Efficient and Incentive-Compatible Liver Exchange: Appendices B–E For Online Publication

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February, 2020

Appendix B Proofs

Proof of Lemma 2:

Parts (1) *and* (2.*b*): For each i_k, we check whether $\mathcal{J}_{k-1} \cup \{i_k\}$ is matchable in G_{k-1} (recall that by construction of G_{k-1}, $\{i_k, j\} \in E_{k-1} \implies j \in \mathcal{E}^\ell(i_k)$). When the answer is affirmative, we include i_k in $\mathcal{J}_k$. Moreover, for all $m > k$, no right-lobe donating match of i_k is ever included in G_m and G_m is constructed from G_{m-1} making sure that $\mathcal{J}_k$ is still matchable. These imply Parts (1) and (2.b) when $k = K$.

If $\mathcal{J}_K = \emptyset$ then $\mathbf{M}_K = \mathbf{M}[G_K] \supseteq \{\varnothing\} \neq \emptyset$. If $\mathcal{J}_K \neq \emptyset$, then we showed that $\mathcal{J}_K$ is matchable in G_K by some matching $M' \in \mathbf{M}[G_K]$ by Part 1. Thus, $M' \in \mathbf{M}_K$. This shows in either, case $\mathbf{M}_K \neq \emptyset$. Suppose $M \in \mathbf{M}_K$ for the remaining parts.

Part (2.*c*): Suppose that there exists some $i_k \in \widetilde{\mathcal{J}}_K$ such that $M(i_k) \in \mathcal{E}^\ell(i_k)$. This and Part (2.b) imply that all of the pairs in $\mathcal{J}_{k-1} \cup \{i_k\}$ are matched in M by donating their left lobes. By construction, $\mathcal{J}_{k-1}$ is matchable in G_{k-1} and $\{i_k\} \cup \mathcal{J}_{k-1}$ is not matchable in G_{k-1}. Again by construction, for all $i \in \{i_{k+1}, \ldots, i_n\}$, $\{i, j\} \in E_{k-1} \implies j \in \mathcal{E}^\ell(i)$. Therefore, $M \notin \mathbf{M}_{k-1}$ and there is some $i \in \mathcal{J}_{k-1} \cup \{i_k\}$ such that $i \in \mathcal{E}^r(M(i))$ and $M(i) \in \{i_{k+1}, \ldots, i_n\}$. Hence, $i \; \Pi_\ell \; M(i)$. We also have $M(i) \in \mathcal{E}^\ell(i)$ as established above. Thus, by definition of the precedence digraph, $\tau(M(i)) \to \tau(i)$. By construction of the topological order, $M(i) \; \Pi_\ell \; i$, which is a contradiction to $i \; \Pi_\ell \; M(i)$.

Part (2.*d*): Let $i_k \in \mathcal{I} \setminus [\mathcal{J}_K \cup \widetilde{\mathcal{J}}_K]$. Thus, $\{i_k, j\} \in E_K \implies j \in \mathcal{E}^\ell(i_k)$. Suppose $M(i_k) \neq \emptyset$. Then $M(i_k) \in \mathcal{E}^\ell(i_k)$. This and Part (2.b) imply that all of the pairs in $\mathcal{J}_{k-1} \cup \{i_k\}$

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are matched in M by donating their left lobes. By construction, $\mathcal{J}_{k-1}$ is matchable in G_{k-1} and $\{i_k\} \cup \mathcal{J}_{k-1}$ is not matchable in G_{k-1}. Thus, $M \notin \mathbf{M}_{k-1}$. Again by construction, for all $i \in \{i_{k+1}, \ldots, i_n\}$, $\{i, j\} \in E_{k-1} \implies j \in \mathcal{E}^\ell(i)$. Therefore, there is some $i \in \mathcal{J}_{k-1} \cup \{i_k\}$ such that $i \in \mathcal{E}^r(M(i))$ and $M(i) \in \{i_{k+1}, \ldots, i_n\}$. Hence, $i \, \Pi_\ell \, M(i)$. We have $M(i) \in \mathcal{E}^\ell(i)$ as established above. Thus, by definition of the precedence digraph, $\tau(M(i)) \to \tau(i)$. By construction of the topological order, $M(i) \, \Pi_\ell \, i$, which is a contradiction to $i \, \Pi_\ell \, M(i)$.

Parts (2.a) *and* (2.e): We prove the following claim to prove these parts:

Claim: For all indices k, all indices $k' \geq k$, and all pairs $i \in \mathcal{J}_k$, the following hold for the induced match sets of pair i at Step 1.(k) and Step 1.(k') reduced compatibility graphs:

1. $E_{k'}(i) \subseteq E_k(i)$, and
2. $\{i, j\} \in E_k \implies$ for all $M' \in \mathbf{M}_{k'}$, $M'(i) \, I_i \, j$.

Proof of Claim:

1. Suppose to the contrary, there exists some $\{i, j\} \in E_{k'} \setminus E_k$. Therefore, j is processed and, in particular, transformed after Step 1.(k). Since $i \in \mathcal{J}_k$, i is not transformed and thus, $j \in \mathcal{E}^\ell(i)$ and $i \in \mathcal{E}^r(j)$ implying that $\tau(j) \to \tau(i)$ in the precedence digraph (by definition). This, in turn, implies $j \, \Pi_\ell \, i$. But this is a contradiction that j is processed after Step 1.(k) while i is processed before or at Step 1.(k).
2. Let $\{i, j\} \in E_k$. Since $i = i_m$ for some index $m \leq k$, by the first part of the claim $\{i, j\} \in E_k \implies \{i, j\} \in E_m$ since $m \leq k$. We delete from E_{m-1} all matches of i but its best achievable matches (while all pairs in $\mathcal{J}_{m-1}$ can simultaneously be matched), i.e., $j \in \mathcal{B}(i|\mathcal{J}_{m-1}, G_{m-1})$. Hence, i is indifferent among all matchings that match it in G_k. Since no new matches of i are added to $E_{k+1}, \ldots, E_{k'}$ by first part of the Claim, for all $M' \in \mathbf{M}_{k'} = \left\{M'' \in \mathbf{M}[G_{k'}] \ : \ M''(h) \neq \emptyset \ \ \forall h \in \mathcal{J}_{k'}\right\}$, $M'(i) \, I_i \, j$. $\diamond$

Pick $i \in \mathcal{J}_K$ and $M' \in \mathbf{M}_K$. Then $M(i) \, I_i \, M'(i)$ by the Claim's second statement. Moreover, by Part (2.d) for all $j \in \mathcal{I} \setminus [\mathcal{J}_K \cup \widetilde{\mathcal{J}}_K]$, $M(j) = M'(j) = \emptyset$. These prove Part (2.a).

Now $i = i_k$ for some k. Since $M \in \mathbf{M}_K$, $M(i_k) \in E_K(i_k) \subseteq E_k(i_k)$ by the Claim's first statement. Since $E_k(i_k) = \left\{\{i_k, j\} \ : \ j \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_{k-1})\right\}$ (by definition of graph G_k) and since $E_K(i_k) \subseteq E_k(i_k)$, we have $E_K(i_k) = \left\{\{i_k, j\} \ : \ j \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_K)\right\}$, this in turn implies $M(i_k) \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_K)$ and $M(i_k) \, I_{i_k} \, j$ for all $j \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_k)$. This proves Part (2.e) and completes the proof of the lemma. ■

Proof of Lemma 3:

Part (1): For all $n = 1, \ldots, N$, $\mathbf{M}^*_n \subseteq \mathbf{M}_K$ follows from the facts that $G^*_0 = G_K$ and the match sets satisfy $E^*_N \subseteq \ldots \subseteq E^*_0 = E_K$; moreover, $\mathcal{J}^*_n \subseteq \widetilde{\mathcal{J}}_K$ follows from the definition of Step 2. Thus, Part (1) is proven when $n = N$.

Part (2): $\mathcal{J}_K \cup \mathcal{J}_n^*$ is matchable in G_n^* follows from the definition of Step 2. Thus, Part (2) follows for $n = N$.

If $\mathcal{J}_K \cup \mathcal{J}_N^* = \emptyset$, then $\mathbf{M}_N^* = \mathbf{M}[G_N^*] \supseteq \{\varnothing\} \neq \emptyset$. If $\mathcal{J}_K \cup \mathcal{J}_N^* \neq \emptyset$, there exists some $M' \in \mathbf{M}[G_N^*]$ such that M' matches all pairs in $\mathcal{J}_K \cup \mathcal{J}_N^*$ as we showed in Part 1. Thus, in either case, $M_N^* \neq \emptyset$.

Let $M \in \mathbf{M}_N^*$ for the rest of the proof.

Parts (3.*b*), (3.*c*), *and* (3.*e*): $\mathbf{M}_N^* \subseteq \mathbf{M}_K$ and Lemma 2 Parts (2.b), (2,c), and (2.e) imply Parts (3.b), (3.c), and (3.e), respectively.

Part (3.*d*): Suppose contrary to the claim there exists $j \in \mathcal{I} \setminus [\mathcal{J}_K \cup \mathcal{J}_N^*]$ such that $M(j) \neq \emptyset$. By Lemma 2 Part (2.d), $j \in \widetilde{\mathcal{J}}_N \setminus \mathcal{J}_N^*$. Thus, $j = i_n^*$ for some $n \leq N$. Since $i_n^* \notin \mathcal{J}_n^* \subseteq \mathcal{J}_N^*$, in substep 2.($n$), $\mathcal{J}_K \cup \mathcal{J}_{n-1}^* \cup \{i_n^*\}$ is not matchable in G_{n-1}^*. Thus, no matching in $\mathbf{M}_{n-1}^*$ matches i_n^*. This contradicts $M(i_n^*) \neq \emptyset$ because $M \in \mathbf{M}_N^* \subseteq \mathbf{M}_{n-1}^*$.

Part (3.*f*): Let $i_n^* \in \mathcal{J}_N^*$. By construction in Step 2.(n),

$$E_n^* = \Big[E_{n-1}^* \setminus E_{n-1}^*(i_n^*)\Big] \cup \Big\{\{i_n^*, j\} \ : \ j \in \mathcal{B}(i_n^* | \mathcal{J}_K \cup \mathcal{J}_{n-1}^*, G_{n-1}^*)\Big\}.$$

That is, while we are obtaining G_n^*, we delete all edges involving i_n^* in G_{n-1}^* except those would match it to one of its best assignments in G_{n-1}^* given that all pairs in $\mathcal{J}_K \cup \mathcal{J}_{n-1}^*$ are simultaneously matched. Since $E_N^* \subseteq E_n^*$, $M \in \mathbf{M}[G_n^*]$. Since i_n^* is matched in M by Part 3.(c), $M(i_n^*) \in \mathcal{B}(i_n^* | \mathcal{J}_K \cup \mathcal{J}_{n-1}^*, G_n^*)$. Since $E_N^* \subseteq E_n^*$, $M(i_n^*) \in \mathcal{B}(i_n^* | \mathcal{J}_K \cup \mathcal{J}_{n-1}^*, G_N^*)$.

Part (3.*a*): For all $j \in \mathcal{J}_K$, the statement holds by Part (3.e). For all $j \in \mathcal{I} \setminus [\mathcal{J}_K \cup \mathcal{J}_N^*]$, the statement holds by Part (3.d). For all $j \in \mathcal{J}_N^*$, the statement holds by Part (3.f).

