Costly Contemplation*

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Abstract

We study preferences over opportunity sets. Such preferences are monotone if every opportunity set is at least as good as its subsets. We prove a representation theorem for monotone preferences. The representation suggests that the decision maker optimally contemplates his mood before making his ultimate choice from his opportunity set. We show that our model reduces to that of Kreps (1979) when contemplation is costless and to the standard rational model if the agent has no preference for flexibility.

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1 Introduction

Consider an individual who needs to buy a car. The alternatives are Mercedes (m), Lexus (l), and Toyota (t). The decision maker can either go to location A where there is a Toyota dealership and a Mercedes dealership or to location B where there is a Lexus dealership and a Toyota dealership or to location C where there are dealerships for all three cars. Hence, $A = \{m, t\}$, $B = \{l, t\}$, and $C = \{m, l, t\}$.

The decision maker realizes that if he were to go to location A, he would surely choose to buy a Toyota. Hence, he is indifferent between the option set $\{t\}$ and the option set $A = \{m, t\}$. In contrast, the decision maker observes that if he were to go to location B, he would probably buy a Toyota but he might be convinced that the additional features offered by the Lexus are worth the additional cost. Hence, he concludes that compared to committing to either a Toyota or a Lexus, going to location $B = \{l, t\}$ offers valuable flexibility; that is, $B = \{l, t\} \succ \{l\}$ and $\{l, t\} \succ \{t\}$.

Finally, the decision maker observes that if he were to go to location C, he might actually end up buying a Mercedes. Hence, $C = \{m, l, t\} \succ \{l, t\} = B$. The fact that he would not buy a Mercedes at location A but may buy one at location C seems strange to him at first. But he realizes that the advantages that a Mercedes offers over a Toyota are unlikely to be worth the very significant price difference between the two cars. Therefore, at location A, he would not consider it worth his while to contemplate how much he would be willing to pay for these additional features.

In contrast, he considers it possible that upon reflection he may find the additional expense of a Lexus justified. Therefore, at location C, some contemplation on the trade-off between quality and price is warranted. But once he starts contemplating, he knows that he may conclude that even the very substantial price of the Mercedes is worth paying. Hence, conditional on spending the effort to decide how he feels about more expensive cars, the Mercedes is a useful option. Since such contemplation is costly, in situations where it is not likely to change his mind (i.e., location A) he chooses not to incur these costs.

The particular ranking of option sets $\{m, l, t\} \succ \{l, t\}$ and $\{m, t\} \sim \{t\}$ above could not come about if contemplation were costless. To see this, note that with costless contemplation $\{m, l, t\} \succ \{l, t\}$ implies that in certain situations (realization of moods or tastes over quality attributes) it is better to buy a Mercedes than to buy a Lexus or Toyota. Hence, in all such situations it must be optimal to buy a Mercedes even if the Lexus option is not available. When contemplation is costly, the agent may choose not to bear this cost if the only alternatives are Mercedes and Toyota but be willing to incur the cost if the option set is richer.

The main result of this paper is a representation theorem for preferences over sets of options. Our representation theorem formalizes the notion of costly contemplation described in the car buyer example above. In our model, we consider a decision maker who in stage 1 must choose a set and in stage 2 must choose an element of that set. The only primitive of our model is a preference over sets describing the decision maker's behavior in the first stage. Our main theorem establishes that any monotone preference, that is, any preference where a set is weakly preferred to all its subsets, can be represented *as if* (i) the decision maker's ultimate utility depends on a subjective state (reflecting his mood or taste), (ii) after choosing his option set, the decision maker determines how and how much to contemplate regarding his second period choice, and (iii) in the second stage, he utilizes his signal (i.e., the result of his contemplation) to choose an option from his set.

Hence, the function describing the decision maker's preferences over option sets is formally identical to the value function of an optimal information acquisition problem familiar from statistical decision theory. The important difference between the usual information models and our analysis is that in our case the *states, signals* and *costs* are all subjective and hence unobservable. The only observables of the problem are behavior in stage 1 and behavior in stage 2. The purpose of our analysis is to identify restrictions on the unobservables based on the decision maker's choices in these two stages and use these restrictions to relate behavior in stage 1 to behavior in stage 2.

We now describe the functional form identified by our representation theorem. Let the finite set S be the decision maker's subjective state space. Let μ be a probability on S and u: $X \times S \rightarrow \mathbb{R}$ be a state dependent utility function over the set of alternatives X. A *contemplation strategy* is a signal about the actual state, modelled by a partition π of the state space S. An agent who implements the contemplation strategy π, uncovers which event of the partition π the actual state lies in. Therefore, each contemplation strategy can be interpreted as *subjective information* about the decision maker's tastes or mood. The value of π when the individual faces the opportunity set A is

$$V(A|\pi) = \sum_{E\in\pi} \mu(E) \max_{x\in A} \sum_{s\in E} \mu(s|E) u(x,s)\,. \tag{1}$$

The individual picks an alternative that yields the highest expected utility conditional on each event E.

In our model, the preferences are over sets of alternatives. The subjective states, the information partitions and costs are "parameters" of the decision maker's preferences. These parameters cannot be observed directly. Hence, our assumptions and our notion of costly contemplation are expressed in terms of the agent's preferences over sets. Our only axiom is *monotonicity*: if A contains B then A should be at least as good as B. We refer to the resulting preferences as *contemplation preferences* and show in Theorem 1 that they can be represented by a function U of the form

$$U(A) = \max_{\pi\in\mathcal{I}} \left[V(A|\pi) - c(\pi)\right] \tag{2}$$

where $c(\pi) \geq 0$ is the subjective cost of the contemplation strategy π and $\mathcal{I}$ is the set of feasible contemplation strategies. The decision maker chooses an optimal level of contemplation by maximizing the value minus the cost of contemplation.

To see how contemplation preferences are consistent with the type of behavior described in the car buyer example above, consider the following description of the state space, state contingent utility function, and contemplation. Suppose that there are three subjective states: low (L), medium (M), and high (H), ranked according to the monetary value that the agent attaches to quality and advanced features. The probabilities of the states are given by $\mu(\{L\}) = 0.6$, $\mu(\{M\}) = 0.3$, and $\mu(\{H\}) = 0.1$. Suppose that the state dependent utility of each car $x \in \{m, l, t\}$ is a function of the car's quality (q_x) and price (p_x), given by

$$u(x,s) = \begin{cases} q_x - p_x & \text{if } s = L \\ 2q_x - p_x & \text{if } s = M \\ 3q_x - p_x & \text{if } s = H \end{cases}$$

and

x	q_x	p_x
m	5	10
l	3	5
t	0	0

.

Then, the state contingent utilities can be summarized as

	L	M	H
$u(m,\cdot)$	-5	0	5
$u(l,\cdot)$	-2	1	4
$u(t,\cdot)$	0	0	0

.

Suppose the agent has two choices of contemplation strategies. He can either not contemplate at all which results in the information partition $\pi_0 = \{\{L, M, H\}\}$ or he can engage in contemplation. The latter option yields the partition $\pi_1 = \{\{L\}, \{M\}, \{H\}\}$ and has a cost of 0.6. Note that if the agent chooses not to contemplate, then the best he can do is to choose the least expensive car from each set. Hence, we can compute the value function as

D	$\{m,l,t\}$	$\{m,l\}$	$\{m,t\}$	$\{l,t\}$	$\{m\}$	$\{l\}$	$\{t\}$
$V(D\|\pi_0)$	0	-0.5	0	0	-2.5	-0.5	0

.

Similarly, if the agent chooses to contemplate, we have

D	$\{m,l,t\}$	$\{m,l\}$	$\{m,t\}$	$\{l,t\}$	$\{m\}$	$\{l\}$	$\{t\}$
$V(D\|\pi_1)$	0.8	-0.4	0.5	0.7	-2.5	-0.5	0

.

