - \theta_x| > \Delta_n\}$ and notice $\mu(\Theta_n) \rightarrow 1$. As $n \rightarrow \infty$, the proportion of types in Θ_{12} which include x in their optimal ballots covers the entire subset Θ_{12} , while the proportion of types in $\Theta \setminus \Theta_{12}$ which include x becomes null. Similarly arguing for $\Theta_1, \Theta_2, \Theta_\emptyset$, we have

$$\sigma_I(P_n) \rightarrow s$$

where $s_A(\theta_A) = A$ for all $\theta_A \in \Theta_A$, for any sequence of selections $P_n \in \mathcal{P}_n$. Then

$$\frac{\mu([\sigma_I(P)](\theta) = \{1, 2\})}{\mu([\sigma_I(P)](\theta) = \emptyset)} - \max \left\{ \frac{\mu([\sigma(P)](\theta) = \{1\})}{\mu([\sigma(P)](\theta) = \{2\})}, \frac{\mu([\sigma_I(P)](\theta) = \{2\})}{\mu([\sigma_I(P)](\theta) = \{1\})} \right\}$$

which is arbitrarily close to

$$\frac{\mu(\Theta_{12})}{\mu(\Theta_\emptyset)} - \max \left\{ \frac{\mu(\Theta_1)}{\mu(\Theta_2)}, \frac{\mu(\Theta_2)}{\mu(\Theta_1)} \right\} > 0$$

for any $P \in \mathcal{P}_n$ as $n \rightarrow \infty$. So, there exists some n_0 such that if $n > n_0$:

$$\frac{\mu([\sigma_I(P)](\theta) = \{1, 2\})}{\mu([\sigma_I(P)](\theta) = \emptyset)} > \max \left\{ \frac{\mu([\sigma_I(P)](\theta) = \{1\})}{\mu([\sigma_I(P)](\theta) = \{2\})}, \frac{\mu([\sigma_I(P)](\theta) = \{2\})}{\mu([\sigma_I(P)](\theta) = \{1\})} \right\}$$

for all $P \in \mathcal{P}_n$.

So, let $n > n_0$ and consider the sequence of strategies $s_I = \sigma_I(P)$ for any $P \in \mathcal{P}_n$. Fix I and let $\mu_A = \mu([\sigma_I(P)](\theta) = A)$ Then:

$$\begin{aligned} \frac{\mu_{12}}{\mu_\emptyset} &> \frac{\mu_1}{\mu_2} \\ \mu_{12}\mu_2 &> \mu_\emptyset\mu_1 \\ \mu_{12}(\mu_2 + \mu_\emptyset) &> \mu_\emptyset(\mu_1 + \mu_{12}) \\ \frac{\mu_{12}}{\mu_1 + \mu_{12}} &> \frac{\mu_\emptyset}{\mu_2 + \mu_\emptyset} \\ \mu(2 \in [\sigma_I(P)](\theta) \mid 1 \in [\sigma_I(P)](\theta)) &> \mu(2 \notin [\sigma_I(P)](\theta) \mid 1 \notin [\sigma_I(P)](\theta)) \end{aligned}$$

By Proposition 5 (see also the remark immediately following the proof), for I sufficiently large, the probability $\pi_I(\sigma_I)$ satisfies

$$[\pi_I(\sigma_I(P))](x, x' \mid x \in C) \geq 1 - \frac{1}{n}$$

for $x = 1, 2$. This means for sufficiently large $I > I_0$, the image $[\pi \circ \sigma_I](\mathcal{P}_n) \subseteq \mathcal{P}_n$. It therefore admits a fixed point $P_I^* \in \mathcal{P}_n$ which defines an equilibrium s_I^* , for all sufficiently large $I > I_0(n)$.