■

We prove Theorem 1 in three parts for each property in Lemmas A-1, A-2, and A-3. Recall that $f^{\mathbf{P}}$ refers to the precedence-induced adaptive-priority mechanism for a fixed (Π_ℓ, Π_r) pair.

Lemma A-1 (IR) *Mechanism $f^{\mathbf{P}}$ is individually rational.*

Proof of Lemma A-1 (IR): In every step of the algorithm, the active reduced compatibility graphs are subgraphs of the IR compatibility graph given the submitted preference profile R. Since $f^{\mathbf{P}}$ chooses a matching of the final graph of the algorithm G_N^*, it is individually rational. ■

Lemma A-2 (PE) *Mechanism $f^{\mathbf{P}}$ is Pareto efficient.*

Proof of Lemma A-2 (PE): Fix $R \in \mathbf{R}$. Recall that $G_{IR}[R] = (\mathcal{I}, E_{IR}[R])$ is the individually rational compatibility graph of the problem induced by R.

Let $M \equiv f^{\mathbf{P}}[R]$. Suppose $M' \in \mathbf{M}_c$ satisfies $M'(i)\ R_i\ M(i)$ for all $i \in \mathcal{I}$. We will show that $M'(i)\ I_i\ M(i)$ for all $i \in \mathcal{I}$ to prove Pareto efficiency of $f^{\mathbf{P}}[R]$. Since $M'(i)\ R_i\ M(i)$ for all $i \in \mathcal{I}$, and M is individually rational by Lemma A-1, we obtain that M' is individually rational, as well.

We consider three separate cases for pairs in $\mathcal{J}_K$, $\mathcal{J}_N^*$, and $\mathcal{I} \setminus [\mathcal{J}_K \cup \mathcal{J}_N^*]$.

1. $\mathcal{J}_K$: By induction we prove that for all $i_p \in \mathcal{J}_K$, $M(i_p)\ I_{i_p}\ M'(i_p)$ and $M'(i_p) \in \mathcal{B}(i_p|\mathcal{J}_{p-1}, G_K)$.
   Fix $k \leq K$. As the inductive assumption, assume that for all $k' \leq k-1$, the following holds:
   $$\text{for all } i_p \in \mathcal{J}_{k'},\ \ M(i_p)\ I_{i_p}\ M'(i_p) \ \text{ and } \ M'(i_p) \in \mathcal{B}(i_p|\mathcal{J}_{p-1}, G_{k'}).$$
   We will prove the same holds for $k' = k$. Two cases hold for i_k: Either $i_k \notin \mathcal{J}_k$ or $i_k \in \mathcal{J}_k$:
   - First, assume $i_k \notin \mathcal{J}_k$. Thus, $\mathcal{J}_k = \mathcal{J}_{k-1}$. Hence, the inductive assumption for $k' = k-1$ implies for all $i_p \in \mathcal{J}_k$, $M(i_p)\ I_{i_p}\ M'(i_p)$.
     – If i_k is not willing, then by definition of Step 1.(k), $G_k = G_{k-1}$, and hence, the inductive assumption for $k' = k-1$ implies $M'(i_p) \in \mathcal{B}(i_p|\mathcal{J}_{p-1}, G_k)$.
     – If i_k is willing, then $i_k \in \widetilde{J}_k$, and by the definition of Step 1.(k), graph $G_k = (\mathcal{I}, E_k)$ satisfies:
     $$E_k = E_{k-1} \cup E_{IR}[R_{\widetilde{\mathcal{J}}_k}, R^0_{-\widetilde{\mathcal{J}}_k}](i_k).$$
     Fix $i_p \in \mathcal{J}_{k-1}$. Since $i_p\ \Pi_\ell\ i_k$, by the definition of the precedence graph and topological order we have $t(i_k, i_p) \neq r$ or $t(i_p, i_k) \neq \ell$. As we have not transformed i_p,
     $$\{i_p, i_k\} \in E_{k-1} \iff \{i_p, i_k\} \in E_k.$$
     Thus, $\mathcal{B}(i_p|\mathcal{J}_{p-1}, G_{k-1}) = \mathcal{B}(i_p|\mathcal{J}_{p-1}, G_k)$. Hence, by the inductive assumption for $k' = k-1$, we still have $M'(i_p) \in \mathcal{B}(i_p|\mathcal{J}_{p-1}, G_k)$.
   - Next, assume $i_k \in \mathcal{J}_k$. Then $\mathcal{J}_k = \mathcal{J}_{k-1} \cup \{i_k\}$. By the definition of Step 1.(k), active graph $G_k = (\mathcal{I}, E_k)$ is obtained from the latest active graph $G_{k-1} = (\mathcal{I}, E_{k-1})$ as follows through deletion of i_k's matches except its best achievable ones:
     $$E_k = \Big[E_{k-1} \setminus E_{k-1}(i_k)\Big] \cup \Big\{\{i_k, j\}\ :\ j \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_{k-1})\Big\}. \qquad (1)$$
     Since $i_k \in \mathcal{J}_K$, $M(i_k) \neq \emptyset$. Since by assumption $M'(i_k)\ R_{i_k}\ M(i_k)$, we have $M'(i_k) \neq \emptyset$, either. Moreover, $M'(i_k) \in \mathcal{E}^\ell(i_k)$, as $M(i_k) \in \mathcal{E}^\ell(i_k)$ and i_k prefers donating left lobe to donating right lobe under any match. Suppose i_q is i_k's assignment under M', i.e., $i_q \equiv M'(i_k)$.
     Two subcases exist for i_q: Either $i_q \notin \mathcal{J}_{k-1}$ or $i_q \in \mathcal{J}_{k-1}$.

(a) First, suppose $i_q \notin \mathcal{J}_{k-1}$.

Observe that it cannot be the case that $q > k$ and yet $t(i_q, i_k) = r$. As otherwise, since i_k donates a left lobe to i_q, we would have $\tau(i_q) \rightarrow \tau(i_k)$ implying that $i_q \; \Pi_\ell \; i_k$, a contradiction to $q > k$. Thus, If $q > k$ then $\{i_k, i_q\}$ is a left-lobe only match, implying that $\{i_k, i_q\} \in E_0$. On the other hand if $q < k$ then $i_q \in \widetilde{\mathcal{J}}_{k-1}$ and it was transformed in Step 1.(q) making the match $\{i_k, i_q\}$ available in E_q and later active graphs, as $t(i_q, i_k) = r$. (Observe that if $t(i_q, i_k) = \ell$, then $i_q \in \mathcal{J}_q$ would be the case.)

Thus, these and the inductive assumption for $k' \leq k-1$ that $\{ \underbrace{i_p}_{\notin\{i_k,i_q\}}, \underbrace{M'(i_p)}_{\notin\{i_k,i_q\}}\} \in E_{k-1}$ for all $i_p \in \mathcal{J}_{k-1}$ imply that the match $\{i_k, i_q\}$ does not conflict with the best achievable match of any $i_p \in \mathcal{J}_{k-1}$ and is still available and has not been deleted from the active graph yet at the end of Step 1.($k-1$), i.e., $\{i_k, i_q\} \in E_{k-1}$. Since $M(i_k) \; I_{i_k} \; j$ for all $j \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_{k-1})$ by Lemma 2 Part (2.e) and $i_q \; R_{i_k} \; M(i_k)$, we have $i_q \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_{k-1})$ and $i_q \; I_{i_k} \; M(i_k)$. As a result $\{i_k, i_q\}$ survives deletion in Step 1.(k) by Equation 1: $M'(i_k) = i_q \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_k)$.

Moreover, in obtaining G_k from G_{k-1}, we do not delete match $\{i_p, M'(i_p)\}$ from E_{k-1} for any $i_p \in \mathcal{J}_{k-1}$ by Equation 1, either. It continues to be the case for the active graph G_k that $M'(i_p)$ is one of the best achievable assignments of i_p, i.e., $M'(i_p) \in \mathcal{B}(i_p|\mathcal{J}_{p-1}, G_k)$. This together with the inductive assumption for $k' = k-1$ that $M'(i_p) \; I_{i_p} \; M(i_p)$ completes this subcase.

(b) Finally, suppose $i_q \in \mathcal{J}_{k-1}$. Then $\{i_k, i_q\}$ is a left-lobe-only match and by the inductive assumption for $k' = k-1$ we have $\{i_q, \underbrace{M'(i_q)}_{=i_k}\} \in E_{k-1}$.

Since $M(i_k) \; I_{i_k} \; j$ for all $j \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_{k-1})$ by Lemma 2 Part (2.e) and $i_q \; R_{i_k} \; M(i_k)$, we do not delete the match $\{i_k, i_q\}$ from the latest active graph G_{k-1} while obtaining G_k by Equation 1. Thus, $M'(i_k) = i_q \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_k)$ and $M'(i_k) \; I_{i_k} \; M(i_k)$.

By the inductive assumption for $k' = k-1$, $M'(i_p) \in \mathcal{B}(i_p|\mathcal{J}_{p-1}, G_{k-1})$ for all $i_p \in \mathcal{J}_{k-1}$.

As the match $\{i_k, i_q\}$ survives deletion in Step 1.(k) by Equation 1, we still have $M'(i_q) \in \mathcal{B}(i_q|\mathcal{J}_{q-1}, G_k)$.

Consider any $i_p \in \mathcal{J}_{k-1} \setminus \{i_q\}$. Since $\{i_p, M'(i_p)\} \in E_{k-1}$ and $M'(i_p) \neq i_k$, this match survives deletion by Equation 1, and we have $M'(i_p) \in \mathcal{B}(i_p|\mathcal{J}_{p-1}, G_k)$.

By the inductive assumption for $k' = k-1$, for all $i_p \in \mathcal{J}_{k-1}$ we have $M'(i_p) \; I_{i_p} \; M(i_p)$, completing the proof of the inductive step for $k' = k$ for this case.