Hence, the utility of each option set can be calculated to be

D	$\{m,l,t\}$	$\{m,l\}$	$\{m,t\}$	$\{l,t\}$	$\{m\}$	$\{l\}$	$\{t\}$
$U(D)$	0.2	-0.5	0	0.1	-2.5	-0.5	0

.

It follows that the agent chooses to contemplate if and only if his set of options D contains both l and t. If $D = \{m, l, t\}$ then while contemplating the choice between l and t, the agent may find out that the best option is actually m. This is true in spite of the fact that had his options consisted just of m and either one of l and t the decision maker would have chosen the less expensive car without contemplating. Clearly, this type of situation can arise only if contemplation is costly.

We say that the decision maker has a *preference for flexibility* if the union of two sets is strictly better than each of them separately. The idea of identifying preference for flexibility through a decision maker's preferences over sets of options and this particular notion of preference for flexibility are due to Kreps (1979). Kreps shows that $\succsim$ is monotone and submodular[1] if and only if it has a representation of type

$$U(A) = \sum_{s \in S} \mu(s) \max_{x \in A} u(x, s) \,. \tag{3}$$

We adopt the same analytical approach and point out a condition that identifies contemplation behavior through choice experiments over opportunity sets. An alternative $x \in A$ is *essential in* A if A just consists of x or $A \succ A \setminus \{x\}$. We say that the agent has *costly contemplation at* A if there are x and B such that $x \in B \subset A$, x is essential in A but not essential in B. We show that given monotonicity, submodularity is equivalent to the agent not having costly contemplation at any set A. Hence, we interpret Kreps' representation as a model of *costless contemplation*, where the individual incurs no disutility from contemplation and optimally resolves his preference uncertainty over the available alternatives until the time of actual choice. Indeed, the representation in (3) corresponds to a special case of the one given by (1) and (2), where perfect information about S is available at no cost.

For a classical rational decision maker who has perfect knowledge of his preferences over the set of alternatives, every opportunity set is only as good as its best singleton subset. Therefore, he never has a preference for flexibility. We show that our model reduces to the standard rational model under no preference for flexibility.

The idea that individuals have to spend (psychologically) costly effort in order to make better decisions is common in a number of bounded rationality models. As in Koopmans (1964) and March (1978), in our representation, the individual is uncertain about his true tastes/preferences over the alternatives, and has to spend costly effort to eliminate this uncertainty. He is boundedly rational in the sense that he does not know his preferences, but a rational Bayesian in the way that he chooses a contemplation strategy.

Ofek, Yildiz, and Haruvy (2002) study a sequential decision problem where in the

[1] *Submodularity* (Kreps' 1.5) requires that if $A \sim A \cup B$ then $A \cup C \sim A \cup C \cup B$, i.e. if B does not add anything to A, then it does not add anything to the larger set $A \cup C$.

first stage the agent ranks two alternatives, and in the second stage he determines his valuation for each of them. They investigate the effects of this two stage procedure on valuations, compared to the case when the agent determines his valuations without going through the first stage. They show that the ranking stage increases the dispersion in valuations and test their results by conducting experiments. They take the costly contemplation model as the primitive of their analysis by assuming that the agent has preference uncertainty and exerts an optimal level of costly cognitive effort in order to resolve this uncertainty. In this paper, we identify joint restrictions on the preference over sets and second period behavior that correspond to the type of behavior that Ofek, Yildiz, and Haruvy (2002) build their analysis on.

After Kreps (1979) there have been a number of papers that have utilized preferences over sets to model phenomena that are typically ruled out by standard assumptions of choice theory. Dekel, Lipman, and Rustichini (2001) and Gul and Pesendorfer (2001) model the set of alternatives as lotteries and make use of the resulting linear structure by imposing variants of von Neumann-Morgenstern axioms over sets. Our formal treatment is closest to Kreps (1979) and Gul and Pesendorfer (2002), in that we do not impose any structure on the set of alternatives.

The organization of the paper is as follows. In the next section, we present our main result. In Section 3, we establish the connection between first period preference and second period behavior. We formally relate our model to Kreps (1979) and to the standard rational model. In Section 4, we extend our representation theorem to the consumption-investment budgeting framework. We present our concluding remarks in Section 5.

2 Contemplation Preferences

For any nonempty set Y, let $\mathcal{P}(Y)$ be the set of all nonempty subsets of Y and let $\Pi(Y)$ be the set of all partitions of Y. For any $\pi, \rho \in \Pi(Y)$ we will write $\pi \geq_f \rho$ to denote that π is (weakly) finer than ρ. In our model, we have a finite set of alternatives X with generic elements $x, y, \ldots$ We will use $\mathcal{X} = \mathcal{P}(X)$ to denote the set of opportunity sets -the choice problems that the decision maker can face. Opportunity sets are generically denoted by $A, B, \ldots$

For any nonempty finite state space S, a *contemplation structure* over S is a collection of partitions $\mathcal{I} \subset \Pi(S)$ that includes $\{S\}$. For any $\pi \in \mathcal{I}$ and $s \in S$, let $\pi(s)$ denote the element of the partition π to which s belongs. We interpret π as a *contemplation strategy*: at state s, the decision maker who implements π exactly knows that the actual state is in $\pi(s)$. Let a *cost function* be a map $c\colon \mathcal{I} \to \mathbb{R}_+$ such that $c(\{S\}) = 0$. A cost function c is *monotone* if $\pi \geq_f \rho$ implies $c(\pi) \geq c(\rho)$ for any $\pi, \rho \in \mathcal{I}$. We interpret the value $c(\pi)$ as the disutility of implementing the contemplation strategy π. Then $c(\{S\}) = 0$ is just a normalization saying that the cost of no contemplation is zero and monotonicity says that more contemplation does not cost less.

Consider an expected utility maximizer with a state dependent utility $u\colon X \times S \to \mathbb{R}$ and a probability distribution μ over S. The values of $u(\cdot, s)$ represent the tastes of the individual at state s where $u(x, s)$ is interpreted as his ex-post utility from choosing the alternative x if the state s is realized. The decision maker's uncertainty about his tastes is therefore represented by the uncertainty about the underlying subjective state space S.

Suppose that the decision maker faces a choice problem $A \in \mathcal{X}$, he implements the contemplation strategy $\pi \in \mathcal{I}$ and the state $\bar{s}$ is realized. Then, he learns that the actual state is in $E = \pi(\bar{s})$ and makes his choice x out of A in order to maximize his expected utility conditional on E:

$$\max_{x \in A} \sum_{s \in E} \mu(s|E) u(x, s) \,.$$

Thus we can define the ex-ante value of the contemplation strategy π to the decision maker facing the decision problem A as

$$V(A|\pi) = \sum_{E \in \pi} \mu(E) \max_{x \in A} \sum_{s \in E} \mu(s|E) u(x, s) = \sum_{E \in \pi} \max_{x \in A} \sum_{s \in E} \mu(s) u(x, s) \,.$$

Note that as in Kreps (1979), the probability distribution μ has no essential role here and it may be omitted by rescaling u. We will therefore omit the probability distribution μ and restrict attention to state dependent utility indices $u(\cdot, s)$ aggregated additively over S. Also note that $V(A|\pi)$ is nondecreasing as A becomes larger and π becomes finer. Therefore the decision maker always values having more alternatives and contemplating more about the state space.

