Efficient and Incentive-Compatible Liver Exchange*

Haluk Ergin† Tayfun Sönmez‡ M. Utku Ünver§

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Abstract

Liver exchange has been practiced in small numbers, mainly to overcome blood-type incompatibility between patients and their living donors. A donor can donate either his smaller left lobe or the larger right lobe, although the former option is safer. Despite its elevated risk, right-lobe transplantation is often utilized due to size-compatibility requirement with the patient. We model liver exchange as a market-design problem, focusing on logistically simpler two-way exchanges, and introduce an individually rational, Pareto-efficient, and incentive-compatible mechanism. Construction of this mechanism requires novel technical tools regarding bilateral exchanges under partial-order-induced preferences. Through simulations we show that not only can liver exchange increase the number of transplants by more than 30%, it can also increase the share of the safer left-lobe transplants.

Keywords: Market design, liver exchange, matching, incentive compatibility, efficiency

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1 Introduction

Following the kidney, the liver is the second most common organ for transplantation worldwide. In 2018 there were 12,720 new additions in the US to waitlists for liver transplants.

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†Department of Economics, University of California at Berkeley; hie@berkeley.edu

‡Department of Economics, Boston College; sonmezt@bc.edu

§Department of Economics, Boston College; unver@bc.edu

While 8,250 patients were removed from waitlists due to receiving liver transplants, 1,159 of them were removed due to death, and 1,315 were removed due to being too sick for a transplant. Transplantation is the only potential treatment for end-stage liver disease, unlike end-stage kidney disease where there is the alternative (although inferior) treatment of dialysis. As in the case of kidneys, transplants from deceased donors and living donors are both possible (and widespread) for liver transplantation.[1] Unlike kidney transplantation, however, a living donor can donate only a part of his liver —henceforth referred to as a *lobe*— going through a liver resection operation called *hepatectomy*. Based on the anatomy of the liver, the main options are donating either the smaller left lobe (normally 30–40% of the liver) with a *left hepatectomy* or the larger right lobe (normally 60–70% of the liver) with a *right hepatectomy*. Following the transplantation, the remnant liver of a living donor typically regenerates within a month. Assuming the donor and the patient are blood-type compatible,[2] which of these two options is preferred (or even feasible) depends on the relative liver volumes of the patient and the donor. In order to provide adequate liver function for the patient, at least 40% of the standard liver volume of the patient is required. The metabolic demands of a larger patient will not be met by the smaller left lobe from a relatively small donor. This phenomenon is known as *small-for-size syndrome*. The primary solution to avoid this syndrome has been harvesting the larger right lobe of the liver for transplantation. This procedure, however, involves considerably higher risks for the donor than harvesting the smaller left lobe. While donor mortality is approximately 0.1% for left hepatectomy, it is in the range of 0.4–0.5% for right hepatectomy (Lee, 2010). Furthermore, other significant risks, referred to as donor *morbidity*, are also much higher under right hepatectomy than left hepatectomy.[3] Mishra et al. (2018) reports that the morbidity rates are 28% for right hepatectomy and 7.5% for left hepatectomy. Hence one of the main challenges for living-donor liver transplantation is that, the much safer left-lobe transplantation is not a viable option for a majority of patients with willing donors. As an implication, many patients with potential donors cannot receive a transplant since either their donors hesitate to go through the higher-risk right hepatectomy, or their doctors recommend against this procedure.

The high risks associated with the right-lobe liver transplantation also affect the public perception of living-donor liver transplantation. The number of annual living-donor liver

[1] The attitude towards living-donor liver transplantation differs considerably between western countries and Asian countries. In contrast to western countries, donations for liver transplantation in much of Asia come from living donors. For example, in 2018, while only 401 of 8,250 liver transplants were from living donors in the US, 1,106 of 1,475 liver transplants in South Korea and 1,150 of 1,588 liver transplants in Turkey were from living donors.

[2] Each individual is of one of the following four blood types: O, A, B, or AB. While a blood-type O donor is blood-type compatible with any blood-type patient, a blood-type A donor is blood-type compatible with patients of blood types A and AB, a blood-type B donor is blood-type compatible with patients of blood types B and AB, and a blood-type AB donor is blood-type compatible with only patients of blood type AB.

[3] Donor morbidity is the medical term for donor complications following the transplantation surgery, and it includes bile leaks, surgery related infectious and gastrointestinal diseases, liver necrosis, wound complications, cardiovascular complications, and being hemorrhagic among others.

transplants in the US peaked in 2001 with 524 transplants, increasing eight-fold in the period from 1996 to 2001. The highly publicized death of a right-lobe liver donor in the US in 2002 brought an end to this remarkable increase, and resulted in a 40–50% reduction from its peak over the next decade.[4] The number of annual living-donor liver transplants in the US have been mostly increasing again since 2012, with 401 transplants in 2018.

As the worldwide shortage of transplant organs keeps increasing annually, living-donor exchanges emerged as an important source for these potentially life-saving resources, especially in the case of kidneys. In its most basic form, a living-donor organ exchange involves two patients with willing donors who exchange donors either because direct donation is not an option due to an immunological barrier, or because one or both patients receive a more favorable outcome through the exchange. The concept was originally proposed for kidneys by Rapaport (1986), and it became widespread over the last 15 years with the introduction of optimization and market-design techniques to kidney exchange (Roth, Sönmez, and Ünver, 2004, 2005, 2007). A vast majority of these exchanges are conducted between incompatible kidney patient-donor pairs, where a donor cannot directly donate to his patient due to immunological barriers.[5] Liver exchanges between incompatible patient-donor pairs are also conducted in modest numbers in several Asian countries, most notably in South Korea. Our focus in this paper is the design of a liver-exchange mechanism that not only includes incompatible pairs, but also a subset of compatible pairs, such as those whose only direct-donation possibility to their patients is through a much higher-donor-risk right hepatectomy. Under an efficient and incentive-compatible mechanism we introduce, compatible pairs participate in exchange only if they strictly benefit by doing so, most notably by reducing the risks to their donors through a left hepatectomy. As such, our proposed mechanism not only increases the number of living-donor liver transplants, but also increases the reliance on the lower-risk left-lobe liver transplantation in the spirit of the central tenet of the hippocratic oath "first do no harm."

While the practice of kidney exchange has flourished worldwide over the last fifteen years, inclusion of compatible pairs in exchange pools has proved to be a challenge since benefits to these pairs from joining kidney-exchange pools are either not present or weak. In contrast, the benefits from joining liver-exchange pools can be considerable for a significant fraction of compatible pairs, if it means their donors can have a left hepatectomy rather than a right hepatectomy. And the welfare gains from their inclusion can be potentially very high. Consider a large, blood-type A liver patient, who in the absence of exchange has to receive a right liver lobe from his small, blood-type O donor. While this is a feasible medical procedure, an alternative arrangement of an exchange of donors with a small, blood-type O patient with a large, blood-type A donor will not only significantly reduce the risks to

[4] See Grady (2002).

[5] For the case of kidney transplantation, these immunological barriers are blood-type incompatibility and tissue-type incompatibility.

his donor (by replacing the donor's right hepatectomy with a left hepatectomy), but also enable a second patient to receive a potentially life-saving liver transplant. The possibility of offering a less risky procedure to such pairs provides an opportunity to increase the size of the liver-exchange pool in a way that includes the much-needed blood-type O donors.