2. $\mathcal{J}_N^*$: By induction, we prove that for all $i_p^* \in \mathcal{J}_N^*$, $M(i_p^*) \; I_{i_p^*} \; M'(i_p^*)$ and $M'(i_p^*) \in \mathcal{B}(i_p^*|\mathcal{J}_K \cup \mathcal{J}_{p-1}^*, G_N^*)$, and for all $i_k \in \mathcal{J}_K$, $M(i_k) \; I_{i_k} \; M'(i_k)$ and $M'(i_k) \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_N^*)$. Fix $n \leq N$. As the inductive assumption, assume that for all $n' \leq n-1$, the following

holds:

$$\text{for all } i_p^* \in \mathcal{J}_{n'}^*, \;\; M(i_p^*)\ I_{i_p^*}\ M'(i_p^*) \;\text{ and }\; M'(i_p^*) \in \mathcal{B}(i_p^* | \mathcal{J}_K \cup \mathcal{J}_{p-1}^*, G_{n'}^*), \text{ and}$$

$$\text{for all } i_k \in \mathcal{J}_K, \;\; M(i_k)\ I_{i_k}\ M'(i_k) \;\text{ and }\; M'(i_k) \in \mathcal{B}(i_k | \mathcal{J}_{k-1}, G_{n'}^*).$$

(Initial step $n' = 0$ is implied by Part 1 for $\mathcal{J}_K$ above.) We will prove the same holds for $n' = n$. Two cases hold for i_n^*: Either $i_n^* \notin \mathcal{J}_n^*$ or $i_n^* \in \mathcal{J}_n^*$:

- First, assume $i_n^* \notin \mathcal{J}_n^*$. Thus, $\mathcal{J}_n^* = \mathcal{J}_{n-1}^*$ and $G_n^* = G_{n-1}^*$. Hence, the inductive assumption for $n' = n-1$ implies the same holds for n.
- Next, assume $i_n^* \in \mathcal{J}_n^*$. Thus, $\mathcal{J}_n^* = \mathcal{J}_{n-1}^* \cup \{i_n^*\}$. Recall that by the definition of Step 2.(n), active graph $G_n^* = (\mathcal{I}, E_n^*)$ is obtained from the latest active graph $G_{n-1}^* = (\mathcal{I}, E_{n-1}^*)$ as follows through deletion of i_n^*'s matches except its best achievable ones:

$$E_n^* = \Big[E_{n-1}^* \setminus E_{n-1}^*(i_n^*)\Big] \cup \Big\{\{i_n^*, j\} \;:\; j \in \mathcal{B}(i_n^* | \mathcal{J}_K \cup \mathcal{J}_{n-1}^*, G_{n-1}^*)\Big\}. \tag{2}$$

  We first prove the inductive statement for i_n^*, then for pairs in $\mathcal{J}_K \cup \mathcal{J}_{n-1}^*$:

  - Since $i_n^* \in \mathcal{J}_N^*$, $M(i_n^*) \neq \emptyset$. Since by assumption $M'(i_n^*)\ R_{i_n^*}\ M(i_n^*)$, we have $M'(i_n^*) \neq \emptyset$. The inductive assumption for $n' = n-1$ implies that for all $i \in \mathcal{J}_K \cup \mathcal{J}_{n-1}^*$, $M'(i)\ I_i\ M(i)$ and $\{i, M'(i)\} \in E_{n-1}^*$. That is to say that M' is a feasible matching in the active graph G_{n-1}^* of the pairs processed prior to i_n^*, assigning each of them to its best achievable assignment.

    Let $i \equiv M'(i_n^*)$. We will first show that $\{i_n^*, i\} \in E_{n-1}^*$:

    (a) If $i \notin \mathcal{J}_K \cup \mathcal{J}_{n-1}^*$ then i is not processed in Step 2 before Step 2.(n). Observe that $\{i_n^*, i\}$ is an individually rational right-lobe-only match, as otherwise either i_n^* or i would be included in $\mathcal{J}_K$. Moreover, $M'(i) = i_n^*\ R_i\ M(i)$ imply that i donates its right lobe to $M(i)$ and as a result i is also transformed in Step 1. Therefore, $\{i_n^*, i\} \in E_0^*$. Since by the inductive assumption for $n' \leq n-1$, this match has no conflict with the best achievable matches of any pair in $\mathcal{J}_K \cup \mathcal{J}_{n-1}^*$, this match never gets deleted in the previous substeps of Step 2, i.e. $\{i_n^*, i\} \in E_{n-1}^*$.

    (b) If $i \in \mathcal{J}_K \cup \mathcal{J}_{n-1}^*$ then by the inductive assumption for $n' = n-1$, we have $\{i_n^*, i\} \in E_{n-1}^*$.

    $M'(i_n^*) = i\ R_{i_n^*}\ M(i_n^*)$ implies that match $\{i_n^*, i\}$ survives the deletion in Step 2.(n) by Equation 2, implying $\{i_n^*, i\} \in E_n^*$.

    By the construction of M in the algorithm, we have $M(i_n^*) \in \mathcal{B}(i_n^* | \mathcal{J}_K \cup \mathcal{J}_{n-1}^*, G_{n-1}^*)$. Thus, not only $M'(i_n^*) = i\ I_{i_n^*}\ M(i_n^*)$, but also $M'(i_n^*) \in \mathcal{B}(i_n^* | \mathcal{J}_K \cup \mathcal{J}_{n-1}^*, G_n^*)$, as well.
  - Next, consider any $j \in \mathcal{J}_K \cup \mathcal{J}_{n-1}^*$. By the inductive assumption for $n' = n-1$

$M'(j)\ I_j\ M(j)$ and $\{j, M'(j)\} \in E^*_{n-1}$. If $M'(j) = i^*_n$ then the part for i^*_n (the above paragraph) implies $\{j, i^*_n\}$ survives the deletion in Step 2.(n), and thus, $\{j, i^*_n\} \in E^*_n$. If $M'(j) \neq i^*_n$, then match $\{j, M'(j)\}$ also survives the deletion in Step 2.(n) by Equation 2, and hence, $\{j, M'(j)\} \in E^*_n$. Thus, if $j = i^*_p \in \mathcal{J}^*_{n-1}$ for some p, then the inductive assumption for $n' = n-1$ also implies that $M'(i^*_p) \in \mathcal{B}(i^*_p|\mathcal{J}_{p-1}, G^*_n)$, and if $j = i_k \in \mathcal{J}_K$ for some k, then the inductive assumption for $n' = n-1$ also implies that $M'(i_k) \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G^*_n)$.

3. $\mathcal{I} \setminus [\mathcal{J}_K \cup \mathcal{J}^*_N]$: Part 2 for $\mathcal{J}^*_N$ also establishes that $M' \in \mathbf{M}^*_N$. Lemma 3 Part (2.d) implies for both M and M',

$$M'(i) = M(i) = \emptyset \quad \text{for all } i \in \mathcal{I} \setminus [\mathcal{J}_K \cup \mathcal{J}^*_N]$$

finishing the induction and showing that $M'(i)\ I_i\ M(i)$ for all $i \in \mathcal{I}$, and hence, $M = f^{\mathbf{P}}[R]$ is Pareto efficient.

■

Lemma A-3 (IC) *Mechanism $f^{\mathbf{P}}$ is incentive compatible.*

Proof of Lemma A-3 (IC): Fix $R \in \mathbf{R}$. Let $M \equiv f^{\mathbf{P}}[R]$. Consider the algorithm executed to find M under R, and let $\mathcal{J}_K$ and $\widetilde{\mathcal{J}}_K$ be the corresponding sets of pairs determined in Step 1.

Consider pair $i \in \mathcal{I}$. Let its preference relation be denoted as $R_i^{a/v} \equiv R_i$ for some participation type $a \in \{d, m\}$ and for some willingness type $v \in \{u, w\}$. Three mutually exclusive cases are possible: $i \in \mathcal{J}_K$, $i \in \widetilde{\mathcal{J}}_K$, and $i \in \mathcal{I} \setminus [\mathcal{J}_K \cup \widetilde{\mathcal{J}}_K]$:

1. If $i \in \mathcal{J}_K$: Then $M(i) \in \mathcal{E}^\ell(i)$. Since it is never transformed,

$$M(i)\ I_i\ f^{\mathbf{P}}[R_i^{a/x}, R_{-i}](i) \ \text{ for } x \in \{u, w\} \setminus \{v\}. \tag{3}$$

There are two subcases for its participation type a:

- If $a = d$, i.e., it is direct-transplant biased: If it is also a left-lobe compatible pair, then $M(i) = i$ and this is its first choice. Thus, it cannot benefit by misreporting. On the other hand, if it is not left-lobe compatible then $R_i = R_i^{d/v} = R_i^{m/v}$. Thus, $M(i)\ I_i\ f^{\mathbf{P}}[R_i^{b/x}, R_{-i}](i)$ for any participation type $b \in \{d, m\}$ and willingness type $x \in \{u, w\}$ by previous statement and Equation 3.
- if $a = m$, i.e., it is transplant maximizer: By individual rationality of $f^{\mathbf{P}}$, we have

$$M(i)\ R_i \left\{ \begin{array}{cl} i & \text{if } i \text{ is left-lobe compatible} \\ M(i) & \text{if } i \text{ is left-lobe incompatible} \end{array} \right\} I_i\ f[R_i^{d/x}, R_{-i}](i) \quad \text{for all } x \in \{u, w\}.$$

This together with Equation 3 establishes that i cannot benefit from misreporting.

2. If $i \in \widetilde{\mathcal{J}}_K$: Then $M(i) = \emptyset$ or $M(i) \in \mathcal{E}^r(i)$. Moreover, by individual rationality of M, i is not a left-lobe-compatible pair and $R_i = R_i^{a/v} \in \{R_i^{d/w}, R_i^{m/w}\}$.
   Let $i_k \equiv i$ and was transformed in Step 1.(k) for some k. It was not matchable by left-lobe donation in addition to pairs in $\mathcal{J}_{k-1}$ in G_{k-1}. Thus, reporting $R_i^{d/u}$ (or $R_i^{m/u}$, which has the same individually rational portion as $R_i^{d/u}$, because i is unwilling and left-lobe incompatible under both) instead of R_i will not change the fact that i is not matchable by left-lobe donation in addition to pairs in $\mathcal{J}_{k-1}$ in G_{k-1}, as the same active graph will occur under both revelations of preferences (as it is not left-lobe compatible, the individually rational options of i in which it donates a left lobe are the same under all preferences). Thus, $M(i)\ R_i\ \emptyset = f^{\mathbf{P}}[R_i^{d/u}, R_{-i}](i)$. Finally consider the remaining manipulation possibility by revealing $R_i^{b/x} \in \{R_i^{d/w}, R_i^{m/w}\} \setminus \{R_i\}$:
   - If $R_i = R_i^{d/w}$, then the remaining manipulation is $R_i^{b/x} = R_i^{m/w}$. If i is not right-lobe-only compatible then $R_i = R_i^{b/x}$, so we are done. On the other hand, if i is right-lobe-only compatible, then $M(i) = i$ by individual rationality. Moreover, $M(i) = i\ R_i\ j$ for all $j \in \mathcal{E}^r(i)$ and $M(i) = i\ P_i\ \emptyset$ by individual rationality again. Since $f^{\mathbf{P}}[R_i^{b/x}, R_{-i}](i) \notin \mathcal{E}^\ell(i)$, we obtain $M(i)\ R_i\ f^{\mathbf{P}}[R_i^{b/x}, R_{-i}](i)$.
   - If $R_i = R_i^{m/w}$, then the remaining manipulation is $R_i^{b/x} = R_i^{d/w}$. If i is not right-lobe-only compatible then $R_i = R_i^{b/x}$, so we are done. On the other hand, if i is right-lobe-only compatible $M(i)\ R_i\ i = f^{\mathbf{P}}[R_i^{b/x}, R_{-i}](i)$ by individual rationality of $f^{\mathbf{P}}$.
3. If $i \in \mathcal{I} \setminus [\mathcal{J}_K \cup \widetilde{\mathcal{J}}_K]$: Then $M(i) = \emptyset$ and it is unwilling and left-lobe incompatible, i.e., $R_i = R_i^{d/u} = R_i^{m/u}$ and $\emptyset\ P_i\ j$ for all $j \in \mathcal{E}^r(i)$. Suppose $i_k \equiv i$ for some k and thus, i is not left-lobe matchable in addition to $\mathcal{J}_{k-1}$ in G_{k-1} (as otherwise $i \in \mathcal{J}_k$, a contradiction). When it announces $R_i^{b/x}$, the same active graph G_{k-1} occurs at the end of Step 1.($k-1$). That is because, as it is not left-lobe compatible, its individually rational left-lobe donation options are the same under all preferences available to it. Thus, it is still not matchable in addition to $\mathcal{J}_{k-1}$ and $f^{\mathbf{P}}[R_i^{b/x}, R_{-i}](i) \notin \mathcal{E}^\ell(i)$, implying $M(i) = \emptyset\ R_i\ f^{\mathbf{P}}[R_i^{b/x}, R_{-i}](i)$ as R_i is an unwilling preference relation.