In our representation, contemplation is not exogenous to the decision maker: faced with a decision problem $A \in \mathcal{X}$, he implements an optimal contemplation strategy that would help him make the "right choice". Contemplation costs enter his utility additively and he rationally chooses his contemplation strategy fully aware of the cost and value of all feasible strategies. Therefore, his ex-ante utility when he faces the decision problem $A \in \mathcal{X}$ is given by

$$\max_{\pi \in \mathcal{I}} \left[V(A|\pi) - c(\pi) \right] .$$

Definition 1 A utility function U over $\mathcal{X}$ is a *contemplation utility* if there exists a quadruple $(S, u, \mathcal{I}, c)$ where S is a nonempty finite set, u: $X \times S \to \mathbb{R}$, $\{S\} \in \mathcal{I} \subset \Pi(S)$, and c: $\mathcal{I} \to \mathbb{R}_+$ with $c(\{S\}) = 0$ such that

$$U(A) = \max_{\pi \in \mathcal{I}} \left[V(A|\pi) - c(\pi) \right]$$

where $A \in \mathcal{X}$ and for any $\pi \in \mathcal{I}$

$$V(A|\pi) = \sum_{E \in \pi} \max_{x \in A} \sum_{s \in E} u(x, s).$$

A preference $\succsim$ over $\mathcal{X}$ is a *contemplation preference* if it is represented by a contemplation utility U.

Note that $(S, u, \mathcal{I}, c)$ above are not primitives of our model but just parameters of a contemplation utility representation. For any such representation, the value function $V(A|\pi)$ is nondecreasing as A gets larger, therefore every option set is weakly better than its subsets. This leads to the following necessary condition for contemplation preferences:

Definition 2 A preference relation $\succsim$ over $\mathcal{X}$ is *monotone* if $A \succsim B$ whenever $A \supset B$.

We next present our main result which shows that monotonicity is also a sufficient condition for a contemplation utility representation. This ties down the costly contemplation model to the first period behavior through monotonicity.

Theorem 1 *A preference relation $\succsim$ over $\mathcal{X}$ is monotone if and only if any representation U of $\succsim$ is a contemplation utility.*

Proof: See Appendix.

The parameters $(S, u, \mathcal{I}, c)$ in a contemplation utility representation are unobservable and therefore not an objective part of our model, they are derived subjectively from the preference over sets. Whenever the preference over sets is consistent with monotonicity, the decision maker behaves *as if* he optimally contemplates about a subjective state space underlying the uncertainty about his tastes. However, the representation may not be unique since a contemplation utility U may be represented by more than one group of parameters $(S, u, \mathcal{I}, c)$.

We next give a sketch of the proof of this result. In the proof, we construct a quadruple $(S, u, \mathcal{I}, c)$ that represents U in the sense of Definition 1, *for every* utility function U that represents $\succsim$.

2.1 Sketch of the Proof

Let $\succsim$ be a monotone preference over $\mathcal{X}$ represented by the utility function U. We will assume that singleton opportunity sets are indifferent. The proof for this case is considerably simpler while it captures the main idea behind our construction. The readers interested in the proof for the general case are referred to the Appendix.

We will first define S, u, and $\mathcal{I}$ *independently* of U, then we will construct c such that $(S, u, \mathcal{I}, c)$ represents a positive affine transformation of U. That is, all the adjustment for the representation is made by using the cost function c.

Let the state space S be the set of strict rankings of the alternatives. For any $s \in S$ and $x, y \in X$, let xR_sy denote that x is ranked weakly above y in state s. Let $u(x, s)$ be the rank of the alternative x from the bottom, in state s, i.e.

$$u(x, s) = |\{y \in X : xR_sy\}|, \quad x \in X, s \in S.$$

For any opportunity set A and $x \in A$, let

$$E_{x,A} = \{s \in S \mid \forall y \in A\colon xR_sy\}.$$

Then $E_{x,A}$ is the event that x is the top element in A. Define the partition π_A by

$$\pi_A = \{E_{x,A} : x \in A\}.$$

The decision maker who implements the contemplation strategy π_A *exactly* knows the top element in the opportunity set A. Let the contemplation structure be the set of partitions of the above type:

$$\mathcal{I} = \{\pi_A \,:\, A \in \mathcal{X}\}.$$

Note again that none of the constructions so far (i.e. S, u, and $\mathcal{I}$) depend on U. Moreover, all the alternatives are treated symmetrically in S, u, and $\mathcal{I}$.

Remark 1 *If* $|B| \geq 2$, $x \in B$, *and* $y \neq x$, *then*

$$\sum_{s \in E_{x,B}} u(x,s) > \sum_{s \in E_{x,B}} u(y,s)\,.$$

The intuition behind this result is quite clear. If $y \in B$, then x does better than y conditional on every state in $E_{x,B}$, so the remark follows. On the other hand if $y \notin B$, then the event $E_{x,B}$ brings favorable news about x and contains no information about y. Since x and y are entirely symmetric prior to this signal, x does strictly better than y conditional on the news that it is the best element in B.

Remark 2 *The opportunity set* B *maximizes* $V(\cdot|\pi_B)$ *on* $\mathcal{X}$. *Moreover if* $|B| \geq 2$, *then* $V(B|\pi_B) > V(A|\pi_B)$ *for any* A *that does not contain* B.

By Remark 1, x is the unique best element in X conditional on $E_{x,B}$, for any $x \in B$. Therefore, given the partition π_B, the best that one can do is to always choose the best element in B. This is possible when the agent faces the opportunity set B, therefore the first part follows. If A does not contain B, then there is $x \in B \setminus A$, so again by Remark 1 the best element in A conditional on $E_{x,B}$ does strictly worse than x. Conditional on other events of π_B, the top element in A does weakly worse than the top element in B. Therefore given π_B, the aggregate value generated by A is strictly less than that generated by B.

Let us define the scalar

$$\gamma = \min_{|B|\geq 2, B\setminus A \neq \emptyset} \left[V(B|\pi_B) - V(A|\pi_B)\right].$$

Then, $\gamma > 0$ by the second part of Remark 2. For any B with at least two elements and A that does not contain B, the value of the contemplation strategy π_B when the agent faces B is at least γ more than the value of π_B when he faces A. In an informal sense, when the agent faces B, he would be willing to incur at least γ more disutility to implement π_B, than what he would be willing to sacrifice when he faces A.

To complete the proof, we conjecture and verify that we can construct a cost function c such that (i) for any opportunity set A, π_A is an optimal partition when the decision maker faces A and (ii) $(S, u, \mathcal{I}, c)$ represents a positive affine transformation $\tilde{U} = \alpha U + \beta$ of U.

Then (i) and (ii) would imply that c is defined by

$$c(\pi_A) = V(A|\pi_A) - \tilde{U}(A)\,, \quad A \in \mathcal{X}. \tag{4}$$

Condition (i) is satisfied if and only if for any $A, B \in \mathcal{X}$

$$V(A|\pi_A) - c(\pi_A) \geq V(A|\pi_B) - c(\pi_B)$$

which, under (4), is equivalent to

$$\tilde{U}(A) - \tilde{U}(B) + V(B|\pi_B) - V(A|\pi_B) \geq 0\,. \tag{5}$$

If $A \succsim B$, then $\tilde{U}(A) \geq \tilde{U}(B)$ and $V(B|\pi_B) - V(A|\pi_B) \geq 0$ since B maximizes $V(\cdot|\pi_B)$, therefore (5) is satisfied.

If $B \succ A$, then by monotonicity and indifference of all singletons, B can neither be a singleton nor a subset of A. Then $V(B|\pi_B) - V(A|\pi_B) \geq \gamma$, so (5) is satisfied if we can guarantee that the variations of $\tilde{U}$ are less than γ, i.e. if we fix $\alpha > 0$ small enough such that

$$\max_{B', A' \in \mathcal{X}} \left[\tilde{U}(B') - \tilde{U}(A')\right] = \alpha \max_{B', A' \in \mathcal{X}} [U(B') - U(A')] < \gamma.$$

Given the above α, one can go back and find β such that c defined in (4) is a cost function (i.e. $c(\{S\}) = 0$ and $c \geq 0$) and verify that conditions (i) and (ii) are satisfied.