Finally, define $n(I) = \max\{n : I > I_0(n)\} \vee I$. Observe that as $I \rightarrow \infty$, we have $n(I) \rightarrow \infty$. For each I , select a fixed point $P_I^* \in [\pi \circ \sigma](\mathcal{P}_{n(I)})$. The induced equilibrium strategy s_I^* satisfies

$$\mu(\sigma_I^*(\theta) \mid 1 \in [\sigma_I(P)](\theta)) > \mu(2 \notin [\sigma_I(P)](\theta) \mid 1 \notin [\sigma_I(P)](\theta))$$

By Proposition 5, the set $\{1, 2\}$ is conditionally certain. Also, recalling the construction, $s_I^* \rightarrow s^*$ where $s^*(\Theta_A) = A$.

A.7 Proof of Proposition 9

As argued in the proof of Proposition 8, it suffices to prove the case $A = \{1, 2\}$. Suppose each issue is conditionally certain to pass. In particular, 2 is conditionally certain to pass at 1. By Corollary 7

$$\mu[2 \in s^*(\theta_i) \mid 1 \in s^*(\theta_i)] \geq \mu[2 \notin s^*(\theta_i) \mid 1 \notin s^*(\theta_i)].$$

For notational convenience, let $\mu_A^* = \mu(\{\theta_i : s^*(\theta_i) = A\})$. Since 2 is conditionally certain to pass at 1, we have:

$$\begin{aligned} \frac{\mu_{12}^*}{\mu_{12}^* + \mu_1^*} &\geq \frac{\mu_\emptyset^*}{\mu_\emptyset^* + \mu_2^*} \\ \mu_{12}^*(\mu_\emptyset^* + \mu_2^*) &\geq \mu_\emptyset^*(\mu_{12}^* + \mu_1^*) \\ \mu_{12}^*\mu_2^* &\geq \mu_\emptyset^*\mu_1^* \\ \frac{\mu_{12}^*}{\mu_\emptyset^*} &\geq \frac{\mu_1^*}{\mu_2^*} \end{aligned}$$

Symmetrically, since 1 is conditionally certain to pass at 2:

$$\frac{\mu_{12}^*}{\mu_{\emptyset}^*} \geq \frac{\mu_2^*}{\mu_1^*}$$

The prior two inequalities imply

$$\frac{\mu_{12}^*}{\mu_{\emptyset}^*} \geq \max \left\{ \frac{\mu_1^*}{\mu_2^*}, \frac{\mu_2^*}{\mu_1^*} \right\} \quad (8)$$

By Proposition 5, we have

$$x \in s^*(\theta_i) \iff \theta_i(\{1, 2\}) \geq \theta(\{x'\})$$

Thus:

$$s^*(\theta) = \begin{cases} \{1, 2\} & \text{if } \theta_{12} \geq \max\{\theta_1, \theta_2\} \\ \{1\} & \text{if } \theta_1 \geq \theta_{12} \geq \theta_2 \\ \{2\} & \text{if } \theta_2 \geq \theta_{12} \geq \theta_1 \\ \emptyset & \text{if } \theta_{12} \leq \min\{\theta_1, \theta_2\} \end{cases}$$

Substituting these cases into condition (8) delivers the result.

A.8 Proof of Proposition 10

Without loss of generality, consider the case $x = 2$. Let

$$\alpha_I = \mathbf{P} \left(\#\{j \neq i : 2 \in s_I^*(\theta_k)\} > \frac{I-1}{2} \mid \#\{j \neq i : 1 \in s_I^*(\theta_k)\} = \frac{I-1}{2} \right).$$

and

$$\beta_I = \mathbf{P} \left(\#\{j \neq i : 2 \in s_I^*(\theta_k)\} < \frac{I-1}{2} \mid \#\{j \neq i : 1 \in s_I^*(\theta_k)\} = \frac{I-1}{2} \right).$$

By the full support assumption, the conditional probability of being pivotal on issue 2 when pivotal on issue 1 vanishes, so $\alpha_I + \beta_I \rightarrow 1$.