■

Appendix C Additional Results

C.1 Impossibilities

Proposition 1 *Consider an exchange pool $(\mathcal{I}, \tau)$ with $\mathcal{I} = \{i_1 \ldots, i_K\}$ in which the underlying precedence digraph $(\mathbf{T} \times \mathbf{T^D}, D^\tau)$ is a cycle $\tau(i_1) \to \tau(i_2) \to \cdots \to \tau(i_K) \to \tau(i_1)$ for $|\mathcal{I}| = K \geq 3$ such that for all k and all $n \notin \{k-1, k+1\}$ in modulo K, $i_n \notin \mathcal{E}(i_k)$. There exists no individually rational, Pareto-efficient, and incentive-compatible mechanism for this exchange pool.*

Proof of Proposition 1: Let f be an individually rational, Pareto-efficient, and incentive-compatible mechanism for this pool. We will show that this will lead to a contradiction. In the proof, all indices are meant in modulo K (i.e., $i_K \equiv i_0$).

Let

$$R^{(K+1)} \equiv (R^{m/w}_{i_1}, R^{m/w}_{i_2}, \ldots, R^{m/w}_{i_K})$$

be the preference profile in which all pairs are willing (and transplant maximizers[1]). Since f is Pareto efficient and individually rational, there exists some $\{i_k, i_{k+1}\} \in f[R^{(K+1)}]$. Without loss of generality, subject to reindexing of the pairs

- if K is odd, suppose $\{i_{K-1}, i_K\} \in f[R^{(K+1)}]$, and
- if K is even, suppose $\{i_K, i_1\} \in f[R^{(K+1)}]$.

Define for any $k \in \{1, 2, \ldots, K\}$, under profile $R^{(k)}$, pairs i_k to i_K have unwilling preferences, i.e.,

$$R^{(k)} \equiv (R^{m/w}_{\{i_1, i_2, \ldots, i_{k-1}\}}, R^{m/u}_{\{i_k, i_{k+1}, \ldots, i_K\}}).$$

We prove the following claim:

Claim: For all $k = K, K-1, \ldots, 3$,

- if k is odd, $\{i_{k-1}, i_k\} \in f[R^{(k)}]$, and
- if k is even, $\{i_{k-2}, i_{k-1}\} \in f[R^{(k)}]$

Proof of Claim: We prove the claim by induction on decreasing k. Fix $k \in \{1, \ldots, K\}$. As the inductive assumption, suppose the Claim is true for $k+1$ if $k < K$. We will prove it also holds for k (the initial step will be handled for $k = K$ below).

Consider the preference profile $R^{(k)}$ as defined above. It satisfies

$$R^{(k)} = (R^{m/u}_{i_k}, R^{(k+1)}_{-i_k}).$$

Two cases for k:

[1] It does not matter whether they are transplant maximizer or direct-transplant biased as they have the same preferences as each pair is incompatible

$\underline{k \text{ is odd}}$ If $k \neq K$, by the inductive assumption for $k+1$ (which is even), and if $k = K$, by the labeling and choice of i_K, we have $\{i_{k-1}, i_k\} \in f[R^{(k+1)}]$. Observe that $t(i_k, i_{k-1}) = \ell$ by the fact that $\tau(i_{k-1}) \to \tau(i_k)$. Moreover, $t(i_k, i_{k+1}) = r$ as $\tau(i_k) \to \tau(i_{k+1})$. Thus, by incentive compatibility of f for i_k, we still have $\{i_{k-1}, i_k\} \in f[R^{(k)}]$.

$\underline{k \text{ is even}}$ If $k \neq K$, by the inductive assumption for $k+1$ (which is odd), and if $k = K$, by the labeling and choice of i_K, we have $\{i_k, i_{k+1}\} \in f[R^{(k+1)}]$. Since $\tau(i_k) \to \tau(i_{k+1})$, we have $t(i_k, i_{k+1}) = r$. By reporting $R^{m/u}_{i_k}$ instead of $R^{m/w}_{i_k}$, the match $\{i_k, i_{k+1}\}$ becomes individually irrational, and hence, $\{i_k, i_{k+1}\} \notin f[R^{(k)}]$ by individual rationality of f.

Moreover, by incentive compatibility of f for i_k, it should not be able to get a match by donating left lobe, i.e., $\{i_{k-1}, i_k\} \notin f[R^{(k)}]$.

We claim that $\{i_{k-2}, i_{k-1}\} \in f[R^{(k)}]$. Suppose not. Since

$$E_{IR}[R^{(k)}](i_{k-1}) = \big\{\{i_{k-2}, i_{k-1}\}, \{i_{k-1}, i_k\}\big\}$$

is the set of individually rational matches for pair i_{k-1}, then i_{k-1} is unmatched in $f[R^{(k)}]$. Similarly i_k is unmatched in $f[R^{(k)}]$ since

$$E_{IR}[R^{(k)}](i_k) = \big\{\{i_{k-1}, i_k\}\big\}$$

is the set of individual matches for pair i_k. Then the following is an individually rational matching,

$$f[R^{(k)}] \cup \{i_{k-1}, i_k\},$$

and it Pareto dominates $f[R^{(k)}]$ under $R^{(k)}$ contradicting f's Pareto efficiency. Thus, $\{i_{k-2}, i_{k-1}\} \in f[R^{(k)}]$, completing the induction. $\diamond$

By the Claim, we are left with the following preference profile and chosen match (as $k = 3$, the last step index of the induction, is odd):

$$R^{(3)} = (R^{m/w}_{\{i_1, i_2\}}, R^{m/u}_{\{i_3, \ldots, i_K\}}) \text{ and}$$

$$\{i_2, i_3\} \in f[R^{(3)}].$$

As $E_{IR}[R^{(3)}] = \big\{\{i_1, i_2\}, \{i_2, i_3\}\big\}$ is the set of individually rational matches and f is individually rational, we have $f[R^{(3)}] = \big\{\{i_2, i_3\}\big\}$.

Consider the preference profile $R^{(2)} = (R^{m/u}_{i_2}, R^{(3)}_{-i_2})$. We have $E_{IR}[R^{(2)}] = \big\{\{i_1, i_2\}\big\}$. Thus, by individual rationality and Pareto efficiency of f, $f[R^{(2)}] = \big\{\{i_1, i_2\}\big\}$. Since $\tau(i_1) \to \tau(i_2)$, $t(i_2, i_1) = \ell$. On the other hand, since $\tau(i_2) \to \tau(i_3)$, $t(i_2, i_3) = r$. Thus, pair i_2 benefits from reporting its type m/u instead of m/w, contradicting the incentive compatibility of f.

■

Example A-1 *In this example we show that, if a pair's willingness to donate a right lobe is allowed to be contingent on the specific compatible liver lobe its patient receives, then a Pareto efficient, individually rational, and incentive compatible mechanism may not exist.*

Consider a liver-exchange pool with four incompatible pairs $\mathcal{I} = \{i_1, i_2, i_3, i_4\}$ *with the following types:*

$$\tau_P(i_1) = \tau_P(i_3) = (1,0,1) \qquad \tau_D(i_1) = \tau_D(i_3) = (0,1,0,1)$$
$$\tau_P(i_2) = \tau_P(i_4) = (0,1,1) \qquad \tau_D(i_2) = \tau_D(i_4) = (1,0,0,1)$$

The set of mutually compatible exchanges are given as

$$E_c = \big\{\{i_1, i_2\}, \{i_2, i_3\}, \{i_3, i_4\}, \{i_4, i_1\}\big\}.$$

Observe that, since the left lobe of each donor is too small for any patient, each donor donates his right lobe under each of these exchanges.

The public information received-graft preference relation over the set of compatible grafts is given as follows for each pair:

$$i_2 \succ_{i_1} i_4$$
$$i_3 \succ_{i_2} i_1$$
$$i_4 \succ_{i_3} i_2$$
$$i_1 \succ_{i_4} i_3$$

Suppose that each pair is willing to donate a right lobe regardless of which graft its patient receives, and thus the preference profile R *is given as follows:*

$$i_2 \; P_{i_1} \; i_4 \; P_{i_1} \; \emptyset$$
$$i_3 \; P_{i_2} \; i_1 \; P_{i_2} \; \emptyset$$
$$i_4 \; P_{i_3} \; i_2 \; P_{i_3} \; \emptyset$$
$$i_1 \; P_{i_4} \; i_3 \; P_{i_4} \; \emptyset$$

The mutual compatibility graph is depicted in Figure A-1.

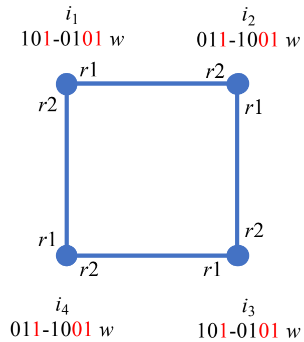

Figure A-1: The mutual compatibility graph for Example A-1. The right-lobe donations are denoted by letter r and preferences are denoted by numbers $1, 2$ next to the donated lobe for each exchange.

Suppose f is a Pareto efficient, individually rational, and incentive compatible mechanism. By Pareto efficiency of f, there exists some $\{i_k, i_{k+1}\} \in f[R]$ (all indices in modulo $K = 4$). Without loss of generality suppose $\{i_1, i_2\} \in f[R]$ (i.e., subject to relabeling of pairs).

Next consider the preference relations for pairs i_2, i_3, and i_4, where each of these pairs is willing to donate a right lobe only if their patient receives their first choice graft under the public information received-graft preference relation. In this case, the preferences R'_{i_2}, R'_{i_3}, and R'_{i_4}, are given as follows:

$$i_3 \; P'_{i_2} \; \emptyset \; P'_{i_2} \; i_1$$
$$i_4 \; P'_{i_3} \; \emptyset \; P'_{i_3} \; i_2$$
$$i_1 \; P'_{i_4} \; \emptyset \; P'_{i_4} \; i_3$$

We next show that, the mechanism f cannot satisfy all three of our axioms in the presence of preference relations R'_{i_2}, R'_{i_3}, and R'_{i_4}:

By assumption, $\{i_1, i_2\} \in f[R]$.

By incentive compatibility of f for i_2,

1. *$\{i_2, i_3\} \notin f[R_{i_1}, R'_{i_2}, R_{i_3}, R_{i_4}]$, and thus*
2. *pair i_2 remains unmatched under $f[R_{i_1}, R'_{i_2}, R_{i_3}, R_{i_4}]$ since only pair i_3 is acceptable under R'_{i_2}.*

Then $\{i_3, i_4\} \in f[R_{i_1}, R'_{i_2}, R_{i_3}, R_{i_4}]$: Otherwise both i_2 and i_3 would be unmatched in $f[R_{i_1}, R'_{i_2}, R_{i_3}, R_{i_4}]$, and $f[R_{i_1}, R'_{i_2}, R_{i_3}, R_{i_4}] \cup \big\{\{i_2, i_3\}\big\}$ would Pareto dominate $f[R_{i_1}, R'_{i_2}, R_{i_3}, R_{i_4}]$ contradicting mechanism f's Pareto efficiency.

By incentive compatibility of f for i_3, $\{i_3, i_4\} \in f[R_{i_1}, R'_{i_2}, R'_{i_3}, R_{i_4}]$.

By Pareto efficiency and individual rationality of f, $\{i_1, i_4\} \in f[R_{i_1}, R'_{i_2}, R'_{i_3}, R'_{i_4}]$.