3 Contemplation Costs and Second Period Behavior

In this section, we relate the preference over sets to the parameters of our representation. We first show that we can obtain a restriction on the cost function without imposing additional assumptions on the preference over sets. We then establish a connection between the first period preference and the second period behavior implied by our representation. We formulate two properties of the preference over sets: costly contemplation and preference for flexibility. We discuss their implications on the second period and use them to distinguish our model from existing theories.

Given a contemplation utility representation, let $\mathcal{O}(A) \subset \mathcal{I}$ denote the set of optimal contemplation strategies when the decision maker faces A, i.e. the set of solutions of

$$\max_{\pi \in \mathcal{I}} \left[V(A|\pi) - c(\pi) \right].$$

In our representation result, we have not required the cost function to be monotone. As we now show, this has no loss of generality since any contemplation utility can be associated with a monotone cost function.

Proposition 1 *Let U be a contemplation utility represented by $(S, u, \mathcal{I}, c)$. Then, the cost function $\tilde{c}\colon \mathcal{I} \to \mathbb{R}_+$ defined by*

$$\tilde{c}(\pi) = \min_{\rho \in \mathcal{I} : \rho \geq_f \pi} c(\rho), \qquad \pi \in \mathcal{I}$$

is monotone and $(S, u, \mathcal{I}, \tilde{c})$ represents U.

Proof: First note that $\tilde{c}(\{S\}) = \min_{\rho \in \mathcal{I}} c(\rho) = 0$. Let $\pi \geq_f \pi'$, then for any $\rho \in \mathcal{I}$ with $\rho \geq_f \pi$ we have $\rho \geq_f \pi'$. Therefore $\tilde{c}(\pi) \geq \tilde{c}(\pi')$ proving monotonicity of $\tilde{c}$.

Let $\mathcal{O}(A)$ and $\tilde{\mathcal{O}}(A)$ denote the set of optimal partitions for A in $(S, u, \mathcal{I}, c)$ and $(S, u, \mathcal{I}, \tilde{c})$ respectively. By definition, $\tilde{c}(\pi) \leq c(\pi)$ for all $\pi \in \mathcal{I}$. Let $\pi \in \mathcal{O}(A)$, then for any $\rho \in \mathcal{I}$ with $\rho \geq_f \pi$, $V(A|\rho) \geq V(A|\pi)$ and $V(A|\pi) - c(\pi) \geq V(A|\rho) - c(\rho)$ implying $c(\rho) \geq c(\pi)$. Minimizing the left hand side over all $\rho \in \mathcal{I}$ with $\rho \geq_f \pi$ we obtain $\tilde{c}(\pi) \geq c(\pi)$, therefore $\tilde{c}(\pi) = c(\pi)$. Now for any $\pi', \rho' \in \mathcal{I}$ with $\rho' \geq_f \pi'$, $V(A|\pi) - \tilde{c}(\pi) = V(A|\pi) - c(\pi) \geq V(A|\rho') - c(\rho') \geq V(A|\pi') - c(\rho')$. Maximizing the right hand side over all $\rho' \in \mathcal{I}$ with $\rho' \geq_f \pi'$ we obtain $V(A|\pi) - \tilde{c}(\pi) \geq V(A|\pi') - \tilde{c}(\pi')$, thus $\pi \in \tilde{\mathcal{O}}(A)$. Then $U(A) = V(A|\pi) - c(\pi) = V(A|\pi) - \tilde{c}(\pi)$, proving that $(S, u, \mathcal{I}, \tilde{c})$ represents U. □

In our representation, state dependence of u makes it impossible to pin down the exact probabilities of the states from the preference over sets. For the same reason, it is also not possible to derive second period choice probabilities from the first period preference. However as we demonstrate next, from the preference over sets, we can identify whether or not an alternative has to be chosen with positive probability from a particular option set.

Let $\succsim$ be a monotone preference over $\mathcal{X}$. When the decision maker faces A, an alternative $x \in A$ is essential (Puppe, 1996) if he suffers from not having x as an available option in the second period:

Definition 3 Let $x \in A$, then x is *essential in* A if $A = \{x\}$ or $A \succ A \setminus \{x\}$. Let $e(A) \subset A$ denote the set of essential elements in A.

The following proposition shows that x is essential in A if and only if for any optimal contemplation strategy π for A, there is an event of π in which x is the unique best element in A.

Proposition 2 *Let $\succsim$ be represented by some $(S, u, \mathcal{I}, c)$ and $x \in A \in \mathcal{X}$. Then the following are equivalent:*

(i) *x is essential in A.*

(ii) *For any $\pi \in \mathcal{O}(A)$ there is $E \in \pi$ such that*

$$\sum_{s\in E} u(x, s) > \sum_{s\in E} u(y, s)$$

for any $y \in A \setminus \{x\}$.

Proof: To prove (i) $\Rightarrow$ (ii), let x be an essential element of A and suppose that (ii) is false. Then $|A| \geq 2$ and there is $\pi \in \mathcal{O}(A)$ such that for any $E \in \pi$, $\sum_{s\in E} u(x, s) \leq \max_{y\in A\setminus\{x\}} \sum_{s\in E} u(y, s)$. This implies that $V(A \setminus \{x\}|\pi) = V(A|\pi)$ therefore $U(A) = V(A \setminus \{x\}|\pi) - c(\pi) \leq U(A \setminus \{x\})$, a contradiction to $A \succ A \setminus \{x\}$.

To prove (ii) $\Rightarrow$ (i), suppose that x is not essential in A, i.e. $|A| \geq 2$ and $A \sim A\setminus\{x\}$. Let $\pi \in \mathcal{O}(A\setminus\{x\})$. Then $U(A) = U(A\setminus\{x\}) = V(A\setminus\{x\}|\pi) - c(\pi) \leq V(A|\pi) - c(\pi) \leq U(A)$, therefore $\pi \in \mathcal{O}(A)$ and $V(A\setminus\{x\}|\pi) = V(A|\pi)$. The latter equality can hold only if for any $E \in \pi$, $\max_{z\in A\setminus\{x\}} \sum_{s\in E} u(z, s) = \max_{z\in A} \sum_{s\in E} u(z, s)$. Then for any $E \in \pi$, there is $y \in A \setminus \{x\}$ such that $\sum_{s\in E} u(y, s) = \max_{z\in A} \sum_{s\in E} u(z, s) \geq \sum_{s\in E} u(x, s)$, i.e. (ii) is false. □

In order to clarify what Proposition 2 implies about the second period, we will make explicit reference to μ, i.e. the underlying probabilities of the states. If $(S, \mu, u', \mathcal{I}, c)$ represents U, then as we pointed out earlier, we can rescale the state dependent utilities to $u(x, s) = \mu(s)u'(x, s)$ so that $(S, u, \mathcal{I}, c)$ represents U in the sense of Definition 1. Then for any event E and $x, y \in X$ we have

$$\sum_{s\in E} u(x,s) > \sum_{s\in E} u(y,s) \iff \mu(E) > 0 \text{ and } \sum_{s\in E} \mu(s|E)u'(x,s) > \sum_{s\in E} \mu(s|E)u'(y,s)\,.$$

Note that the left hand side is the inequality in condition (ii) of Proposition 2. We can therefore restate Proposition 2 as follows: x is essential in A if and only if for any optimal contemplation strategy π for A, there is an event $E \in \pi$ such that $\mu(E) > 0$ and x is the unique best element in A conditional on E. This relates the preference over sets to the second period choice behavior *independently* of the particular contemplation utility representation: x is essential in A if and only if for any representation of $\succsim$, and any optimal contemplation strategy, x has to be chosen from A with positive probability.

Remember that a contemplation utility is derived from a two stage maximization problem. In the first stage, the decision maker chooses a contemplation strategy π that maximizes the value minus cost of contemplation. In the second stage, conditional on every event $E \in \pi$, he chooses an alternative in his option set that maximizes his expected utility conditional on E. There may be ties in either stage. That is, in the contemplation stage there may be more than one optimal contemplation strategy and conditional on an optimal contemplation strategy π and an event realization following contemplation, there may be more than one conditional expected utility maximizing alternative within the available options.