Fix a voter i with type θ and consider a ballot $A \subseteq \{1\}$ which does not include 2. The incentive condition for $\{2\}$ being a better reply than $\{2\} \cup A$ to the strategy s_I^* is:

$$\alpha_I \theta_{12} + \beta_I \theta_2 + (1 - \alpha_I - \beta_I) \theta_{2 \cup A} \geq \alpha_I \theta_1 + \beta_I \theta_{\emptyset} + (1 - \alpha_I - \beta_I) \theta_A.$$

Passing to a subsequence if necessary, there exists an α such that $\alpha_I \rightarrow \alpha$. The incentive inequality can be rewritten as

$$\alpha \theta_{12} + (1 - \alpha) \theta_2 \geq \alpha \theta_1 + (1 - \alpha) \theta_{\emptyset} + \Delta_I$$

where

$$\Delta_I = [\alpha_I - \alpha](\theta_1 - \theta_{12}) + [\beta_I - (1 - \alpha)](\theta_{\emptyset} - \theta_2) + [1 - \alpha_I - \beta_I](\theta_A - \theta_{2 \cup A}).$$

However, $\Delta_I \rightarrow 0$. From here, we can replicate the arguments which conclude the proof of Proposition 5 to conclude that, at the limit, the set of types which support issue 2 is characterized by the inequality

$$\alpha \theta_{12} + (1 - \alpha) \theta_x \geq \alpha \theta_{x'} + (1 - \alpha) \theta_{\emptyset}.$$

A.9 Proof of Propostion 11

Suppose $\varepsilon < \frac{1}{16}$. Let μ be any density in the class described in Example 2. We first prove that there must exist be at least a single issue which exhibits conditional uncertainty.

Lemma 20. *For any density in \mathcal{C} , there is no sequence of equilibria that exhibits conditional certainty.*

Proof. To see that $\{1, 2\}$ cannot be conditionally certain, observe that

$$\frac{\mu(\theta_{12} \geq \max\{\theta_1, \theta_2\})}{\mu(\theta_{12} \leq \min\{\theta_1, \theta_2\})} \leq \frac{\frac{1}{4} + \varepsilon}{\frac{1}{4} - \varepsilon}.$$

For small ε , this ratio approximates one. On the other hand,

$$\frac{\mu(\theta_2 \geq \theta_{12} \geq \theta_1)}{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)} \geq \frac{\frac{1}{2} - \varepsilon}{\varepsilon}.$$

For small ε , this ratio becomes arbitrarily large. This precludes the required inequality in Proposition 9 for $A = \{1, 2\}$. An entirely similar argument proves that the inequality also fails for $A = \{1\}, \{2\}, \emptyset$. By Proposition 9, there cannot be an equilibrium with conditional certainty. \square

By Lemma 20, we can assume that there is some issue with conditional uncertainty. We now prove that this implies the other issue must also be conditionally uncertain. We demonstrate that if issue 2 is conditionally uncertain, then it cannot be the case that issue 1 is conditionally certain to pass. The other cases can be argued symmetrically.

So, suppose issue 1 is conditionally certain to pass. Recall μ_A^* denotes the probability that an anonymous voter submits the ballot A when playing the limit strategy s^* . If 1 is conditionally certain to pass, then $\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \frac{\mu_2^*}{\mu_1^*}$. Since 2 is conditionally uncertain, it is not conditionally certain to fail. We therefore conclude $\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \frac{\mu_1^*}{\mu_2^*}$. So, when issue 2 is conditionally uncertain, the following inequality:

$$\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \max \left\{ \frac{\mu_1^*}{\mu_2^*}, \frac{\mu_2^*}{\mu_1^*} \right\}, \quad (9)$$

is still necessary for issue 1 to be conditionally certain to pass.