However, the last statement contradicts incentive compatibility of f for i_4: Pair i_4 reports R'_{i_4} instead of R_{i_4} and benefits, gets matched to pair i_1, which is more preferable than i_3 under its preference R_{i_4}. $\diamond$

Example A-2 *In this example we show that, if a pair is allowed to prefer a direct transplant to some (but not all) of the strictly better-fit grafts based on its public information received-graft preferences,*[2] *then a Pareto efficient, individually rational, and incentive compatible mechanism may not exist.*

Consider a liver-exchange pool with three left-lobe compatible pairs $\mathcal{I} = \{i_1, i_2, i_3\}$ with for all i_k

$$\tau_P(i_k) = (0,1,0) \qquad\qquad \tau_D(i_k) = (0,1,0,1).$$

The set of mutually compatible exchanges are given as

$$E_c = \big\{\{i_1\}, \{i_2\}, \{i_3\}, \{i_1, i_2\}, \{i_1, i_3\}, \{i_2, i_3\}\big\}.$$

Observe that, since the left lobe of each donor is sufficiently large for any patient, each donor donates his left lobe under each of these exchanges. Hence whether the pairs are willing to donate their right lobes or not is immaterial in this example.

The public information received-graft preference relation over the set of compatible grafts is given as follows for each pair:

$$\begin{array}{ccccc} i_2 & \succ_{i_1} & i_3 & \succ_{i_1} & i_1 \\ i_3 & \succ_{i_2} & i_1 & \succ_{i_2} & i_2 \\ i_1 & \succ_{i_3} & i_2 & \succ_{i_3} & i_3 \end{array}$$

Suppose no pair is direct-transplant biased, and thus the preference profile R is given as follows:

$$\begin{array}{c} i_2 \; P_{i_1} \; i_3 \; P_{i_1} \; i_1 \; P_{i_1} \; \emptyset \\ i_3 \; P_{i_2} \; i_1 \; P_{i_2} \; i_2 \; P_{i_2} \; \emptyset \\ i_1 \; P_{i_3} \; i_2 \; P_{i_3} \; i_3 \; P_{i_3} \; \emptyset \end{array}$$

The mutual compatibility graph for this problem is depicted in Figure A-2.

[2] This can be interpreted as a "mild" direct transplant bias.

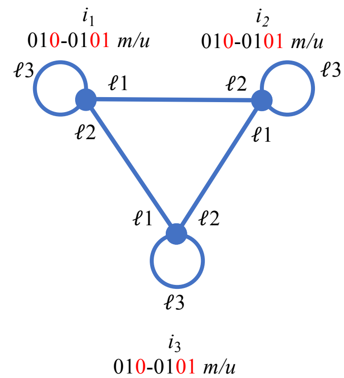

Figure A-2: The mutual compatibility graph for Example A-2. The left-lobe donations are denoted by letter ℓ and preferences are denoted by numbers $1, 2, 3$ next to the donated lobe for each match.

Suppose f is a Pareto efficient, individually rational, and incentive compatible mechanism. By Pareto efficiency of f, there exists some $\{i_k, i_{k+1}\} \in f[R]$ (all indices are in modulo $n = 3$). Without loss of generality suppose $\{i_1, i_2\} \in f[R]$ (i.e., subject to relabeling of pairs).

Consider the following preferences R'_{i_1}, R'_{i_2}, where pairs i_1 and i_2 have a mild direct-transplant bias that allows them to improve the ranking of direct transplant above some of the public information better-fit grafts but not all of them:

$$i_2 \; P'_{i_1} \; i_1 \; P'_{i_1} \; i_3 \; P'_{i_1} \; \emptyset$$
$$i_3 \; P'_{i_2} \; i_2 \; P'_{i_1} \; i_1 \; P'_{i_2} \; \emptyset$$

We next show that, the mechanism f cannot satisfy all three of our axioms in the presence of preference relations R'_{i_1} and R'_{i_2}:

By assumption, $\{i_1, i_2\} \in f[R]$.

By incentive compatibility of f for i_1, $\{i_1, i_2\} \in f[R'_{i_1}, R_{i_2}, R_{i_3}]$.

By Pareto efficiency and individual rationality of f, $\{i_2, i_3\} \in f[R'_{i_1}, R'_{i_2}, R_{i_3}]$.

However, this contradicts incentive compatibility of f for i_2: Pair i_2 reports R'_{i_2} instead of R_{i_2} and benefits, gets matched to pair i_3, which is more preferable than i_1 under its preference R_{i_2}.

Observe that a similar example can be generated for right-lobe donation decision, by changing all patients' sizes to 1 instead of 0 and making all pairs willing. $\diamond$

C.2 Computation

We give a polynomial-time method in $K = |\mathcal{I}|$ to find our mechanism outcome.

The precedence digraph and a topological order can be constructed in polynomial time (for example see Kahn, 1962). There are at most $2K$ substeps for the algorithm, K in Step 1 and K in Step 2. We can check matchability, construct reduced compatibility graphs, and find an outcome matching in the final reduced compatibility graph in polynomial time. Thus, overall the algorithm runs in polynomial time.

Checking matchability: We can use the following method in each substep for checking matchability of a set $\mathcal{J}$ in the active reduced compatibility graph $G = (\mathcal{I}, E)$:

Define pair weights $\pi^{\mathcal{I}}(j)$ for all $j \in \mathcal{I}$ such that

- $\pi^{\mathcal{I}}(j) \neq \pi^{\mathcal{I}}(i)$ for any $i \neq j$, and
- $\pi^{\mathcal{I}}(j) > \pi^{\mathcal{I}}(i)$ for all $j \in \mathcal{J}$ and $i \in \mathcal{I} \setminus \mathcal{J}$.

Define match weights

$$\pi^E(\varepsilon) \equiv \sum_{j \in \varepsilon} \pi^{\mathcal{I}}(j) \quad \text{for all } \varepsilon \in E.$$

Find an outcome matching $\hat{M}$ of the (polynomial-time) edge-weighted matching algorithm of Edmonds (1965) for edge weights π^E on G. This solves the integer-programming problem

$$\max_{M \in \mathbf{M}[G]} \sum_{\varepsilon \in M} \pi^E(\varepsilon) = \max_{M \in \mathbf{M}[G]} \sum_{i: M(i) \neq \emptyset} \pi^{\mathcal{I}}(i).$$

All pairs in $\mathcal{J}$ are matched in $\hat{M}$ if and only if $\mathcal{J}$ is matchable in G.[3]

Finding the outcome matching: In the final substep of Step 2, Substep 2.(N), by setting $\mathcal{J} \equiv \mathcal{J}_K \cup \mathcal{J}_N^*$ and $G \equiv G_K^*$, we can use the outcome of this above procedure to find the outcome of our mechanism.

Construction of the set of best achievable assignments: In each subset of the algorithm, while pair i is being processed, $\mathcal{J}$ is the set of already committed pairs, and G is the active reduced compatibility graph, first we check using the above method whether $\mathcal{J} \cup \{i\}$ is matchable in G. If so, we can construct $\mathcal{B}(i|\mathcal{J}, G)$ as follows in polynomial time:

Let $\mathcal{I}_1$ be the set of pairs that are best individually rational assignments of i in

[3]The equality follows from Okumura (2014) when there are no direct transplants. This determines a priority matching by Proposition 2 Roth, Sönmez, and Ünver (2005) because of the matroid property of matchings on a graph and this algorithm finds a priority matching with respect to priority induced by pair weights $\pi^{\mathcal{I}}$. Since the weights of the pairs in $\mathcal{J}$ are higher than any other pair in $\mathcal{I} \setminus \mathcal{J}$, it will match pairs in $\mathcal{J}$ whenever it can. Extension with direct transplants is straightforward after showing that matroid property extends with direct transplants (also see Sönmez and Ünver, 2014).

$E(i)$ with respect to R_i:

$$\mathcal{I}_1 \equiv \max_{R_i} \big\{ j \in \mathcal{I} \ : \ \{j, i\} \in E(i) \big\}.$$

For each $j \in \mathcal{I}_1$, we form the reduced compatibility graph $G^j = (\mathcal{I}, E^j)$ such that

$$E^j \equiv \big[E \setminus E(i) \big] \cup \big\{ \{i, j\} \big\},$$

in which the only match of i is with j and all other matches are as in E.

- If $\mathcal{J} \cup \{i\}$ is matchable in G^j then we include j in $\mathcal{B}(i|\mathcal{J}, G)$, we continue with the next pair in $\mathcal{I}_1$.
- Otherwise, j is not included in $\mathcal{B}(i|\mathcal{J}, G)$, we continue with the next pair in $\mathcal{I}_1$.

After we process all pairs in $\mathcal{I}_1$, if we placed at least one pair in $\mathcal{B}(i|\mathcal{J}, G)$, then $\mathcal{B}(i|\mathcal{J}, G)$ is constructed at the end of the above process. Otherwise, we consider the next indifference class of i among matches in $E(i)$, $\mathcal{I}_2$, with respect to R_i, similarly, and continue so on until $\mathcal{B}(i|\mathcal{J}, G)$ is constructed. Then, we obtain a new active reduced compatibility graph using $\mathcal{B}(i|\mathcal{J}, G)$.

Appendix D An Illustration of the Algorithm

Example A-3 *Consider a liver exchange pool with 12 pairs with the following types:*

$$\begin{array}{ll} \textit{type } (0,0,0)-(1,0,0,1): 2 \textit{ pairs} & \textit{type } (0,1,1)-(1,1,0,1): 3 \textit{ pairs} \\ \textit{type } (0,1,0)-(1,0,0,1): 2 \textit{ pairs} & \textit{type } (1,0,0)-(0,0,0,1): 1 \textit{ pair} \\ \textit{type } (1,0,0)-(0,1,0,1): 1 \textit{ pair} & \textit{type } (1,0,1)-(0,1,0,1): 1 \textit{ pair} \\ \textit{type } (1,1,0)-(0,1,1,2): 1 \textit{ pair} & \textit{type } (1,1,1)-(0,1,0,1): 1 \textit{ pair} \end{array}$$

The precedence digraph over pair types of the problem is given in Figure A-3.