If there are no ties, then an inessential element in A is never chosen from A. On the other hand when there is multiple optimality in either the optimal contemplation strategy or the optimal choice given a particular signal, then it is possible that an inessential element is selected with positive probability, depending on how the decision maker breaks ties when he is indifferent. For example if under a contemplation utility representation of $\succsim$, two alternatives x and y yield the same utility at each state (they are virtually identical), then neither x nor y is essential in $\{x, y\}$, however one of them will eventually be chosen out of $\{x, y\}$ depending on the tie-breaking rule. In general for any inessential element z, there is always an optimal contemplation strategy π such that conditional on every event in π, there is an alternative among the remaining options which is weakly better than z, i.e. the inessential element *does not* have to be chosen.

We next present two properties of the first period preference: costly contemplation and preference for flexibility. We discuss the implications of these on the second period behavior. Using these two properties, we also establish the relationship of our model with Kreps (1979) and the standard rational model. We show that our model reduces to that of Kreps (1979) under no costly contemplation and to the standard rational model under no preference for flexibility.

Definition 4 The preference $\succsim$ has *costly contemplation at* A if there is $B \subset A$ such that $(e(A) \cap B) \setminus e(B) \neq \emptyset$. The preference $\succsim$ has a *costly contemplation* if it has costly contemplation at some $A \in \mathcal{X}$.

The first period preference has costly contemplation at A if and only if there are x and B such that $x \in B \subset A$, the alternative x is not essential in B, and it becomes essential in the larger set A when more options are made available. Costly contemplation is ruled out by Kreps' second assumption submodularity (condition (i) in the next Proposition, Kreps calls it 1.5). We next show that the converse is also true, i.e. under monotonicity, submodularity is equivalent to no costly contemplation. Therefore in comparison to Kreps, the distinguishing feature of our approach is to allow for costly contemplation.

Proposition 3 *Let* $\succsim$ *be monotone. Then the following are equivalent:*

(i) $\succsim$ *is submodular:* $A \sim A \cup B \Rightarrow A \cup C \sim A \cup C \cup B$.

(ii) $\succsim$ *has no costly contemplation, i.e.* $B \subset A \Rightarrow e(A) \cap B \subset e(B)$.

Proof: The (i) $\Rightarrow$ (ii) part is easily verified. To prove (ii) $\Rightarrow$ (i), suppose that $\succsim$ has no costly contemplation and let A, B, and C be opportunity sets with $A \sim A \cup B$. If $B \subset A$ the conclusion easily follows, so without loss of generality let $B \setminus A \neq \emptyset$. Enumerate the elements of $B \setminus A$ as $x_1, \ldots, x_k$ and let $B_m = \{x_i : 1 \leq i \leq m\}$ for $0 \leq m \leq k$. Let $0 \leq l \leq k-1$. Since $A \sim A \cup B$ and $A \subset A \cup B_l \subset A \cup B_{l+1} \subset A \cup B$, by monotonicity we have that $A \cup B_l \sim A \cup B_{l+1}$. The latter indifference implies that x_{l+1} is not essential in $A \cup B_{l+1}$, therefore from no costly contemplation x_{l+1} is not essential in $A \cup C \cup B_{l+1}$. Reapplying monotonicity, we have $A \cup C \cup B_l \sim A \cup C \cup B_{l+1}$. By induction on l, we conclude that $A \cup C \sim A \cup C \cup B$. $\square$

By Proposition 2, we can express costly contemplation in terms of the second period choice behavior: there is costly contemplation at A if there are x and B such that $x \in B \subset A$, x is not chosen from B but is always chosen with positive probability from the larger set A. It is evident from Kreps' representation that if x is chosen from the set A in an event E, then it also has to be chosen from the smaller set B on the same event E. Therefore in Kreps' representation, if x is always chosen with positive probability from the set A then it also has to be chosen with positive probability from the smaller set B.

For deterministic choice functions, Sen's condition α requires that if an alternative x is chosen from a set A then it also has to be chosen from any subset of A that contains x. Hence, we can interpret no costly contemplation (condition (ii) in Proposition 3) as the stochastic version of Sen's condition α.

The first period preferences may not satisfy what Kreps (1979) calls "revealed preference," i.e. the union of two sets can be strictly preferred to each one separately. We will adopt the latter as our definition of preference for flexibility.

Definition 5 The preference $\succsim$ has a *preference for flexibility at* A if there exist B and C such that

$$A = B \cup C, \;\; A \succ B, \;\text{ and } A \succ C.$$

The preference $\succsim$ has a *preference for flexibility* if it has a preference for flexibility at some $A \in \mathcal{X}$.

Proposition 4 relates preference for flexibility and the second period behavior. It establishes that, if there is a preference for flexibility at A, then one can find $B \subset A$ indifferent to A such that in the second period all the alternatives in B have to be chosen with positive probability when the decision maker faces B. Note that all of B's elements are essential in B, therefore none of them can be dispensed without decreasing the ex-ante utility associated with B. Since restricting the decision maker's options at B in *any* way makes him strictly worse off, he has a stronger form of preference for flexibility at B.

Proposition 4 *Let $\succsim$ be monotone. If $\succsim$ has a preference for flexibility at A, then there is $B \subset A$ such that $|B| \geq 2$, $B \sim A$, and $e(B) = B$.*

Proof: Let $\succsim$ have a preference for flexibility at A. We will show that if A has an inessential element, then there is preference for flexibility at a proper subset A' of A with $A' \sim A$. Suppose that $x \in A$, $A \sim A \setminus \{x\}$, $A = B \cup C$, $A \succ B$, and $A \succ C$. Let $A' = A \setminus \{x\}$, $B' = A' \cap B$, and $C' = A' \cap C$. Then A' is a proper subset of A with $A' \sim A$. Moreover $B', C' \neq \emptyset$, otherwise without loss of generality if $B' = \emptyset$, then $A' \subset C$ and hence $C \succsim A' \sim A$ by monotonicity, a contradiction. Then $A' = B' \cup C'$, $A' \sim A \succ B \succsim B'$, and $A' \sim A \succ C \succsim C'$, i.e. there is preference for flexibility at A'.

If every element of A is essential in A, then let $B = A$. Otherwise, we can find a proper subset A' of A such that there is preference for flexibility at A' and $A' \sim A$. If every element of A' is essential in A', then let $B = A'$. Otherwise, we can find a proper subset A'' of A' such that there is preference for flexibility at A'' and $A'' \sim A' \sim A$. Proceeding like this, we will find the desired B in at most $|A| - 2$ steps, since preference for flexibility at a two element set implies that both elements are essential. □

An agent is rational if he has a preference $\succsim^*$ over X such that (i) in period 1, his preference over sets is induced from $\succsim^*$ through

$$A \succsim B \iff \text{ for all } y \in B \text{ there is } x \in A \text{ such that } x \succsim^* y \tag{6}$$

and (ii) in period 2, faced with any opportunity set A, he chooses a $\succsim^*$-maximal element from A. In particular, (6) suggests that the agent already knows $\succsim^*$ in period 1 and ranks option sets according to their top elements with respect to $\succsim^*$. Hence a rational agent behaves *as if* he has no subjective uncertainty about his tastes over X.

Kreps (1979) introduces the following condition:

$$A \succsim B \Rightarrow A \sim A \cup B\,. \tag{7}$$

Kreps points out that the preference $\succsim$ over sets satisfies (7) if and only if there is a preference $\succsim^*$ over X such that (6) holds. For such an agent, a set A is only as good as its $\succsim^*$-best element x^*, therefore no element other than x^* can be essential in A. Moreover he never has a preference for flexibility at A, because if $A = B \cup C$, then x^* belongs to B or to C implying that $A \sim B$ or $A \sim C$. We next show that our model reduces to the rational one under no preference for flexibility or if every set has at most one essential element.