In view of Proposition 5 and Proposition 10, there exists some $\alpha \in (0, 1)$ such that the following describes the limit strategy in terms of types:

$$s^*(\theta) = \begin{cases} \{1, 2\} & \text{if } \theta_{12} \geq \theta_2 \text{ and } \alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \\ \{1\} & \text{if } \theta_{12} \geq \theta_2 \text{ and } \alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \\ \{2\} & \text{if } \theta_{12} \leq \theta_2 \text{ and } \alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \\ \emptyset & \text{if } \theta_{12} \leq \theta_2 \text{ and } \alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \end{cases}$$

Let

$$\begin{aligned} \varphi_{12}(\alpha) &= \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \mid \theta_{12} \geq \theta_1 \geq \theta_\emptyset \geq \theta_2) \\ \varphi_\emptyset(\alpha) &= \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \mid \theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1). \end{aligned}$$

Observe that, since μ admits has full support and admits a density, φ_{12} and φ_\emptyset are increasing and continuous functions with $\varphi_{12}(0) = \varphi_\emptyset(0) = 0$ and $\varphi_{12}(1) = \varphi_\emptyset(1) = 1$.

Now we can rewrite the limit probability of voting for both issues as:

$$\begin{aligned}
\mu_{12}^* &= \mu(\theta_{12} \geq \theta_2) \cdot \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_0 \mid \theta_{12} \geq \theta_2) \\
&\leq \mu(\theta_{12} \geq \theta_1 \geq \theta_0 \geq \theta_2) \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_0 \mid \theta_{12} \geq \theta_1 \geq \theta_0 \geq \theta_2) + \varepsilon \\
&= \frac{1}{4} \varphi_{12}(\alpha) + \varepsilon.
\end{aligned}$$

Likewise, the limit probability of voting down on both issues can be rewritten as:

$$\begin{aligned}
\mu_0^* &= \mu(\theta_{12} \leq \theta_2) \cdot \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_0 \mid \theta_{12} \geq \theta_2) \\
&\geq \mu(\theta_1 \geq \theta_0 \geq \theta_2 \geq \theta_{12}) \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_0 \mid \theta_1 \geq \theta_0 \geq \theta_2 \geq \theta_{12}) \\
&\quad + \mu(\theta_0 \geq \theta_2 \geq \theta_{12} \geq \theta_1) \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_0 \mid \theta_0 \geq \theta_2 \geq \theta_{12} \geq \theta_1) \\
&\quad + \mu(\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_0) \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_0 \mid \theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_0) \\
&= \mu(\theta_1 \geq \theta_0 \geq \theta_2 \geq \theta_{12}) + \mu(\theta_0 \geq \theta_2 \geq \theta_{12} \geq \theta_1)(1 - \varphi_0(\alpha)) \\
&\geq \frac{1}{4} + \frac{1}{4}(1 - \varphi_0(\alpha)) - \varepsilon
\end{aligned}$$

So

$$\frac{\mu_{12}^*}{\mu_0^*} \leq \frac{\frac{1}{4} \varphi_{12}(\alpha) + \varepsilon}{\frac{1}{4} + \frac{1}{4}(1 - \varphi_0(\alpha)) - \varepsilon} \quad (10)$$

Since (9) provides that $\frac{\mu_{12}^*}{\mu_0^*} \geq \frac{\mu_1^*}{\mu_2^*}$. So (10) implies:

$$\frac{\mu_1^*}{\mu_2^*} \leq \frac{\frac{1}{4} \varphi_{12}(\alpha) + \varepsilon}{\frac{1}{4} + \frac{1}{4}(1 - \varphi_0(\alpha)) - \varepsilon} \quad (11)$$

Again by (9), $\frac{\mu_{12}^*}{\mu_0^*} \geq 1$. Therefore (10) also implies:

$$\begin{aligned}
\frac{1}{4} \varphi_{12}(\alpha) + \varepsilon &\geq \frac{1}{4} + \frac{1}{4}(1 - \varphi_0(\alpha)) - \varepsilon \\
\varphi_{12}(\alpha) &\geq 2 - \varphi_0(\alpha) - 8\varepsilon \\
\varphi_{12}(\alpha) &> 1 - 8\varepsilon.
\end{aligned} \quad (12)$$