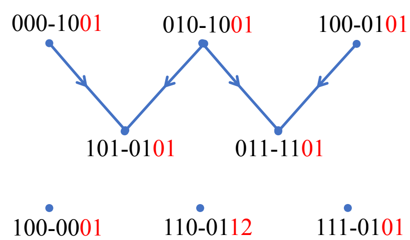

Figure A-3: Precedence digraph for Example A-3

Based on this digraph, we need to order pairs of types $(0,0,0)-(1,0,0,1)$ *and* $(0,1,0)-(1,0,0,1)$ *before the pair of type* $(1,0,1)-(1,0,0,1)$*, and pairs of types* $(0,1,0)-(1,0,0,1)$ *and* $(1,0,0)-(0,1,0,1)$ *before pairs of type* $(0,1,1)-(1,1,0,1)$ *in any topological order; otherwise, we are free to order pairs in any way we want. Let the left-lobe matching priority order* $\Pi_\ell = i_1 - i_2 - \ldots - i_{12}$ *be a topological order of this digraph such that pairs are reindexed as in Figure A-4.*

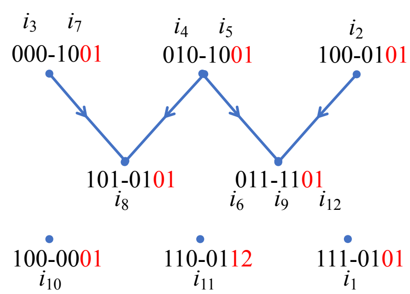

Figure A-4: Left-lobe matching topological order $\Pi_\ell = i_1 - i_2 - \ldots - i_{12}$ for Example A-3

Suppose pairs report the preferences such that all pairs except i_2*,* i_4*,* i_8*, and* i_9 *are willing (w) and the compatible pairs (of types* $(0,0,0)-(1,0,0,1)$ *and* $(0,1,1)-(1,1,0,1)$*) are transplant maximizers (m).*

Some patients have strict preferences over received transplants so that the individually rational portion of their pair preferences becomes:

$$\textit{type } (0,0,0)-(1,0,0,1) \begin{cases} R^{m/w}_{i_3}: & i_{10}\ P_{i_3}\ i_7\ P_{i_3}\ i_3 \\ R^{m/w}_{i_7}: & i_{10}\ P_{i_7}\ i_3\ P_{i_7}\ i_7 \end{cases}$$

$$\textit{type } (0,1,1)-(1,1,0,1) \begin{cases} R^{m/w}_{i_6}: & i_5\ P_{i_6}\ i_{11}\ P_{i_6}\ i_1\ P_{i_6}\ i_6 \\ R^{m/u}_{i_9}: & i_5\ P_{i_9}\ i_{11} \\ R^{m/w}_{i_{12}}: & i_5\ P_{i_{12}}\ i_{11}\ P_{i_{12}}\ i_1\ P_{i_{12}}\ i_{12} \end{cases}$$

Other patients are indifferent over received grafts. Let R be the pair preference profile. The individually rational compatibility graph $G_{IR}[R]$ is given in Figure A-5. Only four pairs, i_2, i_4, i_8, and i_9 are unwilling to donate their right lobes. Only those four are marked with u in the figure, while willing pairs are not marked.

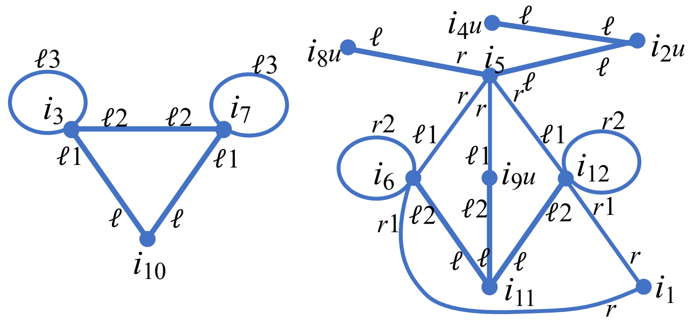

Figure A-5: Individually rational compatibility graph $G_{IR}[R]$ in Example A-3. If a pair has strict preferences, then the matches in which it donates left lobe are indexed as $\ell 1, \ell 2, \ldots$ and the matches in which it donates right lobe are indexed as $r1, r2, \ldots$ in the order of its preferences.

Suppose that the right-lobe matching priority order is $\Pi_r = i_{12} - i_{11} - \ldots - i_1$, which reverses Π_ℓ.

The execution of the precedence-adjusted priority algorithm for Π_ℓ and Π_r is as follows:

***Step 1**: The active reduced compatibility graph G_0 includes all left-lobe-only individually rational matches and is given in Figure A-6. Initially, the set of left-lobe-committed pairs is $\mathcal{J}_0 \equiv \emptyset$ and the set of transformed pairs is $\widetilde{\mathcal{J}}_0 \equiv \emptyset$.*

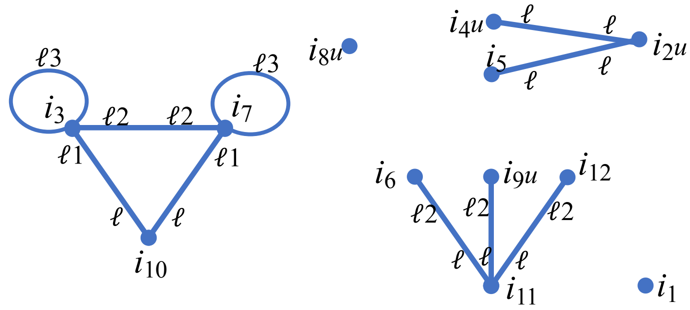

Figure A-6: G_0 in Example A-3

Step 1.(1): $\mathcal{J}_0 \cup \{i_1\}$ *is not matchable in* G_0 *as* i_1 *has no matches. As* i_1 *is willing, we transform it and leave the set of left-lobe-committed pairs unchanged as* $\mathcal{J}_1 \equiv \mathcal{J}_0 = \emptyset$. *The set of transformed pairs becomes* $\widetilde{\mathcal{J}}_1 \equiv \widetilde{\mathcal{J}}_0 \cup \{i_1\} = \{i_1\}$. *After transformation of* i_1, *no new matches become available (yet), as all possible such matches involve only right-lobe transplants and no other pair is transformed yet. Thus,* $G_1 \equiv G_0$.

Step 1.(2): $\mathcal{J}_1 \cup \{i_2\}$ *is matchable in* G_1: $M = \Big\{\{i_2, i_5\}\Big\}$ *is such a matching. Thus,* $\mathcal{J}_2 \equiv \mathcal{J}_1 \cup \{i_2\} = \{i_2\}$ *and* $\widetilde{\mathcal{J}}_2 \equiv \widetilde{\mathcal{J}}_1 = \{i_1\}$. *Moreover,* i_2 *is indifferent between its achievable assignments* i_4 *and* i_5. *Thus, we keep all associated matches in the graph:* $G_2 \equiv G_1 = G_0$.

Step 1.(3): $\mathcal{J}_2 \cup \{i_3\}$ *is matchable in* G_2: $M = \Big\{\{i_2, i_5\}, \{i_3, i_{10}\}\Big\}$ *is such a matching. We commit to match* i_3 *as a left-lobe donating pair and set* $\mathcal{J}_3 \equiv \mathcal{J}_2 \cup \{i_3\} = \{i_2, i_3\}$. *Transformed set does not change:* $\widetilde{\mathcal{J}}_3 \equiv \widetilde{\mathcal{J}}_2 = \{i_1\}$. *Pair* i_3 *strictly prefers* i_{10} *to* i_7 *and to itself, which are its achievable assignments in* G_2. *Thus, we only keep match* $\{i_3, i_{10}\}$ *and delete* $\{i_3\}$ *and* $\{i_3, i_7\}$ *from* G_2. *The active reduced compatibility graph* G_3 *is given in Figure A-7.*

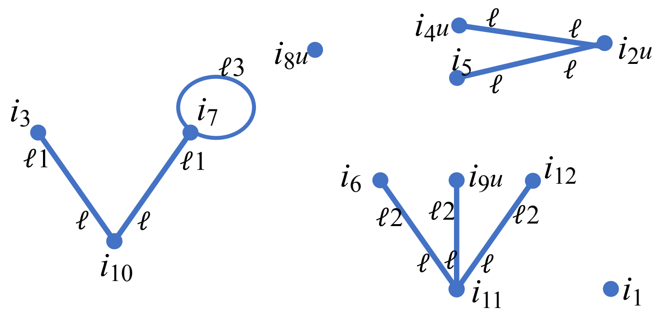

Figure A-7: G_3 in Example A-3

Step 1.(4): $\mathcal{J}_3 \cup \{i_4\}$ *is matchable in* G_3: $M = \Big\{\{i_2, i_4\}, \{i_3, i_{10}\}\Big\}$ *is such a matching. Thus, we set* $\mathcal{J}_4 \equiv \mathcal{J}_3 \cup \{i_4\} = \{i_2, i_3, i_4\}$ *and* $\widetilde{\mathcal{J}}_4 \equiv \widetilde{\mathcal{J}}_3 = \{i_1\}$. *Pair* i_4 *has only one possible assignment in* G_3, i_2. *Thus,* $G_4 \equiv G_3$.

Step 1.(5): $\mathcal{J}_4 \cup \{i_5\}$ *is not matchable in* G_4: *Pair* i_5*'s only possible match is with* i_2, *but i_2 has to be matched with i_4 in G_4 as $i_4 \in \mathcal{J}_4$ and i_4 has only one achievable match, i.e., $\{i_2, i_4\}$, in G_4. Since i_5 is willing, we transform it and the set of transformed pairs becomes $\widetilde{\mathcal{J}}_5 \equiv \widetilde{\mathcal{J}}_4 \cup \{i_5\} = \{i_1, i_5\}$ while the set of left-lobe-committed pairs does not change: $\mathcal{J}_5 \equiv \mathcal{J}_4 = \{i_2, i_3, i_4\}$. Transforming i_5 leads to 4 new matches $\{i_5, i_6\}$, $\{i_5, i_8\}$, $\{i_5, i_9\}$, and $\{i_5, i_{12}\}$ in all of which only i_5 donates right lobe while the other pairs donate left lobe. By adding these matches to G_4, the active graph becomes G_5 that is given in Figure A-8.*

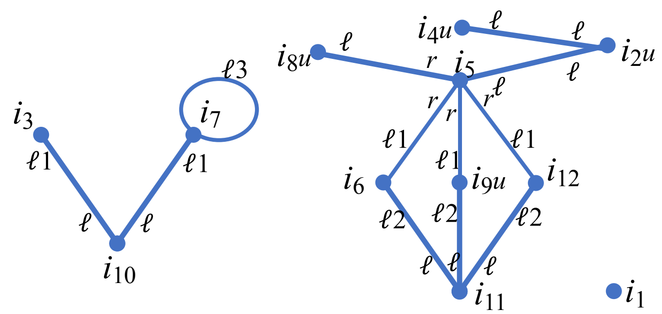

Figure A-8: G_5 in Example A-3

<u>Step 1.(6):</u> $\mathcal{J}_5 \cup \{i_6\}$ is matchable in G_5: $M = \Big\{\{i_2, i_4\}, \{i_3, i_{10}\}, \{i_6, i_5\}\Big\}$ is such a matching. Thus, $\mathcal{J}_6 \equiv \mathcal{J}_5 \cup \{i_6\} = \{i_2, i_3, i_4, i_6\}$ and $\widetilde{\mathcal{J}}_6 \equiv \widetilde{\mathcal{J}}_5 = \{i_1, i_5\}$. Moreover, i_6 prefers i_5 to i_{11}, which are its only achievable assignments. Therefore, we remove $\{i_6, i_{11}\}$ from G_5 to obtain G_6 (see Figure A-9).

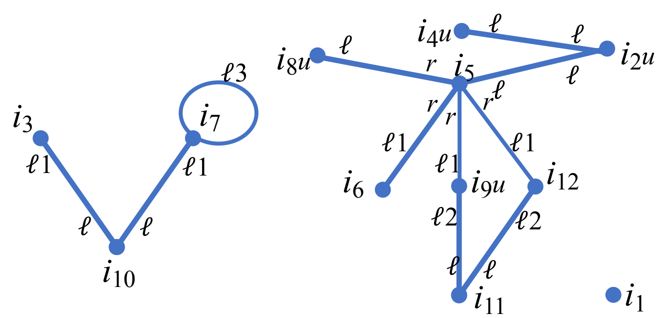

Figure A-9: G_6 in Example A-3

<u>Step 1.(7):</u> $\mathcal{J}_6 \cup \{i_7\}$ is matchable in G_6: $M = \Big\{\{i_2, i_4\}, \{i_3, i_{10}\}, \{i_6, i_5\}, \{i_7\}\Big\}$ is such a matching. Thus, we add i_7 to the left-lobe-committed set of pairs: $\mathcal{J}_7 \equiv \mathcal{J}_6 \cup \{i_7\} = \{i_2, i_3, i_4, i_6, i_7\}$ and set of transformed pairs remains the same: $\widetilde{\mathcal{J}}_7 \equiv \widetilde{\mathcal{J}}_6 = \{i_1, i_5\}$. Pair i_7 has one achievable match, which is with itself. Its other feasible match is with i_{10}, which it prefers to itself. However, i_{10} is not achievable, as $i_3 \in \mathcal{J}_6$ has to be matched with i_{10} in all possible matchings that also match i_3. Thus, we delete match $\{i_7, i_{10}\}$ from G_6 to obtain G_7 in Figure A-10.