Proposition 5 *Let $\succsim$ be represented by the contemplation utility U. Then the following are equivalent:*

(i) *$\succsim$ has no preference for flexibility.*

(ii) *$|e(A)| \leq 1$ for all $A \in \mathcal{X}$.*

(iii) *$U(A) = \max_{x \in A} U(\{x\})$ for all $A \in \mathcal{X}$.*

(iv) *There is a preference $\succsim^*$ over X such that (6) is satisfied.*

Proof: Condition (7) is equivalent to monotonicity and no preference for flexibility together. Therefore by Kreps (1979), (i) $\Leftrightarrow$ (iv) . The (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) parts are easily verified. To see that (ii) $\Rightarrow$ (i), suppose that $\succsim$ has a preference for flexibility. Then by Proposition 4, there is a set B such that $|B| \geq 2$ and $e(B) = B$, i.e. (ii) is false. $\square$

4 Consumption-Investment Budgeting

There are two finite sets X_1 and X_2, representing first and second period consumption choices. The decision maker's consumption choices in period 1 may restrict his available alternatives in period 2. He now has a preference over $X_1 \times \mathcal{P}(X_2)$, i.e. pairs of consumption choices in period 1 and opportunity sets in period 2. Our construction can be extended to this framework. We begin by giving a definition of contemplation utility in this setup.

Definition 6 In the consumption-investment budgeting model, a utility function U over $X_1 \times \mathcal{P}(X_2)$ is a *contemplation utility* if there exists a quadruple $(S, u, \mathcal{I}, c)$ where S is a nonempty finite set, u: $X_1 \times X_2 \times S \to \mathbb{R}$, $\{S\} \in \mathcal{I} \subset \Pi(S)$, c: $\mathcal{I} \to \mathbb{R}_+$ with $c(\{S\}) = 0$ such that

$$U(x_1, A_2) = \max_{\pi \in \mathcal{I}} [V(x_1, A_2|\pi) - c(\pi)]$$

where $(x_1, A_2) \in X_1 \times \mathcal{P}(X_2)$ and for any $\pi \in \mathcal{I}$

$$V(x_1, A_2|\pi) = \sum_{E \in \pi} \max_{x_2 \in A_2} \sum_{s \in E} u(x_1, x_2, s) \, .$$

A preference $\succsim$ over $X_1 \times \mathcal{P}(X_2)$ is a *contemplation preference* if it is represented by a *contemplation utility* U.

The natural counterpart of monotonicity in this setup is the following.

Definition 7 A preference $\succsim$ over $X_1 \times \mathcal{P}(X_2)$ is *monotone* if $(x_1, A_2) \succsim (x_1, B_2)$ whenever $A_2 \supset B_2$.

Note that a contemplation preference satisfies monotonicity since the value function $V(x_1, A_2|\pi)$ in Definition 6 is nondecreasing as A_2 becomes larger.

Let $\succsim$ be a monotone preference and let U be a utility function that represents it. For each $x_1 \in X_1$, U_{x_1} defined by $U_{x_1}(A_2) = U(x_1, A_2)$ is a monotone utility function over $\mathcal{P}(X_2)$. Therefore by Theorem 1, there exist $(S, u_{x_1}, \mathcal{I}, c_{x_1})$ such that U_{x_1} is the contemplation utility represented by $(S, u_{x_1}, \mathcal{I}, c_{x_1})$. Note that we may guarantee that S and $\mathcal{I}$ here do not depend on x_1, because in the proof of Theorem 1 they are constructed independently of the preference. If in Definition 6 we had allowed contemplation costs to depend on the first period consumption, then by setting $u(x_1, x_2, s) = u_{x_1}(x_2, s)$, we would have that $(S, u, \mathcal{I}, (c_{x_1})_{x_1 \in X_1})$ represents U, as an easy corollary of Theorem 1. With some more work, we can show that monotonicity is sufficient for a representation where contemplation costs do not depend on x_1.

Theorem 2 *A preference relation $\succsim$ over $X_1 \times \mathcal{P}(X_2)$ is monotone if and only if any representation U of $\succsim$ is a contemplation utility.*

Proof: See Appendix.

5 Conclusion

We have provided a model of preference for flexibility where the decision maker's preferences are represented by a function that is similar to the value function of costly information acquisition problems familiar from statistical decision theory. Our model differs from the standard statistical decision theory problem in that the underlying states, the cost of information, and the decision maker's signal are all unobservable, hypothetical constructs.

We have offered choice experiments that relate these constructs to observed behavior and relate behavior in stage 1 to behavior in stage 2. In spite of the fact that we can offer no uniqueness theorem, we have been able to relate definitions of preference for flexibility and costly contemplation based on the agent's preference over sets to the appropriate terms in the utility function. We have shown that in the presence of monotonicity, which is assumed both in our model and in Kreps (1979), the remaining axiom of Kreps' theorem is equivalent to the agent not having costly contemplation at any set. Hence, we have been able to interpret Kreps' theorem as a characterization of preference for flexibility with costless contemplation.

Extending our model to choices over lotteries, the way Dekel, Lipman, and Rustichini have extended Kreps' costless contemplation, may yield additional restrictions and make it easier to identify and distinguish between the elements of the representation.

A Appendix

A.1 Proof of Theorem 1

Let $n = |X|$. Assume that $\succsim$ is monotone and U represents $\succsim$. Let S be the set of linear orders over X. For $s \in S$ and $x, y \in X$, we will write xR_sy to denote that x is ranked weakly above y with respect to s. For any $A \in \mathcal{X}$ and $x \in A$, let $E_{x,A} = \{s \in S \mid \forall y \in A \colon xR_sy\}$ denote the set of states in which x is the top element in A. Then the partition $\pi_A = \{E_{x,A} : x \in A\} \in \Pi(S)$ contains exactly all the necessary information to make the best choice out of A. Let $\mathcal{I} = \{\pi_A : A \in \mathcal{X}\}$. Note that for any singleton set A, π_A is the trivial partition $\{S\}$. In particular $\{S\} \in \mathcal{I}$, i.e. $\mathcal{I}$ is a contemplation structure.

Let the canonical index be the map $u \colon X \times S \to \mathbb{R}$ defined by

$$u(x, s) = |\{y \in X : xR_sy\}| , \quad x \in X, s \in S.$$

Then $u(x, s)$ denotes the rank of x from the bottom at state s. For any $\epsilon \in \mathbb{R}^X$, let the perturbed index be the map $u^\epsilon \colon X \times S \to \mathbb{R}$ defined by

$$u^\epsilon(x, s) = u(x, s) + \epsilon_x , \quad x \in X, s \in S.$$

Let V and V^ϵ denote the value functions associated with u and u^ϵ respectively. Note that $u = u^0$ and $V = V^0$.

For any $\pi \in \mathcal{I}$ and $A \in \mathcal{X}$, let $\mathcal{M}(\pi, A)$ denote the set of π-measurable functions $f \colon S \to A$. We can rewrite the V^ϵ as

$$(*) \qquad V^\epsilon(A|\pi) = \max_{f \in \mathcal{M}(\pi,A)} \sum_{s \in S} u^\epsilon(f(s), s) , \qquad A \in \mathcal{X}, \pi \in \mathcal{I}.$$

By finiteness of the model, we can perceive the above value functions as elements of some finite dimensional Euclidean space. The following Lemma shows that for any $A \in \mathcal{X}$ and $\pi \in \mathcal{I}$, $V^\epsilon(A|\pi)$ is continuous in ϵ at $\epsilon = 0$.