Symmetrically,

$$\varphi_0(\alpha) > 1 - 8\varepsilon. \quad (13)$$

We can rewrite the limit probability of voting only for issue 1 as:

$$\begin{aligned}
\mu_1^* &= \mu(\theta_{12} \leq \theta_2) \cdot \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_0 \mid \theta_{12} \geq \theta_2) \\
&\geq \mu(\theta_1 \geq \theta_0 \geq \theta_2 \geq \theta_{12}) \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_0 \mid \theta_1 \geq \theta_0 \geq \theta_2 \geq \theta_{12}) \\
&\quad + \mu(\theta_0 \geq \theta_2 \geq \theta_{12} \geq \theta_1) \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_0 \mid \theta_0 \geq \theta_2 \geq \theta_{12} \geq \theta_1) \\
&\quad + \mu(\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_0) \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_0 \mid \theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_0) \\
&= \mu(\theta_0 \geq \theta_2 \geq \theta_{12} \geq \theta_1) \varphi_0(\alpha) + \mu(\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_0) \\
&\geq \frac{1}{4} \varphi_0(\alpha) + \frac{1}{4} - \varepsilon.
\end{aligned}$$

Similarly, rewriting the probability of voting only for issue 2:

$$\mu_2^* \leq \frac{1}{4}(1 - \varphi_{12}(\alpha)) + \varepsilon.$$

So,

$$\frac{\mu_1^*}{\mu_2^*} \geq \frac{\frac{1}{4}\varphi_0(\alpha) + \frac{1}{4} - \varepsilon}{\frac{1}{4}(1 - \varphi_{12}(\alpha)) + \varepsilon} \quad (14)$$

Combining (11) and (14):

$$\frac{\frac{1}{4}\varphi_{12}(\alpha) + \varepsilon}{\frac{1}{4} + \frac{1}{4}(1 - \varphi_\theta(\alpha)) - \varepsilon} \geq \frac{\frac{1}{4}\varphi_0(\alpha) + \frac{1}{4} - \varepsilon}{\frac{1}{4}(1 - \varphi_{12}(\alpha)) + \varepsilon}.$$

This can be rewritten as

$$(\varphi_{12}(\alpha) + 4\varepsilon)(1 - \varphi_{12}(\alpha) + 4\varepsilon) \geq (\varphi_0(\alpha) + 1 - 4\varepsilon)(2 - \varphi_0(\alpha) - 4\varepsilon). \quad (15)$$

At the same time, recalling earlier inequalities:

$$\begin{aligned} \varphi_{12}(\alpha) + 4\varepsilon &< 1 + 4\varepsilon \\ &< 2 - 12\varepsilon, \text{ since } \varepsilon < \frac{1}{16} \\ &< \varphi_0(\alpha) + 1 - 4\varepsilon, \text{ by (13)}. \end{aligned}$$

And:

$$\begin{aligned} 1 - \varphi_{12}(\alpha) + 4\varepsilon &< 1 - (1 - 8\varepsilon) + 4\varepsilon, \text{ by (12)} \\ &= 12\varepsilon \\ &< 1 - 4\varepsilon, \text{ since } \varepsilon < \frac{1}{16} \\ &< 2 - \varphi_0(\alpha) - 4\varepsilon. \end{aligned}$$

But the prior two series of inequalities contradict (15), since the left hand side is the product of strictly smaller positive quantities than those in the product on the right hand side.

A.10 Proof of Proposition 12

For notational ease, we define the following. Let $\mu^I(x|x') = \mu(x \in s_I(\theta_i) \mid x' \in s_I(\theta_i))$ and $\mu^I(x|\neg x') = \mu(x \in s_I(\theta_i) \mid x' \notin s_I(\theta_i))$. Let $\mu^I(x) = \mu(x \in s_I(\theta_i))$.