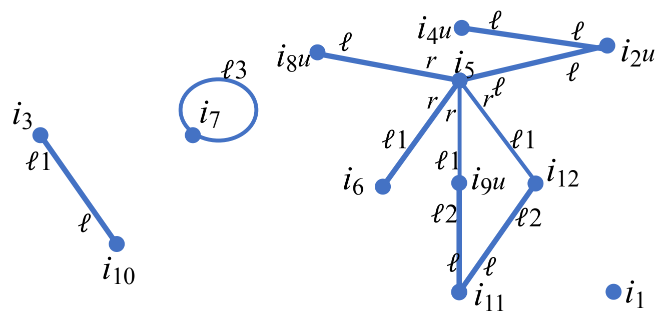

Figure A-10: G_7 in Example A-3

Step 1.(8): $\mathcal{J}_7 \cup \{i_8\}$ *is not matchable in* G_7*: Pair* i_8*'s only feasible assignment* i_5 *has to be matched with* $i_6 \in \mathcal{J}_7$*, to keep* i_6 *matched. Since pair* i_8 *is unwilling, we will never be able to match it; thus, we skip it. The active graph and committed and transformed pair sets remain the same:* $\mathcal{J}_8 \equiv \mathcal{J}_7 = \{i_2, i_3, i_4, i_6, i_7\}$*,* $\widetilde{\mathcal{J}}_8 \equiv \widetilde{\mathcal{J}}_7 = \{i_1, i_5\}$*, and* $G_8 \equiv G_7$*.*

Step 1.(9): $\mathcal{J}_8 \cup \{i_9\}$ *is matchable in* G_8*:* $M = \Big\{\{i_2, i_4\}, \{i_3, i_{10}\}, \{i_6, i_5\}, \{i_7\}, \{i_9, i_{11}\}\Big\}$ *is such a matching. Thus,* $\mathcal{J}_9 \equiv \mathcal{J}_8 \cup \{i_9\} = \{i_2, i_3, i_4, i_6, i_7, i_9\}$ *and* $\widetilde{\mathcal{J}}_9 \equiv \widetilde{\mathcal{J}}_8 = \{i_1, i_5\}$*. Pair* i_9 *has one achievable assignment* i_{11}*; its other feasible assignment in* G_8 *is* i_5*. However,* i_5 *is not achievable, (although* i_9 *prefers* i_5 *to* i_{11}*) as pair* i_5 *has to be matched with* $i_6 \in \mathcal{J}_8$*. Thus, we delete* $\{i_5, i_9\}$ *from* G_8 *to obtain* G_9 *(see Figure A-11).*

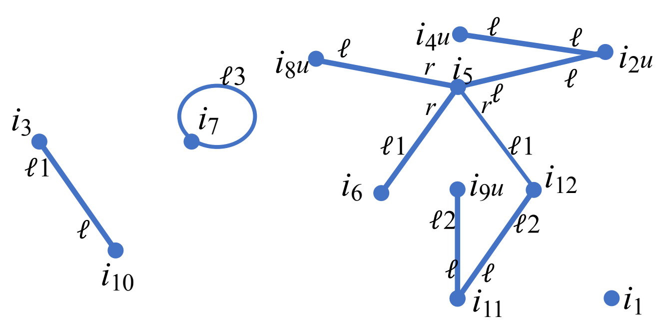

Figure A-11: G_9 in Example A-3

Step 1.(10): $\mathcal{J}_9 \cup \{i_{10}\}$ *is matchable in* G_9*:* $M = \Big\{\{i_2, i_4\}, \{i_3, i_{10}\}, \{i_6, i_5\}, \{i_7\}, \{i_9, i_{11}\}\Big\}$ *is such a matching. Thus, we set* $\mathcal{J}_{10} \equiv \mathcal{J}_9 \cup \{i_{10}\} = \{i_2, i_3, i_4, i_6, i_7, i_9, i_{10}\}$ *and* $\widetilde{\mathcal{J}}_{10} \equiv \widetilde{\mathcal{J}}_9 = \{i_1, i_5\}$*. Pair* i_{10} *has one feasible assignment* i_3 *so the active graph does not change:* $G_{10} \equiv G_9$*.*

Step 1.(11): $\mathcal{J}_{10} \cup \{i_{11}\}$ *is matchable in* G_{10}*:* $M = \Big\{\{i_3, i_{10}\}, \{i_2, i_4\}, \{i_6, i_5\}, \{i_7\}, \{i_9, i_{11}\}\Big\}$ *is such a matching. Thus,* $\mathcal{J}_{11} \equiv \mathcal{J}_{10} \cup \{i_{11}\} = \{i_2, i_3, i_4, i_6, i_7, i_9, i_{10}, i_{11}\}$ *and* $\widetilde{\mathcal{J}}_{11} \equiv \widetilde{\mathcal{J}}_{10} = \{i_1, i_5\}$*. Pair* i_{11} *has one achievable assignment* i_9 *while its other feasible assignment* i_{12} *is not achievable: pair* $i_9 \in \mathcal{J}_{10}$ *has to be matched with* i_{11}*. So graph* G_{11} *is obtained by deleting*

$\{i_{11}, i_{12}\}$ *from* G_{10} *(see Figure A-12).*

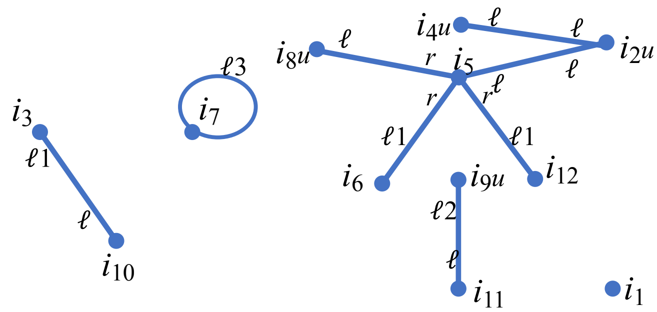

Figure A-12: G_{11} in Example A-3

<u>*Step 1.(12):*</u> $\mathcal{J}_{11} \cup \{i_{12}\}$ *is not matchable in* G_{11}*: Pair* i_{12}*'s only feasible assignment* i_5 *has to be matched with* $i_6 \in \mathcal{J}_{11}$*. Since* i_{12} *is willing, we transform it and add its two matches,* $\{i_{12}\}$ *and* $\{i_1, i_{12}\}$*, involving only right-lobe transplants to* G_{11} *to obtain active graph* G_{12} *(see Figure A-13). Observe that we had transformed* i_1 *earlier in Step 1.(1). While* $\mathcal{J}_{12} \equiv \mathcal{J}_{11} = \{i_2, i_3, i_4, i_6, i_7, i_9, i_{10}, i_{11}\}$*, we update the transformed pair set as* $\widetilde{\mathcal{J}}_{11} \equiv \widetilde{\mathcal{J}}_{10} \cup \{i_{12}\} = \{i_1, i_5, i_{12}\}$*. Step 1 ends with this substep.*

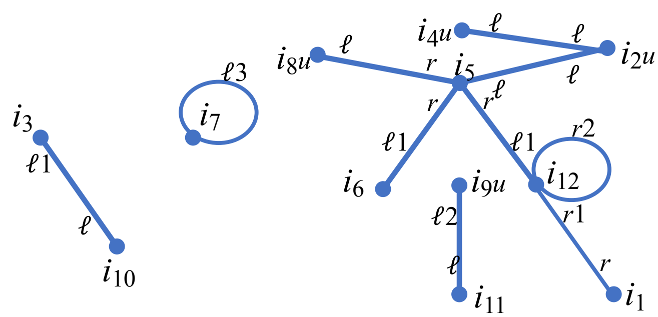

Figure A-13: G_{12} in Example A-3

Step 2*: The active reduced compatibility graph is* $G_0^* \equiv G_{12}$*. Right-lobe matching priority order* Π_r *orders transformed pairs in* $\widetilde{\mathcal{J}}_{12} = \{i_1, i_5, i_{12}\}$ *in reverse order of* Π_ℓ *as* $i_{12} - i_5 - i_1$*. The set of right-lobe-committed pairs is initialized as* $\mathcal{J}_0^* = \emptyset$*.*

<u>*Step 2.(1):*</u> $\mathcal{J}_{12} \cup \mathcal{J}_0^* \cup \{i_{12}\}$ *is matchable in* G_0^**:*
$M = \Big\{\{i_2, i_4\}, \{i_3, i_{10}\}, \{i_6, i_5\}, \{i_7\}, \{i_9, i_{11}\}, \{i_{12}, i_1\}\Big\}$ *is such a matching. We update the right-lobe-committed set of pairs as* $\mathcal{J}_1^* \equiv \mathcal{J}_0^* \cup \{i_{12}\} = \{i_{12}\}$*.* G_1^* *is obtained by removing matches* $\{i_{12}\}$ *(which is achievable, but worse than being matched with* i_1 *for* i_{12}*) and* $\{i_5, i_{12}\}$ *(which is better than being matched with* i_1 *but is not achievable for* i_{12} *as* $i_6 \in \mathcal{J}_{12}$ *has to be matched with* i_5*) (see Figure A-14).*

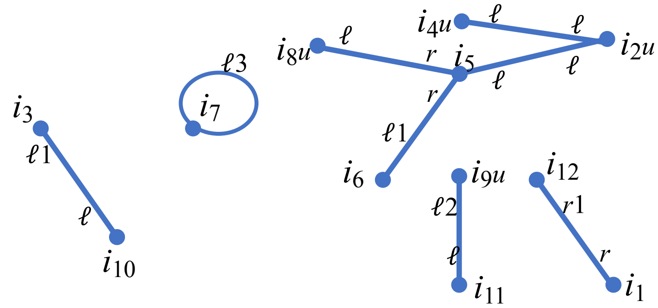

Figure A-14: G_1^* in Example A-3

Step 2.(2): $\mathcal{J}_{12} \cup \mathcal{J}_1^* \cup \{i_5\}$ *is matchable in* G_1^**:*
$M = \Big\{\{i_2, i_4\}, \{i_3, i_{10}\}, \{i_6, i_5\}, \{i_7\}, \{i_9, i_{11}\}, \{i_{12}, i_1\}\Big\}$ *is the unique such matching. We set* $\mathcal{J}_2^* \equiv \mathcal{J}_1^* \cup \{i_5\} = \{i_{12}, i_5\}$*.* G_2^* *is obtained by removing* $\{i_5, i_2\}$ *and* $\{i_5, i_8\}$ *from* G_1^* *(see Figure A-15). These are unachievable matches for* i_5 *as* i_5 *has to be matched with* $i_6 \in \mathcal{J}_{12}$*, whose only feasible assignment is* i_5*.*