In the above example, the large, blood-type A patient with a small, blood-type O donor would likely be motivated to participate in exchange, if the pair benefits from exchange by reducing the donor risk through a much safer procedure of left hepatectomy. However, not all cases are this straightforward. Consider a blood-type A patient with a blood-type B donor. Since this pair is blood-type incompatible to start with, not only can it benefit from exchange through a left-lobe donation, but also through the less-desired right-lobe donation if the pair is willing to expose the donor to the higher mortality and morbidity risks of a right hepatectomy. This possibility is the primary reason why one cannot adopt the mechanisms and techniques developed for kidney exchange directly to liver exchange, unless the higher-donor-risk right hepatectomy is completely ruled out. A liver-exchange mechanism has to determine not only which pairs are to be matched with each other to exchange donors, but it shall also determine which donors have to donate their right lobes rather than their left lobes. Of course, some pairs may not be willing to expose their donors to the more risky procedure of right hepatectomy, but a poorly designed exchange mechanism may also give them incentives to hide their willingness to do so even if they are. As such, our focus is not only the design of an efficient mechanism, but at the same time the design of an *incentive-compatible* liver exchange mechanism where a pair never receives a less favorable outcome by either revealing its willingness to go through the less desired right hepatectomy or by revealing whether it has a direct-transplant bias or not.

The key pairs in the design of an efficient and incentive-compatible mechanism are those who can participate in exchange both through a left-lobe donation as well as through a less-preferred right-lobe donation.[6] The challenge is determining when the donors of a particular pair shall be considered for a right-lobe donation rather than a left-lobe donation. We refer to this process as a *transformation*. To assure incentive compatibility, a pair should be transformed only after their left-lobe-exchange possibilities are exhausted, so that their announcement of whether they are willing for their donors to go through a right hepatectomy does not affect whether or not their donors go through the safer left hepatectomy. One simple approach might be first considering all such pairs for left-lobe donation, and then transforming them simultaneously once their left-lobe-donation possibilities are exhausted. There are two difficulties with this simple approach. First, it is possible that an exchange between two such pairs might be possible with the transformation of only one of these pairs, say pair 1. If so, transforming both pairs and matching them for an exchange results in a Pareto-inferior outcome. Second, this possibility might encourage pair 1 to hide its willingness for a right-lobe donation. Hence, key in our design is determining the order in which pairs are to be

[6]Throughout the paper, Pareto efficiency is intended by the term efficiency.

transformed. We show that there is a well-defined ordering, which assures that the resulting mechanism is not only Pareto efficient, but also incentive compatible. We also illustrate the potential gains from adopting our proposed mechanism on simulated pools based on South Korean population and transplantation characteristics. We show that, through liver exchange, the number of living-donor liver transplants can be increased by more than 30%.

1.1 Double Equipoise and Vancouver Forum

From an ethical perspective, the concept of *double equipoise* was proposed by Cronin et al. (2001) to balance the risk of a healthy donor versus the benefit for a high-risk recipient. A well-designed liver exchange system can not only be an effective tool to achieve this balance, but it also complies with the mainstream approach towards living-donor liver transplantation, summarized in the *Report of the Vancouver Forum* (Barr et al., 2006). Two of the main principles of live liver donation, both related to the concept of double equipoise, are stated as follows in this reference document:

> Live liver donation should only be performed if the risk to the donor is justified by the expectation of an acceptable outcome in the recipient ...
>
> The estimated risk of mortality and morbidity currently associated with live donor right hepatectomy is 0.4% and 35% respectively. Since the risk to the donor is considerable, programs performing live donor liver transplantation should institute procedures and protocols that insure that donor mortality and morbidity is minimized.

Hence, a possible establishment of a liver exchange program is very much in the spirit of the principles outlined by the Vancouver Forum, especially if it gives priority to left-lobe donation.

In part motivated by the theory of double equipoise, there has been some renewed interest in the liver transplantation community in finding ways to replace higher-donor-risk right hepatectomy with left hepatectomy. Suggesting that

1. donor complications are not only 4- to 12-fold lower for left-lobe donors than right-lobe donors, but also
2. complications are less severe under left-lobe donation than under right-lobe donation,

Roll et al. (2013) propose shifting the risk from the donor to the patient by lowering the minimum acceptable liver tissue volume to a less conservative level. They state:

> Although using smaller grafts from LL [left lobe] may decrease recipient benefit absolutely, their double-equipoise analysis suggests that LL [left lobe] is more efficient than RL [right lobe] in converting donor risk into recipient benefit.

Establishment of a liver exchange program and adoption of our mechanism can be seen as part of these efforts to reduce donor-risk through increased use of left hepatectomy. In contrast to Roll et al. (2013) proposal where the increased utilization of left hepatectomy comes at the expense of an increased average risk to patients, under our proposed liver exchange mechanism it is achieved more naturally without any adverse effect.

Finally, there is one additional benefit of liver exchange, reported in Pomfret et al. (2011). There can be situations where a direct liver transplant from a donor to his patient is ethically unacceptable based on the theory of double-equipoise, for example due to old age of the patient that translates to low patient benefit, but an exchange involving the same patient-donor pair may be ethically acceptable due to the additional benefit to the other patient.

1.2 Other Related Literature

Kidney exchange, as an application of market design, was initiated by Roth, Sönmez, and Ünver (2004, 2005, 2007). Recent developments in market design for kidney exchanges include studies on incentivizing compatible pairs to participate in exchange (Nicolò and Rodriguez-Álvarez, 2017; Sönmez, Ünver, and Yenmez, 2018), using kidney exchange along with ABO-blood-type-incompatible kidney transplants (Andersson and Kratz, 2019) or the use of immunosuppressants (Chun et al., 2017), and designing an incentive-compatible participation scheme for transplant centers in kidney exchange (Agarwal et al., 2019).

Unlike the growing literature on kidney exchange, there are only a handful papers on liver exchange. These include Hwang et al. (2010) and Chan et al. (2010), both of which demonstrate the proof of concept for liver exchange, and Mishra et al. (2018), which advocates for organized liver exchange in the US. Dickerson and Sandholm (2014) advocates for trans-organ exchange, where a donor associated with a kidney recipient donates a liver lobe and a donor associated with a liver recipient donates a kidney, whereas Samstein et al. (2018) explores some of the ethical concerns this practice might encounter, including unbalanced donor risks. Ergin, Sönmez, and Ünver (2017) studies dual-donor organ exchange, where each patient receives organs from two living donors. Dual-graft liver exchange, where each patient participates in exchange with two left-lobe donating donors, is an application of this model. Although dual-graft liver transplantation is practiced in a few countries, including South Korea and China, overcoming size incompatibility through a right-lobe transplantation is far more common throughout the world. And while the difference between the mortality and morbidity risks of right lobe vs. left-lobe donation is well established in the transplantation literature, our main focus, the design implications of these two main liver transplantation technologies, is not considered in any of the papers on liver exchange.

In terms of modeling, there is a conceptual similarity between our liver-exchange model and the "matching with contracts" model of Hatfield and Milgrom (2005), which extends two-sided matching problems (Gale and Shapley, 1962) by allowing various contractual arrangements between the two sides. While left-lobe donation and right-lobe donation can be interpreted as two different contractual arrangements, unlike the matching with contracts model, our model is one sided. Hence the cumulative offer mechanisms introduced for the matching with contracts model by Hatfield and Milgrom (2005) and extended by Hatfield and Kojima (2010) is not applicable in our framework.