Lemma 21. *Issue x is conditionally uncertain if and only if*

$$\lim_{I \rightarrow \infty} \left| \sqrt{(I-1)} \left((\mu^I(x|x') + \mu^I(x|\neg x') - 1) \right) \right| < \infty.$$

Proof. Take $x = 2$; the case $x = 1$ is identical. Recall the two arrays defined in the proof of Proposition 6, rowwise independent binary random variables Y^{Ii} and Z^{Ii} whose success probabilities are $\mu(2 \in s_I(\theta_i) \mid 1 \in s_I(\theta_i))$ and $\mu(2 \in s_I(\theta_i) \mid 1 \notin s_I(\theta_i))$. In Lemma 18 of that proof, we demonstrated that the conditional distribution of the vote count on issue 2 is equal to the distribution of $\sum_{i=1}^{\frac{I-1}{2}} Y^{Ii} + \sum_{i=1}^{\frac{I-1}{2}} Z^{Ii}$. Let $W^{Ii} = Y^{Ii} + Z^{Ii}$. As Y^{Ii} and Z^{Ii} are mutually independent, the array W^{Ii} defines a rowwise independent array of

random variables. We can write that

$$\mathbf{P} \left(\#\{j \neq i : 2 \in s_i^*(\theta_j)\} > \frac{I-1}{2} \mid \#\{j \neq i : 1 \in s_i^*(\theta_j)\} = \frac{I-1}{2} \right) = \mathbf{P} \left(\sum_{i=1}^{\frac{I-1}{2}} W^{Ii} > \frac{I-1}{2} \right)$$

Recalling the definition of the binary random variables $Y^{Ii}(\theta)$ and $Z^{Ii}(\theta)$ we have that

$$\mathbf{E}(W^{Ii}) = \mu^I(2|1) + \mu^I(2|-1)$$

and

$$\mathbf{Var}(W^{Ii}) = \mu^I(2|1) [1 - \mu^I(2|1)] + \mu^I(2|-1) [1 - \mu^I(2|-1)].$$

Applying the Central Limit Theorem for triangular arrays:

$$\mathbf{P} \left(\frac{\sum_{i=1}^{\frac{I-1}{2}} W^{Ii} - \left(\frac{I-1}{2}\right) [\mu^I(2|1) + \mu^I(2|-1)]}{\sqrt{\left(\frac{I-1}{2}\right) (\mu^I(2|1) [1 - \mu^I(2|1)] + \mu^I(2|-1) [1 - \mu^I(2|-1)])}} < y \right) = \Phi(y), \quad (16)$$

where Φ denotes the standard normal cumulative distribution function.

As $I \rightarrow \infty$, the probability that $\sum_{i=1}^{\frac{I-1}{2}} W^{Ii} > \frac{I-1}{2}$ converges to expression (16) evaluated at $y = \frac{I-1}{2}$:

$$\Phi \left(\frac{\sqrt{(I-1)} (1 - (\mu^I(2|1) + \mu^I(2|-1)))}{\sqrt{\mu^I(2|1) [1 - \mu^I(2|1)] + \mu^I(2|-1) [1 - \mu^I(2|-1)]}} \right)$$

Therefore $\lim_{I \rightarrow \infty} \left| \sqrt{(I-1)} (\mu^I(2|1) + \mu^I(2|-1) - 1) \right| < \infty$ is necessary and sufficient for issue 2 to be conditional uncertain. \square

Lemma 22. *Issue x is unconditionally uncertain if and only if $\lim_{I \rightarrow \infty} \left| \sqrt{I} (\mu_k^I - \frac{1}{2}) \right| < \infty$*

Proof. Define the binary random variable

$$V^{Ii} = \begin{cases} 1 & \text{with probability } \mu_k^I \\ 0 & \text{with probability } 1 - \mu_k^I \end{cases}$$

with mean $\mu^I(x)$ and variance $\mu^I(x) (1 - \mu^I(x))$. The probability that issue k will pass (fail) is

$$\mathbf{P} \left(\sum_{i=1}^I V_k^{Ii} > (<) \frac{I}{2} \right).$$

Arguing as in the proof of Lemma 21, we have that the asymptotic (unconditional) probability that issue x will pass is equal to

$$\Phi \left(\frac{\sqrt{I} \left(\frac{1}{2} - \mu^I(x) \right) - \frac{1}{2I}}{\sqrt{\mu^I(x) (1 - \mu^I(x))}} \right).$$