Lemma 1

$$\lim_{\|\epsilon\| \to 0} V^\epsilon = V .$$

Proof: Let $\mathbf{1} \in \mathbb{R}^X$ denote the vector of 1's and let $\epsilon \in \mathbb{R}^X$. Define $m(\epsilon) = (\min_{x\in X} \epsilon_x) \mathbf{1} \in \mathbb{R}^X$ and $M(\epsilon) = (\max_{x\in X} \epsilon_x) \mathbf{1} \in \mathbb{R}^X$. Then $u^{m(\epsilon)}$ and $u^{M(\epsilon)}$ are constant shifts of u and therefore $V^{m(\epsilon)}$ and $V^{M(\epsilon)}$ solve the same maximization problem $(*)$ as V, differing by a constant. In particular, $V^{m(\epsilon)} = V + (n!)m(\epsilon)$ and $V^{M(\epsilon)} = V + (n!)M(\epsilon)$. Moreover, since $u^{m(\epsilon)} \leq u^\epsilon \leq u^{M(\epsilon)}$, we have that $V^{m(\epsilon)} \leq V^\epsilon \leq V^{M(\epsilon)}$. Thus,

$$V = \lim_{\|\epsilon\|\to 0} V^{m(\epsilon)} \leq \lim_{\|\epsilon\|\to 0} V^\epsilon \leq \lim_{\|\epsilon\|\to 0} V^{M(\epsilon)} = V$$

as desired. □

Let $\beta = V(\{x\}|\{S\})$ for some $x \in X$. Since all alternatives in X are symmetric with respect to the S and u, β does not depend on the particular choice of x. Moreover

$$V^\epsilon(A|\{S\}) = \max_{x\in A} \sum_{s\in S} u^\epsilon(x,s) = \max_{x\in A} \sum_{s\in S} [u(x,s) + \epsilon_x] = \max_{x\in A} [\beta + (n!)\epsilon_x] = \beta + (n!) \max_{x\in A} \epsilon_x.$$

Lemma 2 *Let $|B| \geq 2$, $x \in B$, and $y \neq x$, then*

$$\Delta_{B,x,y} := \sum_{s\in E_{x,B}} [u(x,s) - u(y,s)] > 0$$

Proof: Let B, x, and y be as in above. If $y \in B$, then $u(x,s) \geq u(y,s) + 1$ for any $s \in E_{x,B}$, so the result follows immediately.

If $y \notin B$ then let $F = \{s \in E_{x,B} \,|\, \forall z \in B \setminus \{x\} : yR_s z\}$ be the set of states in $E_{x,B}$ that rank elements of $B \setminus \{x\}$ below y. Consider the map ψ: $F \to F$, where $\psi(s)$ is the state that has the ranking obtained by switching the places of y and x in s, for $s \in F$. Then ψ is a permutation of F and

$$\sum_{s\in F} u(y,s) = \sum_{s\in F} u(y,\psi(s)) = \sum_{s\in F} u(x,s) \tag{1}$$

where the first equality is a reordering of the summation that is possible since ψ is a permutation of F. Equation 1 implies that

$$\Delta_{B,x,y} = \sum_{s\in E_{x,B}\setminus F} [u(x,s) - u(y,s)].$$

Since $u(x,s) - u(y,s) \geq 2$ for any $s \in E_{x,B} \setminus F$, we have $\Delta_{B,x,y} > 0$ when $E_{x,B} \setminus F \neq \emptyset$, i.e. when $|B| \geq 2$. □

Lemma 3 *If* $|B| \geq 2$ *and* $B \setminus A \neq \emptyset$, *then*

$$V(B|\pi_B) > V(A|\pi_B).$$

Proof: Let $|B| \geq 2$, $B \setminus A \neq \emptyset$, and $x \in B$. By Lemma 2, we have

$$\sum_{s \in E_{x,B}} u(x,s) \geq \max_{z \in A} \sum_{s \in E_{x,B}} u(z,s)$$

where the inequality is strict if $x \notin A$. Since $B \setminus A \neq \emptyset$, there exists $x \in B \setminus A$, therefore

$$V(B|\pi_B) = \sum_{x \in B} \sum_{s \in E_{x,B}} u(x,s) > \sum_{x \in B} \max_{y \in A} \sum_{s \in E_{x,B}} u(y,s) = V(A|\pi_B)$$

as desired. □

Lemma 4 *There is* $\delta > 0$ *such that for any* $\epsilon \in \mathbb{R}^X$ *and* $B \in \mathcal{X}$ *such that* $\|\epsilon\| < \delta$ *and* $|B| \geq 2$, B *maximizes* $V^\epsilon(\cdot|\pi_B)$ *over* $\mathcal{X}$.

Proof: It is easily seen that the lemma is satisfied if $n \leq 2$. Assume that $n \geq 3$ and set

$$\delta = \frac{1}{n!} \min_{|B'| \geq 2, x' \in B', y' \neq x'} \Delta_{B',x',y'}.$$

Then $\delta > 0$ by Lemma 2. Let $\|\epsilon\| < \delta$ and $|B| \geq 2$. We will show that $V^\epsilon(B|\pi_B) \geq V^\epsilon(X|\pi_B)$, which implies the desired result by monotonicity of $V^\epsilon(\cdot|\pi_B)$. If $B = X$, this inequality is trivially satisfied, so in the following concentrate on the case where $B \subsetneq X$. Then, for any $x \in B$ and $y \neq x$,

$$\sum_{s \in E_{x,B}} [u^\epsilon(x,s) - u^\epsilon(y,s)] = \sum_{s \in E_{x,B}} [u(x,s) - u(y,s) + (\epsilon_x - \epsilon_y)]$$

$$= \Delta_{B,x,y} + (\epsilon_x - \epsilon_y)\,|E_{x,B}| > 0$$

since

$$|\epsilon_x - \epsilon_y|\,|E_{x,B}| < 2\|\epsilon\|\,|E_{x,B}| < \delta n! \leq \Delta_{B,x,y}$$

where the first inequality follows from $\|\epsilon\| < \delta$, the second inequality follows from $|E_{x,B}| = \frac{n!}{|B|} \leq \frac{n!}{2}$, and the last inequality follows from the definition of δ.

Since for any $x \in B$

$$\sum_{s \in E_{x,B}} u^\epsilon(x,s) = \max_{y \in X} \sum_{s \in E_{x,B}} u^\epsilon(y,s)$$

we have that $V^\epsilon(B|\pi_B) \geq V^\epsilon(X|\pi_B)$. □

Define

$$\gamma = \min_{|B|\geq 2, B\setminus A\neq\emptyset} \left[V(B|\pi_B) - V(A|\pi_B)\right].$$

Then, $\gamma > 0$ by Lemma 3. By Lemma 1, let $\delta' > 0$ be such that $\|\epsilon\| < \delta' \Rightarrow \|V^\epsilon - V\| < \gamma/3$. Let $\alpha > 0$ be small enough such that

$$\alpha \| \left(U(\{x\})\right)_{x\in X} \| < (n!) \min\{\delta, \delta'\} \quad \text{ and } \quad \alpha \max_{A,B\in\mathcal{X}} [U(A) - U(B)] < \gamma/3.$$

Set $\tilde{U} = \alpha U + \beta$ and $\epsilon = \frac{\alpha}{n!} \left(U(\{x\})\right)_{x\in X} \in \mathbb{R}^X$. Then $\|\epsilon\| < \min\{\delta, \delta'\}$.

Now define $c(\{S\}) = 0$ and

$$c(\pi_A) = V^\epsilon(A|\pi_A) - \tilde{U}(A), \qquad A \in \mathcal{X}.$$

Note that our definition is consistent since $c(\pi_A) = 0$ for any singleton $A \in \mathcal{X}$. In the following let ϵ, $\tilde{U}$, and c be as in above.