More broadly, our paper contributes to a very diverse list of market-design applications, including entry-level labor markets (Roth and Peranson, 1999), spectrum auctions (Milgrom, 2000), internet auctions (Edelman, Ostrovsky, and Schwarz, 2007; Varian, 2007), school choice (Abdulkadiroğlu and Sönmez, 2003), course allocation (Sönmez and Ünver, 2010; Budish and Cantillon, 2012), affirmative action (Kojima, 2012; Hafalir, Yenmez, and Yildirim, 2013; Echenique and Yenmez, 2015), refugee matching (Moraga and Rapoport, 2014; Jones and Teytelboym, 2017; Delacrétaz, Kominers, and Teytelboym, 2017), and assignment of airport landing slots (Schummer and Vohra, 2013; Schummer and Abizada, 2017).

2 A Model of Dual Technology Liver Transplantation

There are two liver transplantation technologies: A donor can donate either his *left liver lobe* or his *right liver lobe* for a transplant, although the latter involves considerably higher risk to the donor. We sometimes refer to a liver lobe as a *graft*, when it is donated for a transplant.

2.1 Size Compatibility

The volume of the left liver is generally between 30% to 40% of the liver volume, and the right lobe makes up the rest. Formally, the **size of a liver lobe** (or a graft) is the volume of the liver lobe.

A patient typically requires a liver graft with a volume of at least 40% of her own liver volume, although for some patients this minimum requirement may differ depending on the details of the patient's disease. Formally, the **size of a patient** is the minimum required volume of the liver graft he needs for a transplant. Therefore, a liver lobe is **size compatible** with a patient if and only if it is as large as the size of the patient.

Let $\mathbf{S} = \{0, 1, \ldots, S-1\}$ denote the set of possible sizes, where $S \geq 1$ is the number of possible sizes. Here the larger numbers correspond to larger sizes.[7]

2.2 Blood-type Compatibility

The blood type of an individual is determined by the availability or the lack of two antigens referred to as antigen A and antigen B. An individual of blood type O has neither antigen,

[7] Set $\mathbf{S}$ can be specific to each liver-exchange pool, allowing for a continuum possibility for sizes generically as long as (1) each pool we analyze is finite and (2) corresponding sizes in a pool–represented by integers–are ordinal.

an individual of blood type A has only antigen A, an individual of blood type B has only antigen B, and an individual of blood type AB has both antigens. A donor (and each of his liver lobes) are **blood-type compatible** with a patient if he does not have a blood antigen the patient lacks. That means a blood-type O donor (having neither antigen) is blood-type compatible with patients of all blood types, a blood-type A donor is blood-type compatible with patients of blood types A and AB, a blood-type B donor is blood-type compatible with patients of blood types B and AB, and a blood type AB donor is blood-type compatible with patients of only blood type AB. Let $\mathbf{B} = \{O, A, B, AB\}$ denote the set of blood types.

2.3 Liver Donation Relation

The blood type and the size of each patient are assumed to be observable physical attributes. Similarly, the blood type and the size of each liver lobe (i.e. graft) are also assumed to be observable. The set of patient types, and the set of liver lobe (or equivalently graft) types are both referred to as $\mathbf{T} \equiv \mathbf{B} \times \mathbf{S}$. Observe that, the type of a liver lobe only specifies its blood type and size, and not whether it is the left lobe or the right lobe.

Consider a donor who wants to donate a liver lobe to a patient. He can do so if and only if he is blood-type compatible and the intended lobe is size compatible with the patient.

We define a liver donation partial order on $\mathbf{T}$ to denote for any $X, X' \in \mathbf{T}$, graft of type X can be feasibly donated to patient of type X' if and only if type X is both blood-type compatible and size compatible with type X'. To do that, we present an equivalent representation for $\mathbf{B}$. We redefine the set of blood types as

$$\mathbf{B} \equiv \{0, 1\}^2,$$

where for any blood type $b = (b_1, b_2) \in \mathbf{B}$, $b_1 = 0$ refers to the existence of A blood antigen and $b_1 = 1$ refers to its non-existence, and $b_2 = 0$ refers to the existence of B blood antigen and $b_2 = 1$ refers to its non-existence. Thus,

$$(0,0) \equiv AB, \ \ (0,1) \equiv A, \ \ (1,0) \equiv B, \text{ and } (1,1) \equiv O$$

denote the four blood types, and a graft of blood type $b \in \mathbf{B}$ is blood-type compatible with a patient of blood type $b' \in \mathbf{B}$ if and only if $b \geq b'$. Hence, $\mathbf{T} = \mathbf{B} \times \mathbf{S} = \{0, 1\}^2 \times \{0, 1, \ldots, S - 1\}$.

A patient/liver lobe type $X = (X_1, X_2, X_3) \in \mathbf{T}$ consists of its blood type $(X_1, X_2) \in \mathbf{B}$ and its size $X_3 \in \mathbf{S}$.

We define the **liver donation partial order** $\geq$ as the standard coordinate-wise comparison partial order over $\mathbf{T}$, and $(\mathbf{T}, \geq)$ as its associated partially ordered set: for any $X, X' \in \mathbf{T}$, a graft of type $X = (X_1, X_2, X_3)$ is **compatible** with a patient of type $X' = (X'_1, X'_2, X'_3)$

if $(X_1, X_2, X_3) \geq (X_1', X_2', X_3')$.[8] A graft can be transplanted to a patient if and only if it is compatible with the patient.

The set of types $\mathbf{T}$ represents both the set of patient types and the set of liver lobe types. The description of a donor, on the other hand, includes the sizes of both his liver lobes. Hence, the set of donor types is represented by

$$\mathbf{T^D} \equiv \mathbf{B} \times \mathbf{S}^2 = \{0,1\}^2 \times \{0,1,\ldots,S-1\}^2$$

where, for any donor type $Y = (Y_1, Y_2, Y_{3\ell}, Y_{3r}) \in \mathbf{T}^D$,

1. the **left-lobe size** is given by $Y_{3\ell}$,
2. the **right-lobe size** is given by Y_{3r}, and
3. $$\left\{ \begin{array}{ll} Y_{3\ell} < Y_{3r} & \text{if } Y_{3\ell} < S-1 \\ Y_{3\ell} = Y_{3r} & \text{if } Y_{3\ell} = S-1 \end{array} \right\}.$$

Given a donor of type $Y = (Y_1, Y_2, Y_{3\ell}, Y_{3r}) \in \mathbf{T^D}$,

- $Y^\ell \equiv (Y_1, Y_2, Y_{3\ell}) \in \mathbf{T}$ denotes the type of his left liver lobe, and
- $Y^r \equiv (Y_1, Y_2, Y_{3r}) \in \mathbf{T}$ denotes the type of his right liver lobe.

Given the human anatomy, the right liver lobe is always larger than the left liver lobe.[9] Therefore, for any pair $X, Y \in \mathbf{T} \times \mathbf{T^D}$, a patient of type X can receive a transplant from a donor of type Y

- through a left-lobe donation (i.e. left hepatectomy) if $X \leq Y^\ell$, and
- through a right-lobe donation (i.e. right hepatectomy) if $X \leq Y^r$.

Observe that these vector inequalities are equivalent to the donor being blood-type compatible with and his liver lobe for transplantation being size compatible with the patient.

For the case of two sizes ($S = 2$), Figure 1 illustrates the liver donation partial order $\geq$ over the corners of the three-dimensional cube $\mathbf{T} = \{0,1\}^3$.