Therefore unconditional uncertainty is equivalent to

$$\lim_{I \rightarrow \infty} \left| \sqrt{I} \left(\frac{1}{2} - \mu^I(x) \right) \right| < \infty. \quad \square$$

To prove the proposition, let

$$\begin{aligned} x^I &= \mu^I(1) & a^I &= \mu^I(1|2) \\ y^I &= \mu^I(2) & b^I &= \mu^I(1|-2) \\ & & c^I &= \mu^I(2|1) \\ & & d^I &= \mu^I(2|-1) \end{aligned}$$

We have a system of two equations with two unknowns, x^I and y^I :

$$x^I = a^I y^I + b^I (1 - y^I) \quad (17)$$

$$y^I = c^I x^I + d^I (1 - x^I) \quad (18)$$

The corresponding solutions for x and y are:

$$x^I = \frac{(a^I - b^I)d^I + b^I}{1 - (a^I - b^I)(c^I - d^I)} \quad (19)$$

$$y^I = \frac{(c^I - d^I)b^I + d^I}{1 - (c^I - d^I)(a^I - b^I)}. \quad (20)$$

We will first prove that if there is conditional uncertainty on both issues, then there must be unconditional uncertainty on both. Subtracting one half from both sides in Equations (19) and (20) yields, after some manipulation:

$$\begin{aligned} x^I - \frac{1}{2} &= \frac{1}{2} \frac{(b^I - a^I)(1 - (c^I + d^I)) + (1 - (a^I + b^I))}{1 - (a^I - b^I)(c^I - d^I)} \\ y^I - \frac{1}{2} &= \frac{1}{2} \frac{(d^I - c^I)(1 - (a^I + b^I)) + (1 - (c^I + d^I))}{1 - (c^I - d^I)(a^I - b^I)}. \end{aligned}$$

By Lemma 21, conditional uncertainty on both issues means

$$\lim_{I \rightarrow \infty} \sqrt{I} |1 - (a^I + b^I)| < \infty$$

and

$$\lim_{I \rightarrow \infty} \sqrt{I} |1 - (c^I + d^I)| < \infty.$$

Since $1 - (a^I - b^I)(c^I - d^I)$ is uniformly bounded away from 0 and $|(a^I - b^I)|$ is bounded by 1, this suffices to show that $\lim_{I \rightarrow \infty} \sqrt{I} |x^I - \frac{1}{2}|$ and $\lim_{I \rightarrow \infty} \sqrt{I} |y^I - \frac{1}{2}|$ are both finite. By Lemma 22, this implies unconditional uncertainty on both issues.

We finally show that unconditional uncertainty on both issues implies conditional uncertainty on both. Equations (17) and (18) imply:

$$\begin{aligned} x^I - y^I &= (a^I + b^I - 1) y^I + 2a^I \left(\frac{1}{2} - y^I\right) \\ y^I - x^I &= (c^I + d^I - 1) x^I + 2c^I \left(\frac{1}{2} - x^I\right). \end{aligned}$$

These can be rewritten as

$$\begin{aligned}(a^I + b^I - 1) y^I &= (x^I - \frac{1}{2}) + 2(\frac{1}{2} - a^I)(\frac{1}{2} - y^I) \\ (c^I + d^I - 1) x^I &= (y^I - \frac{1}{2}) + 2(\frac{1}{2} - c^I)(\frac{1}{2} - x^I).\end{aligned}$$

By Lemma 22, unconditional uncertainty on both issues provides $\lim_{I \rightarrow \infty} \sqrt{I} |\frac{1}{2} - x^I|$ and $\lim_{I \rightarrow \infty} \sqrt{I} |\frac{1}{2} - y^I|$ are both finite. Since both $|\frac{1}{2} - a^I|$ and $|\frac{1}{2} - c^I|$ are bounded by $\frac{1}{2}$, this suffices to show that

$$\lim_{I \rightarrow \infty} \sqrt{I} |a^I + b^I - 1| y^I < \infty.$$

By Lemma 21, this implies issue 1 is conditionally uncertain. Similarly, issue 2 is also conditionally uncertain.

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