Lemma 5 *Let $A, B \in \mathcal{X}$ with $|A| > |B| \geq 1$, then $c(\pi_A) > c(\pi_B)$.*

Proof: Let $A, B \in \mathcal{X}$ with $|A| > |B| \geq 1$ and choose $B' \supset B$ with $|B'| = |A| \geq 2$. Then

$$\begin{aligned} c(\pi_A) - c(\pi_B) &= V^\epsilon(A|\pi_A) - V^\epsilon(B|\pi_B) + \left[\tilde{U}(B) - \tilde{U}(A)\right] \\ &= [V^\epsilon(A|\pi_A) - V(A|\pi_A)] + [V(B'|\pi_{B'}) - V(B|\pi_B)] \\ &\quad + [V(B|\pi_B) - V^\epsilon(B|\pi_B)] + \left[\tilde{U}(B) - \tilde{U}(A)\right] > 0 \end{aligned}$$

where the first equality follows from definition c and the second equality follows from $V(A|\pi_A) = V(B'|\pi_{B'})$ by symmetry. The strict inequality follows from

$$|V^\epsilon(A|\pi_A) - V(A|\pi_A)| < \gamma/3, \quad |V(B|\pi_B) - V^\epsilon(B|\pi_B)| < \gamma/3,$$

$$\text{and } \left|\tilde{U}(B) - \tilde{U}(A)\right| < \gamma/3$$

by our choice of ϵ and $\tilde{U}$, and

$$V(B'|\pi_{B'}) - V(B|\pi_B) \geq V(B'|\pi_{B'}) - V(B|\pi_{B'}) \geq \gamma$$

where the first inequality above follows from $\pi_{B'} \geq_f \pi_B$ and the second inequality follows from the definition of γ, $|B'| \geq 2$, and $B' \setminus B \neq \emptyset$. $\square$

Note that $c(\{S\}) = 0$ so by Lemma 5, c is nonnegative valued. Therefore, c is a cost function.

Lemma 6 *Let $A, B \in \mathcal{X}$, then*

$$V^\epsilon(A|\pi_A) - c(\pi_A) \geq V^\epsilon(A|\pi_B) - c(\pi_B).$$

Proof: Let $A, B \in \mathcal{X}$. Then the above inequality is equivalent to

$$\left[\tilde{U}(A) - \tilde{U}(B)\right] + [V^\epsilon(B|\pi_B) - V^\epsilon(A|\pi_B)] \geq 0$$

where the latter is obtained by using the definition of c. We will consider three different cases:

(i) $\underline{|B| = 1}$: Let $y^* \in \text{argmax}_{x \in A} \epsilon_x$ and $B = \{y\}$. Then $\tilde{U}(A) \geq \tilde{U}(\{y^*\})$ since $\tilde{U}$ represents $\succsim$ and $\succsim$ is monotone. By definition, $\tilde{U}(\{y^*\}) = (n!)\epsilon_{y^*} + \beta$ and $\tilde{U}(\{y\}) = (n!)\epsilon_y + \beta$. Moreover $V^\epsilon(B|\pi_B) = \beta + (n!)\epsilon_y$ and $V^\epsilon(A|\pi_B) = \beta + (n!)\epsilon_{y^*}$. Thus

$$\left[\tilde{U}(A) - \tilde{U}(B)\right] + [V^\epsilon(B|\pi_B) - V^\epsilon(A|\pi_B)] \geq \left[\tilde{U}(\{y^*\}) - \tilde{U}(\{y\})\right] + [(n!)\epsilon_y - (n!)\epsilon_{y^*}] = 0$$

as desired.

(ii) $\underline{|B| \geq 2 \text{ and } A \succsim B}$: Since $|B| \geq 2$, by Lemma 4 $V^\epsilon(B|\pi_B) \geq V^\epsilon(A|\pi_B)$. Since $A \succsim B$ we also have that $\tilde{U}(A) \geq \tilde{U}(B)$. Therefore

$$\left[\tilde{U}(A) - \tilde{U}(B)\right] + [V^\epsilon(B|\pi_B) - V^\epsilon(A|\pi_B)] \geq 0.$$

(iii) $\underline{|B| \geq 2 \text{ and } B \succ A}$: Monotonicity of $\succsim$ and $B \succ A$ imply that $B \setminus A \neq \emptyset$ (i.e. B is not a subset of A). Then

$$\left[\tilde{U}(A) - \tilde{U}(B)\right] + [V^\epsilon(B|\pi_B) - V^\epsilon(A|\pi_B)] = \left[\tilde{U}(A) - \tilde{U}(B)\right]$$

$$+ [V^\epsilon(B|\pi_B) - V(B|\pi_B)] + [V(B|\pi_B) - V(A|\pi_B)] + [V(A|\pi_B) - V^\epsilon(A|\pi_B)] > 0$$

where the above equality is just an identity. The above inequality follows from

$$|V^\epsilon(B|\pi_B) - V(B|\pi_B)| < \gamma/3, \quad |V(A|\pi_B) - V^\epsilon(A|\pi_B)| < \gamma/3,$$

$$\text{and } \left|\tilde{U}(B) - \tilde{U}(A)\right| < \gamma/3$$

by our choice of ϵ and $\tilde{U}$, and

$$V(B|\pi_B) - V(A|\pi_B) \geq \gamma$$

by definition of γ and since $|B| \geq 2$ and $B \setminus A \neq \emptyset$. □

For any $A \in \mathcal{X}$,

$$\max_{\pi \in \mathcal{I}} [V^\epsilon(A|\pi) - c(\pi)] = V^\epsilon(A|\pi_A) - c(\pi_A) = \tilde{U}(A)$$

where the first equality follows from Lemma 6 and the last equality follows from the definition of $c(\pi_A)$, completing the proof of the Theorem. □

A.2 Proof of Theorem 2

Assume that $\succsim$ is a monotone preference over $X_1 \times \mathcal{P}(X_2)$ represented by U. For any $x_1 \in X_1$, U_{x_1} defined by $U_{x_1}(A_2) = U(x_1, A_2)$ represents a monotone preference over $\mathcal{P}(X_2)$. Therefore, by our main theorem, U_{x_1} is a contemplation utility over $\mathcal{P}(X_2)$ represented by some $(\tilde{S}, \tilde{u}_{x_1}, \tilde{\mathcal{I}}, \tilde{c}_{x_1})$. Note that $\tilde{S}$ and $\tilde{\mathcal{I}}$ do not depend on x_1, because in the proof of Theorem 1 they are constructed independently of the preference.

Let $S = X_1 \times \tilde{S}$ and define u: $X_1 \times X_2 \times S \to \mathbb{R}$ by

$$u(x_1, x_2, s) = \begin{cases} \tilde{u}_{x_1}(x_2, s) & \text{if } s = (x_1, \tilde{s}) \text{ for some } \tilde{s} \in \tilde{S} \\ 0 & \text{otherwise} \end{cases}$$

for any $(x_1, x_2, s) \in X_1 \times X_2 \times S$.

Let

$$\mathcal{I} = \big\{\{S\}\big\} \cup \Big\{\{x_1\} \times \tilde{\pi} \cup \big\{\, (X_1 \setminus \{x_1\}) \times \tilde{S}\big\} : x_1 \in X_1, \tilde{\pi} \in \tilde{\mathcal{I}}\Big\}$$

and define c: $\mathcal{I} \to \mathbb{R}$ by $c(\{S\}) = 0$ and

$$c\Big(\{x_1\} \times \tilde{\pi} \cup \big\{(X_1 \setminus \{x_1\}) \times \tilde{S}\big\}\Big) = \tilde{c}_{x_1}(\tilde{\pi})$$

for any $x_1 \in X_1$ and $\tilde{\pi} \in \tilde{\mathcal{I}}$.

It is now straightforward to verify that U is the contemplation utility represented by $(S, u, \mathcal{I}, c)$ in the consumption-investment budgeting model. □

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