A patient of type $X \in \mathbf{T}$ and a donor of type $Y \in \mathbf{T^D}$ are **left-lobe compatible** if $X \leq Y^\ell$. Since the right lobe is larger than the left lobe, the right-lobe-donation technology increases the set of potential exchanges and direct donations. However, because it involves

[8] The relation $\geq$ in $(\mathbf{T}, \geq)$ is the usual coordinate-wise partial order over integer vectors in Euclidean n-dimensional space: for all $a = (a_1, \ldots a_n)$, $b = (b_1, \ldots, b_n) \in \mathbb{Z}_+^n$ we say $a \geq b$ if $a_k \geq b_k$ for all k. Its asymmetric part is denoted as $>$: $a > b$ if $a_k \geq b_k$ for all k and $a_k > b_k$ for some k; its symmetric part is denoted as $=$: $a = b$ if $a_k = b_k$ for all k.

[9] We have the only exception for a donor type with a largest size, $S-1$ left lobe. We assume for such a type left lobe and right lobe are of the largest size $S-1$. This is assumed for notational convenience of defining one size set $\mathbf{S}$ for both patient needs and donor lobe sizes and terminological convenience of saying there are S sizes. For such a donor type, the right lobe is never donated as its left lobe is large enough for all patient types. This assumption is made for notational simplicity as well as in order to avoid introducing two separate sets of sizes, one for the patients and other for the liver lobes.

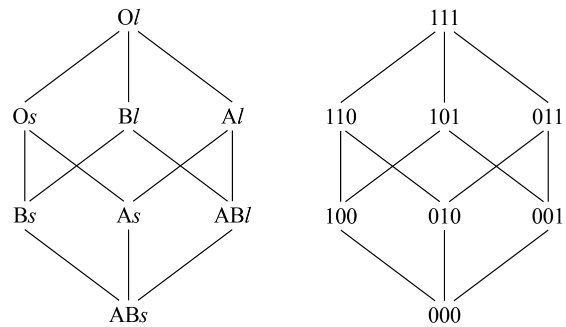

Figure 1: The partially ordered set $(\mathbf{T}, \geq)$ using its blood type/size representation with two sizes small (s) and large (l), and its integer vector representation.

higher risks for the donor, it is less preferred than left-lobe donation. Therefore, for donors who can feasibly donate their left lobes to a patient, we assume that right-lobe donation is not a viable option. A patient of type $X \in \mathbf{T}$ and a donor of type $Y \in \mathbf{T^D}$ are **right-lobe-only compatible** if the donor can donate his right lobe to the patient, but not his left lobe, i.e., $X \leq Y^r$ and $X \not\leq Y^\ell$.

3 Liver Exchange

As in kidney exchange, the number of living-donor liver transplants can be increased through exchange of donors. Living-donor liver transplantation is a more complex medical procedure than living-donor kidney transplantation, in part because only a portion of the donor's liver is transplanted to the patient, and hence a detailed analysis of patient and donor anatomies is required. Hence the logistics of liver exchange gets more complicated as the number of pairs increase in an exchange. Indeed, in a recent paper proposing an organized liver exchange to the members of the transplantation community, Mishra et. al. (2018) suggest:

> As with the initial experience with KPE, it is anticipated that LPE would begin with 2-way swaps, the simplest form of exchange.

Consistent with their suggestion, we assume that only two-way exchanges are feasible.[10]

3.1 Liver-Exchange Pool

Each patient participates in liver exchange with one donor. A patient and her donor are referred to as a **pair**.[11] The observable characteristics of a pair are summarized by an ordered

[10] All liver exchanges reported in the literature as of October 2019 are between two patients and their donors.

[11] We use pronouns "she" for a patient, "he" for a donor, and "it" for a pair.

pair of individual types $X - Y \in \mathbf{T} \times \mathbf{T^D}$, where X denotes the type of the patient and Y denotes the type of the donor; $X - Y$ is called the **pair type**.[12]

A **liver-exchange pool** is a tuple $(\mathcal{I}, \tau)$ where

1. $\mathcal{I} = \{1, 2, \ldots, K\}$ is a finite set of patient-donor pairs, and
2. $\tau : \mathcal{I} \to \mathbf{T} \times \mathbf{T^D}$ is a function, such that, for every pair $i \in \mathcal{I}$, $\tau(i)$ is its pair type.

For every pair $i \in \mathcal{I}$, we denote its type as $\tau(i) = \tau_P(i) - \tau_D(i)$, where $\tau_P(i) \in \mathbf{T}$ is the type of the patient of the pair, and $\tau_D(i) \in \mathbf{T^D}$ is the type of the donor of the pair.

Moreover, given a pair $i \in \mathcal{I}$, let $\tau_D^\ell(i) \in \mathbf{T}$ denote the type of its donor's left lobe, and $\tau_D^r(i) \in \mathbf{T}$ denote the type of its donor's right lobe.

Throughout the paper, we fix a liver-exchange pool $(\mathcal{I}, \tau)$.

3.2 Feasible Grafts and Assignments

Since we rule out the possibility of right-lobe donation when a donor can more safely donate his left lobe, there is a unique donation "mode" between any donor and patient: For any two pairs $(i, j) \in \mathcal{I} \times \mathcal{I}$, the donor of pair j can either feasibly donate his left lobe, or his right lobe, or neither lobe to the patient of pair i. The following function keeps track of which lobe is to be donated (if any) in any potential assignment. Define the **transplant type** function $t : \mathcal{I} \times \mathcal{I} \to \{\ell, r, \emptyset\}$ as follows: For any $(j, i) \in \mathcal{I} \times \mathcal{I}$,

$$t(j,i) = \begin{cases} \ell & \text{if } \tau_P(i) \leq \tau_D^\ell(j) \\ r & \text{if } \tau_P(i) \not\leq \tau_D^\ell(j) \ \& \ \tau_P(i) \leq \tau_D^r(j) \\ \emptyset & \text{otherwise} \end{cases}$$

For any two pairs j and i, the transplant type function $t(\cdot)$ determines whether the donor of the first pair j and the patient of the second pair i are left-lobe compatible (ℓ), right-lobe-only compatible (r), or incompatible ($\emptyset$). Define $t(i) \equiv t(i, i)$ for any $i \in \mathcal{I}$.

For any pair $i \in \mathcal{I}$, define

$$\mathcal{C}(i) \equiv \Big\{ j \in \mathcal{I} \ : \ t(j, i) \neq \emptyset \Big\}$$

be the set of pairs from whom the patient of i can receive a transplant. Since the transplant type function uniquely determines which lobe of a compatible donor is to be transplanted to a patient, the set $\mathcal{C}(i)$ also uniquely defines the set of feasible grafts for the patient of pair i. That is, for the patient of pair i

- the left lobe of the donor of pair j is **feasible** if and only if $j \in \mathcal{C}(i)$ and $t(j, i) = \ell$, and
- the right lobe of the donor of pair j is **feasible** if and only if $j \in \mathcal{C}(i)$ and $t(j, i) = r$.

[12]We refer to a pair type as $X - Y$ instead of (X, Y) as a convention.

With a slight abuse of terminology, we will also refer to set $\mathcal{C}(i)$ as **the set of feasible grafts** for the patient of pair i.

3.3 Preferences

There are three types of possible outcomes for a pair i:

1. The patient of pair i receives her own donor's feasible graft in a **direct transplant**.
2. Pair i exchanges donors with another pair to form a **(two-way) exchange**, so that the patient of each pair receives from the other pair's donor the graft indicated by the transplant type function.
3. Pair i remains **unmatched** and its patient does not receive a transplant. This outcome is denoted as $\emptyset$.