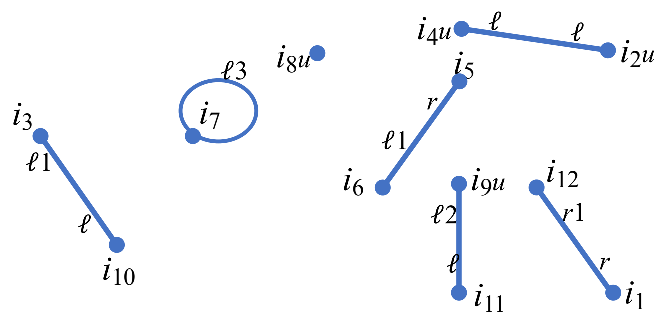

Figure A-15: G_2^* in Example A-3

Step 2.(3): $\mathcal{J}_{12} \cup \mathcal{J}_2^* \cup \{i_1\}$ *is matchable in* G_2^**:*
$M = \Big\{\{i_2, i_4\}, \{i_3, i_{10}\}, \{i_6, i_5\}, \{i_7\}, \{i_9, i_{11}\}, \{i_{12}, i_1\}\Big\}$ *is the unique such matching. We set* $\mathcal{J}_3^* \equiv \mathcal{J}_3^* \cup \{i_1\} = \{i_{12}, i_5, i_1\}$ *and* $G_3^* \equiv G_2^*$*, as* i_1 *does not have any other matches than* $\{i_{12}, i_1\}$ *in* G_2^**.*

Step 2 terminates with the active reduced compatibility graph $G_3^* = G_2^*$*, the set of left-lobe-committed pairs*

$$\mathcal{J}_{12} = \{i_2, i_3, i_4, i_6, i_7, i_9, i_{10}, i_{11}\},$$

and the set of right-lobe-committed pairs

$$\mathcal{J}_3^* = \{i_{12}, i_5, i_1\}.$$

The unique matching in G_3^* *that matches all pairs in* $\mathcal{J}_{12} \cup \mathcal{J}_3^*$ *is the outcome of the algorithm* *and only leaves pair* i_8 *unmatched (note that* $\mathcal{I} \setminus (\mathcal{J}_{12} \cup \mathcal{J}_3^*) = \{i_8\}$*):*

$$M = \Big\{\{i_2, i_4\}, \{i_3, i_{10}\}, \{i_6, i_5\}, \{i_7\}, \{i_9, i_{11}\}, \{i_{12}, i_1\}\Big\}.$$

Appendix E Precedence Digraph Examples

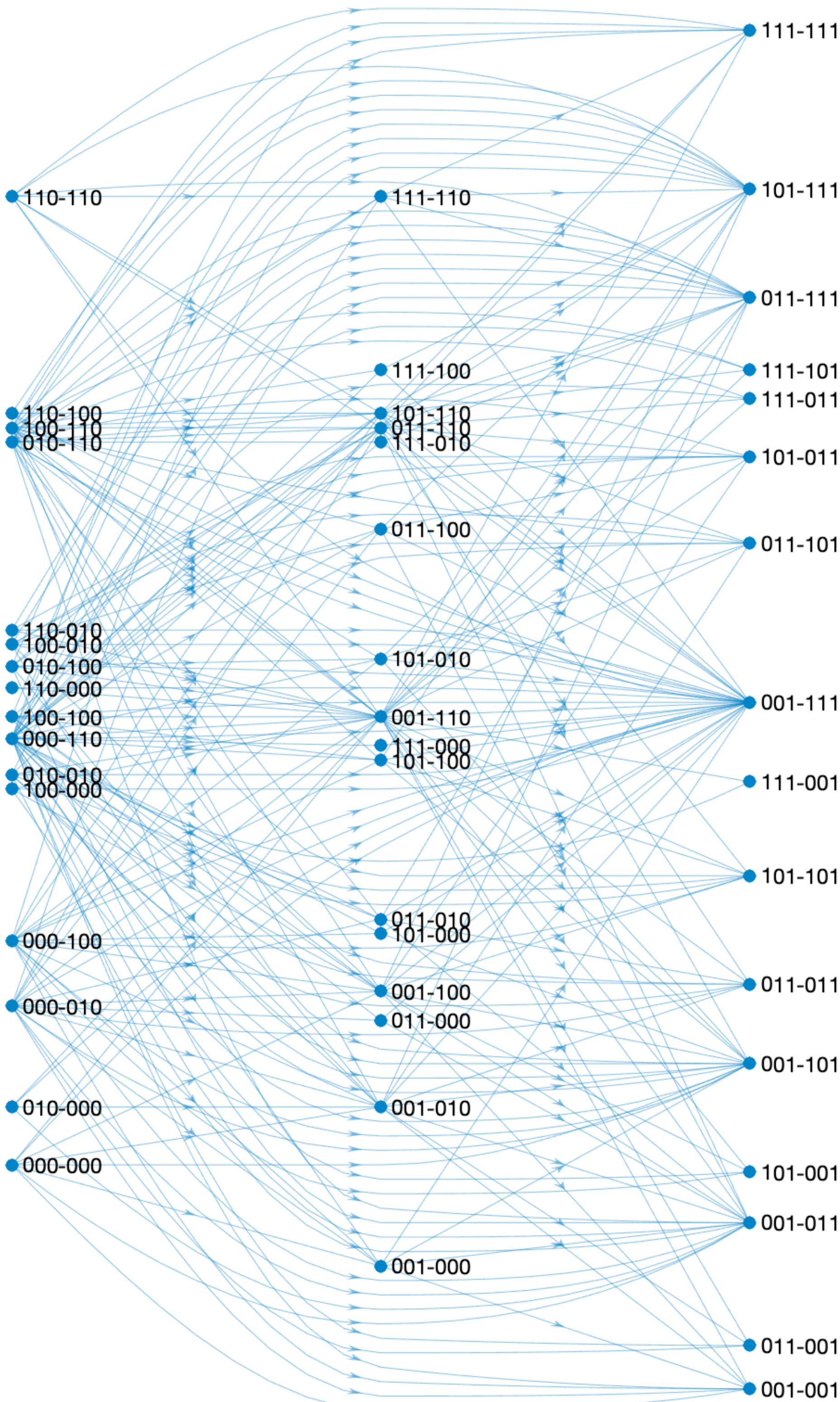

Figure A-16: The precedence digraph with two sizes ($S = 2$). We only denote left-lobe size of the donor types in this depiction, as their right-lobe size is uniquely determined by their left-lobe size. 16 pair types have no adjacent edges in the digraph, so those are not shown.

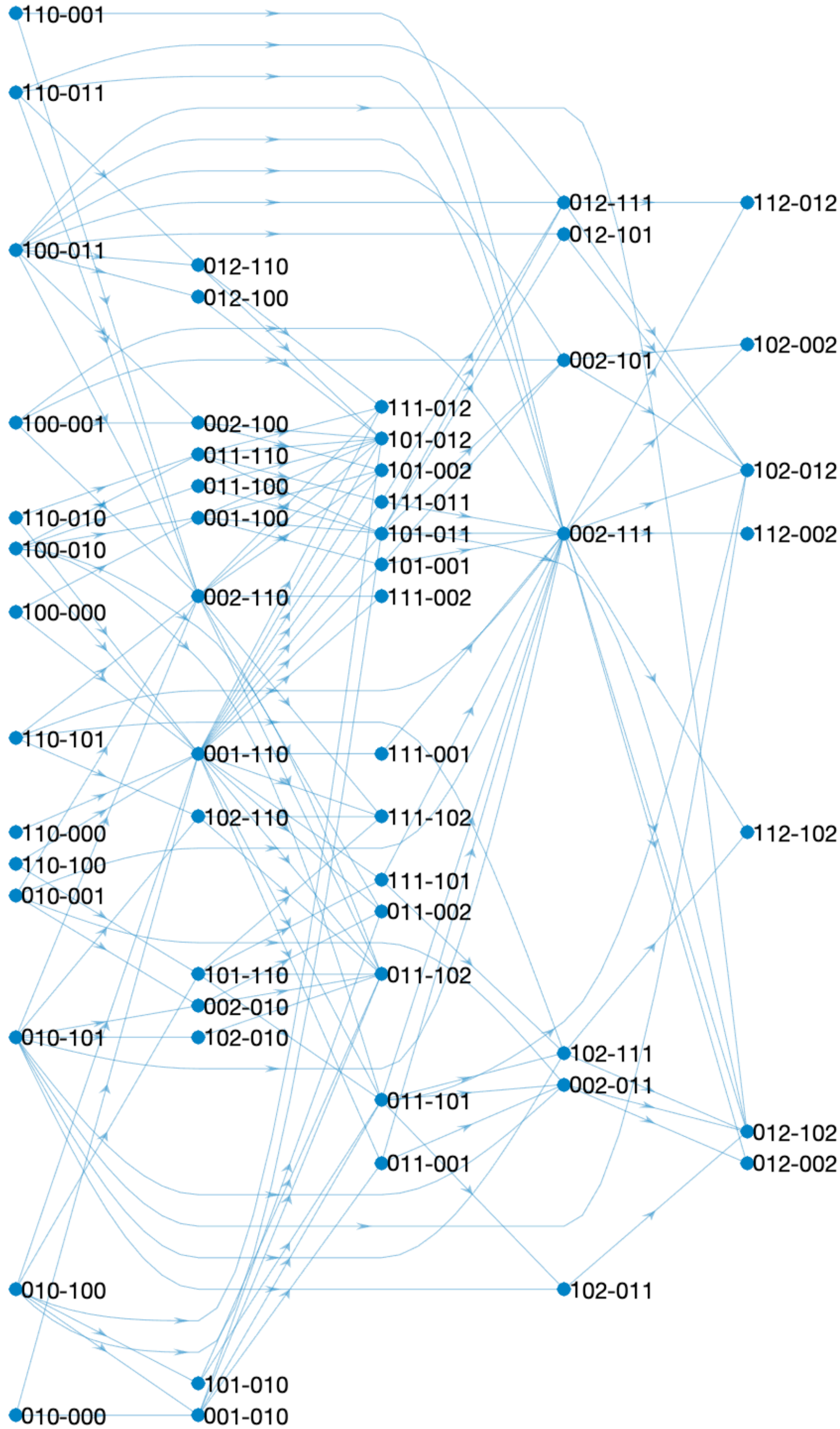

Figure A-17: The precedence digraph with three sizes ($S = 3$) when left-lobe compatible pairs do not participate in exchange. We only denote left-lobe size of the donor types in this depiction, as their right-lobe size is uniquely determined by their left-lobe size. 34 pair types have no adjacent edges in the digraph, so those are not shown.

References

Edmonds, J. (1965): "Paths, Trees, and Flowers," *Canadian Journal of Mathematics*, 17, 449–467.

Kahn, A. B. (1962): "Topological sorting of large networks," *Communications of the ACM*, 5, 558–562.

Okumura, Y. (2014): "Priority matchings revisited," *Games and Economic Behavior*, 88, 242–249.

Roth, A. E., T. Sönmez, and M. U. Ünver (2005): "Pairwise Kidney Exchange," *Journal of Economic Theory*, 125, 151–188.

Sönmez, T. and M. U. Ünver (2014): "Altruistically unbalanced kidney exchange," *Journal of Economic Theory*, 152, 105–129.