In any potential exchange, both patients have to receive a transplant. Hence, define the set of **(feasible) assignments** for pair i as

$$\mathcal{E}(i) \equiv \Big\{ j \in \mathcal{C}(i) \ : \ i \in \mathcal{C}(j) \Big\}.$$

A patient can receive a left lobe of her own paired donor if she is left-lobe compatible with her donor, or a right lobe of her own paired donor if she is right-lobe-only compatible with him. Formally, we say that a pair i is **compatible** if $i \in \mathcal{E}(i)$.

Since the function $t(i, \cdot)$ uniquely identifies which lobe the donor of pair i donates in any assignment in $\mathcal{E}(i)$, we can further partition the set $\mathcal{E}(i)$ based on the donated lobe:

$$\mathcal{E}^{\ell}(i) \equiv \Big\{ j \in \mathcal{C}(i) \ : \ i \in \mathcal{C}(j) \text{ and } t(i,j) = \ell \Big\}, \text{ and}$$
$$\mathcal{E}^{r}(i) \equiv \Big\{ j \in \mathcal{C}(i) \ : \ i \in \mathcal{C}(j) \text{ and } t(i,j) = r \Big\}.$$

We say that pair i is **left-lobe compatible** if $i \in \mathcal{E}^{\ell}(i)$ and it is **right-lobe-only compatible** if $i \in \mathcal{E}^{r}(i)$.

We interpret a patient-donor pair as a single agent in our model, and thus, preferences refer to preferences of the pair. Patients or donors do not have preferences of their own. Preferences of a pair depend on the following three factors:

1. Observable characteristics of the received compatible graft.
2. Whether the donor donates his left lobe or his right lobe.
3. Whether the patient receives a graft through direct transplant or through an exchange.

For any pair $i \in \mathcal{I}$, the preference relation R_i is defined over the set $\mathcal{E}(i) \cup \{\emptyset\}$; that is, $R_i \subseteq \big(\mathcal{E}(i) \cup \{\emptyset\}\big) \times \big(\mathcal{E}(i) \cup \{\emptyset\}\big)$. Let P_i denote the asymmetric part and I_i denote the symmetric part of R_i.

In order to motivate the restrictions we make on preferences, we next discuss the role of each of these factors on liver transplantation.

3.3.1 Preferences on Observable Characteristics of a Graft

While blood-type compatibility and size compatibility are the primary considerations for liver transplantation, to a lesser extent other factors (such as the age of the donor) can also influence the outcome. As a result, other things being equal, a pair may prefer one compatible graft to another, even if they are of the same type. For a given pair i, let the weak order $\succsim_i$ represent the **received-graft preference relation** over the set of feasible grafts $\mathcal{C}(i)$, with its asymmetric part indicated by $\succ_i$ and symmetric part indicated by $\sim_i$.

Since it purely depends on observable donor characteristics and determined based on agreed-upon medical criteria, we assume that the received-graft preference relation $\succsim_i$ is public information.

Given the high risk to donors, their screening is very strict for living-donor liver transplantation, and donation is ruled out unless the donor is in perfect health and the benefit to the patient is sufficiently high. Grafts must be sufficiently large (usually 40% of the patient's liver volume) to minimize the risk of *small-for-size syndrome*, a condition where a patient develops liver dysfunction and ascites when the transplanted graft is too small. As a result, the expected benefit to the patient is "similar" between any two grafts deemed compatible for the patient. It is also possible that the central planner may choose to disclose information on patient-donor compatibility only, and may not make any additional information on pairs available to third parties. Indeed, this is often the case for kidney exchange, where patient-donor pairs do not meet until after the exchange has taken place, and they only meet if each person agrees. In this likely scenario, pairs will be assumed to be indifferent between all compatible grafts under the received-graft preference relation $\succsim_i$.

Based on these observations, any asymmetric part of the received-graft preference relation on compatible grafts will play a secondary role of a "tie-breaker" on the preferences of a pair.

3.3.2 Preferences on Left-Lobe Donation vs. Right-Lobe donation

While the expected benefit to a patient is similar for all compatible grafts, donor mortality and morbidity risks differ considerably between left-lobe donation and right-lobe donation. Based on the 2006 Vancouver Forum report, mortality rate exceeds 0.4% for right-lobe donation in contrast to approximately 0.1% for left-lobe donation.[13] Morbidity rates to the donor, frequency of significant medical complications other than mortality per donation, are also much higher under right-lobe donation. Mishra et al. (2018) report that the morbidity rates are 28% for right-lobe donation and 7.5% for left-lobe donation. As a result, pairs have a strong preference for their donors to donate their left lobes rather than their right lobes. Indeed, pairs may not be willing to have their donors to donate their right lobes at all. We

[13]More recent reports give similar mortality rates.

refer to such pairs as **unwilling ($\boldsymbol{u}$)**. Pairs that are open to the possibility of right-lobe donation from their donors, on the other hand, are referred to as **willing ($\boldsymbol{w}$)**.

Formally,

- for an unwilling pair i, for all $j \in \mathcal{E}^\ell(i)$ and $j' \in \mathcal{E}^r(i)$,

$$j \; P_i \; \emptyset \; P_i \; j', \qquad \text{and}$$

- for a willing pair i, for all $j \in \mathcal{E}(i)$,

$$j \; P_i \; \emptyset.$$

Whether a pair i is willing or unwilling is its private information.

Importantly, willingness of a pair is assumed to be independent of the graft received by its patient and whether it is received through a direct transplant or an exchange. Therefore, which lobe is donated by the pair is the primary consideration in their preferences. We show in Example A-1 in Appendix C.1 that, in the absence of this assumption, a mechanism that is Pareto efficient, individually rational, and incentive compatible may fail to exist. However, it is important to emphasize that, this technical result is not the only justification for our assumption. Even though the risk-benefit ratio, one of the key considerations in living-donor organ transplantation, is highly responsive to whether the left or right lobe donated by the donor, it is relatively irresponsive to the received graft by the patient. Hence the preferences largely depending on whether the left or right lobe is donated is a fairly realistic assumption. Indeed, this assumption is supported by the findings of Molinari et al. (2014), where the authors analyze living liver donors' risk thresholds using decision analysis techniques based on the probability trade-off method. They report that the decision to donate largely depends on various risks and burdens to the donor along with expected life gain for the patient. Based on their analysis, one can infer that the lobe to be donated is key in the decision to donate for a significant percentage of the pairs. For example, the authors report that while more than 95% of the donors are willing to accept a morbidity rate of 10% (i.e. the approximate morbidity rate from left-lobe donation), less than 70% of them are willing to accept a morbidity rate of 30% (i.e. the approximate morbidity rate from right-lobe donation).[14]

[14]Other key factors reported in Molinari et al. (2014) such as donor's mortality risk, donor's risk for decreased physical capacity, duration of donor's hospital stay, donor's time off work and donor's financial burden also suggest that the specific lobe to be donated is key for a significant proportion of pairs in their decision to donate. In contrast, the findings in Molinari et al. (2014) also suggest that decision to donate is unlikely to be affected by the specific liver graft received by the patient, provided that the transplantation is justified based on the medical norms on acceptable risk-benefit ratio for the transplant: The authors report, 88% of the participants would donate for a gain of 1 year, 95% would donate for a gain of 3 years, and 98% would donate for a gain of 5 years. Therefore, given the 3-year survival rate of 83% and the 5-year survival rate of 78% reported by Goldberg et al. (2014), the specific graft the patient receives virtually plays no role in the decision to donate. Therefore, we not only expect heterogeneity in willingness for right-lobe donation, but also the preferences to primarily depend on which lobe is to be donated by the donor.

3.3.3 Direct Donation Bias

In real-life applications of kidney exchange, it is well established that most pairs have a *direct-transplant bias*, which means a compatible pair opts for a direct transplant even when its patient is committed to receive a more-favorable kidney through exchange. This preference can have a time-preference component: the exchange option involves more waiting than direct transplant, which can be realized without finding a suitable exchange partner and expectation of a more-favorible match than a direct transplant may not outweigh this waiting cost. Often it also has an emotional component: direct donation from a loved one may induce a higher utility for the pair than a donation from a third-party.

In an organized liver exchange, we expect the direct donation bias to be also prevalent, unless the risk to the donor can be reduced through a much safer left-lobe donation. While we do not expect many compatible pairs to participate in exchange in the absence of this tangible benefit, we allow for it in our model. We will assume, however that, a possible direct-transplant bias never dominates the safety concerns for the donor, and as such a pair always prefers an exchange with a left-lobe donation to a direct-transplant of the right lobe. We also rule out the possibility of a "mild" direct-transplant bias in the sense that, subject to donating the same lobe, a pair that has a direct-transplant bias strictly prefers direct-transplant to any other graft through exchange. We show in Example A-2 in Appendix C.1 that, in the absence of this assumption, a mechanism that is Pareto efficient, individually rational, and incentive compatible may fail to exist.

To capture this private-information component of the preference relation, we introduce two *participation types* for left-lobe or right-lobe-only compatible pairs.

A left-lobe compatible or a right-lobe-only compatible pair i is a **transplant maximizer** (***m***) if,

$$\begin{array}{rc} \text{for all } j, j' \in \mathcal{E}^\ell(i) & j \; R_i \; j' \iff j \;\succsim_i\; j' \\ \text{for all } j, j' \in \mathcal{E}^r(i) & j \; R_i \; j' \iff j \;\succsim_i\; j' \\ \text{for all } j \in \mathcal{E}^\ell(i) \text{ and } j' \in \mathcal{E}^r(i) & j \; P_i \; j' \end{array}$$

A right-lobe-only compatible pair is **direct-transplant biased** (***d***) if,

$$\begin{array}{rc} \text{for all } j \in \mathcal{E}^\ell(i) & j \; P_i \; i \\ \text{for all } j \in \mathcal{E}^r(i) \setminus \{i\} & i \; P_i \; j \end{array}$$

A left-lobe compatible pair i is **direct-transplant biased** (***d***) if,

$$\text{for all } j \in \mathcal{E}(i) \setminus \{i\} \qquad i \; P_i \; j$$

Observe that, a direct-transplant-biased left-lobe compatible pair has no reason to participate in exchange.

3.3.4 Preference Domain

For each pair $i \in \mathcal{I}$, fix the public information received-graft preference relation $\succsim_i$. Given the restrictions described in Sections 3.3.1-3.3.3, each pair has one of the four possible preference relations $R_i^{m/w}$, $R_i^{m/u}$, $R_i^{d/w}$, $R_i^{d/u}$, depending on whether it is willing or unwilling, and whether it is direct-transplant biased or transplant maximizer:

- For a willing transplant maximizer pair i, the possible outcomes are ranked as follows under its **transplant maximizer/willing preference relation** $R_i^{m/w}$:
  1. Feasible direct transplant and exchanges in which the pair donates a left lobe are ranked in order of its received-graft preferences $\succsim_i$.
  2. Feasible direct transplant and exchanges in which the pair donates a right lobe are ranked in order of its received-graft preferences $\succsim_i$.
  3. The pair is unmatched.
- For an unwilling transplant maximizer pair i, the possible outcomes are ranked in the following order under its **transplant maximizer/unwilling preference relation** $R_i^{m/u}$**:**
  1. Feasible direct transplant and exchanges in which the pair donates a left lobe are ranked in order of its received-graft preferences $\succsim_i$.
  2. The pair is unmatched.
  3. Feasible direct transplant and exchanges in which the pair donates a right lobe are ranked in order of its received-graft preferences $\succsim_i$.
- For a willing direct-transplant-biased pair i, the possible outcomes are ranked in the following order under its **direct-transplant biased/willing preference relation** $R_i^{d/w}$**:**
  1. Feasible left-lobe direct transplant if i is left-lobe compatible.
  2. Feasible exchanges in which the pair donates a left lobe are ranked in order of its received-graft preferences $\succsim_i$.
  3. Feasible right-lobe direct transplant if i is right-lobe-only compatible.
  4. Feasible exchanges in which the pair donates a right lobe are ranked in order of its received-graft preferences $\succsim_i$.
  5. The pair is unmatched.
- For an unwilling direct-transplant-biased pair i, the possible outcomes are ranked in the following order under its **direct-transplant biased/unwilling preference relation** $R_i^{d/u}$**:**
  1. Feasible left-lobe direct transplant if i is left-lobe compatible.
  2. Feasible exchanges in which the pair donates a left lobe are ranked in order of its received-graft preferences $\succsim_i$.
  3. The pair is unmatched.

4. Feasible right-lobe direct transplant if i is right-lobe-only compatible.
5. Feasible exchanges in which the pair donates a right lobe are ranked in order of its received-graft preferences $\succsim_i$.

Let $\mathbf{R}_i = \{R_i^{m/w}, R_i^{m/u}, R_i^{d/w}, R_i^{d/u}\}$ denote the set of possible preference relations for pair i.

An assignment $j \in \mathcal{E}(i)$ is **individually rational** for a pair i under a preference relation R_i if

$$\begin{array}{lll} j \; R_i \; \emptyset & \text{whenever} & i \notin \mathcal{E}(i), \text{ and} \\ j \; R_i \; \emptyset \text{ and } j \; R_i \; i & \text{whenever} & i \in \mathcal{E}(i). \end{array}$$

That is, an assignment is individually rational if it is not only weakly preferred to remaining unmatched, but also to a direct transplant whenever the pair is compatible. Individual rationality is important because, no donor can be enforced for a donation he does not volunteer for. At any time during the donation process a living donor may change his or her mind. Throughout our analysis, we focus on individually rational outcomes.[15]

Observe that

- if pair i is *incompatible*, i.e., $i \notin \mathcal{E}(i)$, then the two willing preferences coincide, i.e., $R_i^{m/w} = R_i^{d/w}$, and the two unwilling preferences also coincide, i.e., $R_i^{m/u} = R_i^{d/u}$;
- if pair i is *left-lobe compatible*, i.e., $i \in \mathcal{E}^\ell(i)$, then the individually rational assignments are ranked the same way under the two transplant maximizer preferences, $R_i^{m/w}$ and $R_i^{m/u}$, and also the same way under the two direct-transplant-biased preferences, $R_i^{d/w}$ and $R_i^{d/u}$; and
- if pair i is *right-lobe-only compatible*, i.e., $i \in \mathcal{E}^r(i)$, then the individually rational assignments are ranked the same way under the two unwilling preferences, $R_i^{m/u}$ and $R_i^{d/u}$.

Let $\mathbf{R} = \mathbf{R}_1 \times \ldots \times \mathbf{R}_K$ denote the set of preference profiles.

We refer to a triple $(\mathcal{I}, \tau, R)$, i.e., the liver-exchange pool together with the pair preference profile, as a **liver-exchange problem**. Since we fix the exchange pool, it is often denoted by the preference profile R.

[15] As we focus on individually rational assignments in our analysis, by slight abuse of the definition of preference types, we sometimes assume that the ranking of assignments inferior to remaining unmatched or a direct transplant (for a compatible pair) is arbitrary.

3.4 Outcome of the Problem: A Matching

We are ready to define the outcome of a liver-exchange problem. The set of all mutually compatible **matches** E_c is given as follows: [16]

$$E_c \equiv \Big\{\{i,j\} \subseteq \mathcal{I} : j \in \mathcal{E}(i)\Big\}.$$

A match $\{i,j\} \in E_c$ is a (two-way) exchange if $i \neq j$ and a direct transplant if $i = j$.

The **compatibility graph** is defined as the undirected graph, with the pairs as its nodes and the mutually compatible matches as its edges, $G_c = (\mathcal{I}, E_c)$.[17]

Given a compatibility graph $G_c = (\mathcal{I}, E_c)$, a **matching** $M \subseteq E_c$ is a collection of compatible matches such that

$$\text{for all } \varepsilon, \varepsilon' \in M, \qquad \varepsilon \cap \varepsilon' \neq \emptyset \implies \varepsilon = \varepsilon'.$$

That is, no pair participates in two distinct exchanges or both in a direct transplant and in an exchange at the same time. Let $\mathbf{M}_c$ be the set of matchings supported by the compatibility graph G_c.

We denote the assignment of pair $i \in \mathcal{I}$ in matching $M \in \mathbf{M}_c$ as $M(i)$. If $M(i) = i$ (i.e., $\{i\} \in M$), then the pair participates in a direct transplant. If $M(i) = j$ for some $j \neq i$ (i.e., $\{i,j\} \in M$), then pairs i and j participate in an exchange. If $M(i) = \emptyset$ (i.e., there is no $\varepsilon \in M$ such that $i \in \varepsilon$), then pair i remains unmatched.

Consider a match $\{i,j\}$ (possibly with $i = j$) in a matching M. Since a donor only donates a right lobe when his left lobe is too small for the intended receiver, which lobe is donated by either donor is uniquely determined by the match $\{i,j\}$. The same argument also holds for the entire matching M.

To summarize, each matching M is a collection of direct transplants and exchanges, and together with the function $t(\cdot)$, it also uniquely specifies which liver lobe is donated by each donor: For any $\{i\} \in M$, the pair i engages in a *direct left-lobe transplant* if $t(i) = \ell$ and in a *direct right-lobe transplant* if $t(i) = r$. Similarly, for any $\{i,j\} \in M$ such that $i \neq j$,

- the pairs i and j engage in a two-way exchange,
- the donor of i donates his left lobe if $t(i,j) = \ell$ and his right lobe if $t(i,j) = r$, and
- the donor of j donates his left lobe if $t(j,i) = \ell$ and his right lobe if $t(j,i) = r$.

The preferences introduced in Subsection 3.3 can be directly extended to the set of matchings. We slightly abuse the notation, and let R_i also denote the resulting preference relation

[16] Observe that this definition allows for a loop $\{i,i\} = \{i\}$ to be in E_c. This depicts that the donor of pair i can donate to the patient of the pair.

[17] Graph theoretical preliminaries are stated formally in Appendix A. Some of our current definitions are restated for general graphs in this appendix, as well.

over all matchings $\mathbf{M}_c$ defined as:

$$M \; R_i \; M' \quad \Longleftrightarrow \quad M(i) \; R_i \; M'(i).$$

3.5 Mechanisms and Axioms

Although the types of the participating pairs and their received-graft preferences are observable, their pair preferences (or equivalently their willingness for a right-lobe donation and whether they have direct-transplant bias) are not. A (direct) mechanism determines a matching as a function of the reported preference profile.

Since we fix an exchange pool $(\mathcal{I}, \tau)$ and a received-graft preference profile $\succsim = (\succsim_i)_{i \in \mathcal{I}}$ throughout, we define a **mechanism** as a function $f : \mathbf{R} \to \mathbf{M}_c$.

A matching $M \in \mathbf{M}_c$ is **individually rational (IR)** at a preference profile $R \in \mathbf{R}$, if for every pair $i \in \mathcal{I}$, either

1. $M(i)$ is an individually rational assignment at R, or
2. $M(i) = \emptyset$ and a direct transplant is not an individually rational assignment for i.

A mechanism f is **individually rational (IR)** if $f[R]$ is individually rational at R for any $R \in \mathbf{R}$.

A matching $M \in \mathbf{M}_c$ is **Pareto efficient (PE) at** a preference profile $R \in \mathbf{R}$ if there does not exist a matching $M' \in \mathbf{M}_c$ such that $M' \; R_i \; M$ for all $i \in \mathcal{I}$ and $M' \; P_i \; M$ for some $i \in \mathcal{I}$. A mechanism f is **Pareto efficient (PE)** if $f[R]$ is Pareto efficient at R for any $R \in \mathbf{R}$.

A mechanism f is **incentive compatible (IC)** if for all $i \in \mathcal{I}$, $R_{-i} \in \prod_{j \neq i} \mathbf{R}_j$ and $R_i, \hat{R}_i \in \mathbf{R}_i$,

$$f[R_i, R_{-i}] \; R_i \; f[\hat{R}_i, R_{-i}].$$

4 An Efficient and Incentive-Compatible Mechanism

In this section we introduce a mechanism that is individually rational, Pareto-efficient, and incentive-compatible. In order to do so, we will make a number of key observations about the structure of the problem, and develop the tools required to equip a priority matching mechanism with the “safeguards” to avoid two potential complications due to availability of dual transplantation technologies for liver exchange.

4.1 Challenges to Overcome

While the starting point of our design is a priority mechanism, it has to be considerably modified to maintain Pareto efficiency and incentive compatibility. Under the most basic form of a *priority mechanism*, agents are committed (to the extent it is feasible) to be matched with one of their best possible matches –one at a time– following a fixed priority ordering. When there is a single transplantation technology (as in kidney exchange), and assuming agents are indifferent among all their individually rational assignments, this simple mechanism is not only Pareto efficient, but also incentive compatible when individually rational matches of agents are of private information (Roth, Sönmez, and Ünver (2005) and Sönmez and Ünver (2014)). As we show in our next example, the priority mechanism in this basic form is no longer incentive compatible in our model due to the presence of dual transplantation technologies.

Example 1 *There is a set of three incompatible pairs* $\mathcal{I} = \{i_1, i_2, i_3\}$ *with the following types:*

$$\begin{aligned} \tau_P(i_1) &= (0,1,1) & \tau_D(i_1) &= (1,0,1,2) \\ \tau_P(i_2) &= (1,0,1) & \tau_D(i_2) &= (0,1,0,1) \\ \tau_P(i_3) &= (0,1,0) & \tau_D(i_3) &= (1,0,0,1) \end{aligned}$$

This pool of pairs result in the following set of feasible matches

$$E_c = \big\{\{i_1, i_2\}, \{i_2, i_3\}\big\},$$

and the following transplant type function

$$t(i_1, i_2) = \ell,\;\; t(i_1, i_3) = \emptyset,\;\; t(i_2, i_1) = r,\;\; t(i_2, i_3) = \ell,\;\; t(i_3, i_1) = \emptyset,\;\; t(i_3, i_2) = r.$$

Suppose all pairs are willing and none of them have a direct-transplant bias. That is, each pair $i \in \mathcal{I}$ *has the preference relation* $R_i^{m/w}$*. The compatibility graph of this problem is depicted in Figure 2.*

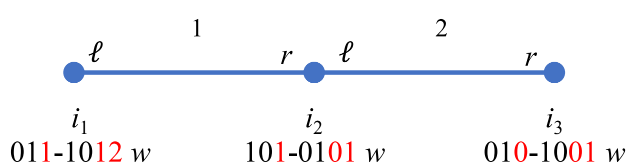

Figure 2: The compatibility graph for Example 1. All pairs are willing. The left- and right-lobe donations are denoted by letters ℓ and r, respectively. There are two individually rational exchanges.

Let us find the outcome of the priority mechanism for the priority order

$$\Pi = i_1 - i_2 - i_3.$$

First, we process pair i_1*. Since exchange* $\{i_1, i_2\}$ *is the only feasible match for pair* i_1*, the* *priority mechanism commits for the exchange* $\{i_1, i_2\}$. *Next, we process pair* i_2. *Pair* i_2 *is already committed for a specific match, i.e. with pair* i_1. *Finally, we process pair* i_3. *There is no exchange that can match* i_3 *in addition to* i_1 *and* i_2. *So the outcome of the priority mechanism is the matching*

$$M = \{\{i_1, i_2\}\},$$

where the donor of pair i_1 *donates a left lobe, whereas the donor of pair* i_2 *donates a right lobe.*

On the other hand, if pair i_2 *declares itself as unwilling reporting its preferences as* $R_i^{m/u}$, *the exchange* $\{i_1, i_2\}$ *ceases to be individually rational, and thus pair* i_2 *enforces the priority mechanism to pick the unique individually rational exchange resulting with the matching*

$$M' = \{\{i_2, i_3\}\}.$$

The donor of pair i_2 *donates a left lobe under this alternative exchange, and thus the pair benefits from this manipulation.*

Given the priority order Π, the priority mechanism enforces pair i_2 in Example 1 for a right-lobe donation, even though this pair is in a position to enforce an outcome where not only its patient receives a transplant, but also its donor donates the much safer left lobe. This example motivates our first departure from a basic priority mechanism: Under the modified mechanism, while we order pairs in a priority list and sequentially commit to matching them –one at a time– whenever it is feasible, we consider pairs for right-lobe donation only after their left-lobe donation possibilities are exhausted. Importantly, a pair's left-lobe donation possibilities include those exchanges where other pairs donate right lobes. Under our modification, right-lobe donation from a pair is to be considered only when these possibilities are exhausted as well. It is easy to see that, this modification restores incentive compatibility of the mechanism. However, restoring incentive compatibility with this simple fix may come at the expense of losing Pareto efficiency. We illustrate this possibility with the following example:[18]

Example 2 *There is a set of three incompatible pairs* $\mathcal{I} = \{i_1, i_2, i_3\}$ *with the following set of feasible matches*

$$E_c = \{\{i_1, i_2\}, \{i_2, i_3\}, \{i_3, i_1\}\},$$

and the transplant type function

$$t(i_1, i_2) = r,\ t(i_1, i_3) = \ell,\ t(i_2, i_1) = \ell,\ t(i_2, i_3) = r,\ t(i_3, i_1) = r,\ t(i_3, i_2) = \ell.$$

[18]We will later show that the compatibility graph given in Figure 3 of Example 2 cannot be generated by a liver-exchange problem. That is why the types of pairs are not given in Example 2.

Suppose all pairs are willing and none of them have a direct-transplant bias. That is, each pair $i \in \mathcal{I}$ has the preference relation $R_i^{m/w}$. The compatibility graph of this problem is depicted in Figure 3.

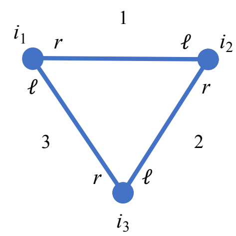

Figure 3: The compatibility graph for Example 2. All pairs are willing. The left- and right-lobe donations are denoted by letters ℓ and r. There are three individually rational exchanges.

The two critical observations in this example are:

1. *each of the three feasible exchanges involves one left-lobe donation and one right-lobe donation, and*
2. *each of the three pairs is part of two feasible exchanges, one with a left-lobe donation, and another with a right-lobe donation.*

What this means is, the only left-lobe donation possibility for each pair depends on a right-lobe donation from another pair. As such, our proposed modification of "right-lobe donation from a pair is to be considered only when its left-lobe donation possibilities are exhausted" simply means, no exchange can be picked by the modified priority mechanism, no matter how pairs are priority ordered.

For example, suppose the pairs are processed following the priority order

$$\Pi = i_1 - i_2 - i_3.$$

Pair i_1 is processed first. There is no left-lobe-only exchange in the problem, and at this initial stage of the modified process no pair is to be considered for right-lobe donation. Hence pair i cannot be committed for an assignment just yet. On the other hand, since $\{i_1, i_3\}$ is a feasible match with $t(i_1, i_3) = \ell$ and $t(i_3, i_1) = r$, pair i_1 can donate a left lobe if pair i_3 donates a right lobe. Thus, pair i_1 cannot be made available –just yet– for right-lobe donation either. Hence, under our proposed modification, the process has to bypass pair i_1, neither committing it an assignment, nor making it available for a right-lobe donation. Moreover, exactly analogous situation occurs for pairs i_2 and i_3 in steps 2 and 3, respectively. Therefore, no exchange becomes permissible under our proposed modification, and thus no exchange is conducted.

This is clearly a Pareto inefficient outcome since picking any of the individually rational *exchanges, each involving one left-lobe donation and one right-lobe donation, results in a Pareto improvement.*

The failure of the priority matching approach in Example 2 is directly linked to the existence of a cycle of exchanges, in which

- each pair donates a right lobe clockwise in the cycle, and
- a left lobe counter-clockwise in the cycle.

As it turns out, this difficulty is not unique to our priority matching approach. In Proposition 1 of Appendix C.1, we show that the existence of a *left-lobe–right-lobe exchange cycle* of any number of pairs in the underlying compatibility graph, rules out the existence of a mechanism that is individually rational, Pareto-efficient, and incentive compatible.

This negative result, however, does not imply that the priority matching approach has to be fruitless in our model. Observe that, unlike in Example 1, we have not given the types of pairs in Example 2. We have only given a set of (allegedly) feasible exchanges, along with a transplant type function that gives rise to a compatibility graph with a left-lobe–right-lobe exchange cycle. As we show in Lemma 1 in Section 4.2, one of the key observations in our paper, the existence of such a cycle is ruled out in our model.

We next formalize the structure that rules out a left-lobe–right-lobe exchange cycle in our model.

4.2 The Precedence Digraph

Consider two pairs of types $X - Y$, $U - V \in \mathbf{T} \times \mathbf{T^D}$. Suppose that, while the two pairs cannot form a left-lobe-only exchange, they can form an exchange where the donor of type Y donates his right lobe to the patient of type U, and the donor of type V donates his left lobe to the patient of type X. Observe that, the two pairs cannot form an exchange where the donor of type V donates his right lobe to the patient of type X, and the donor of type Y donates his left lobe to the patient of type U, for otherwise they could have formed a left-lobe-only exchange as well. Therefore, just focusing on these two pairs for the moment, it would be plausible to avail the pair of type $X - Y$ for right-lobe donation prior to the pair of type $U - V$, because the left-lobe-exchange possibilities of the pair of type $U - V$ expands with the availability of type $X - Y$ pair for right-lobe donation.

We can extend this line of reasoning to problems with more than two pairs as well, provided that there are no left-lobe–right-lobe exchange cycles (of the sort we have seen in Example 2) in the following directed graph.

Definition 1 *The* **precedence digraph** $(\mathbf{T} \times \mathbf{T^D}, D^\tau)$ *is a directed graph where,*

1. *each pair type in* $\mathbf{T} \times \mathbf{T^D}$ *is a node,*

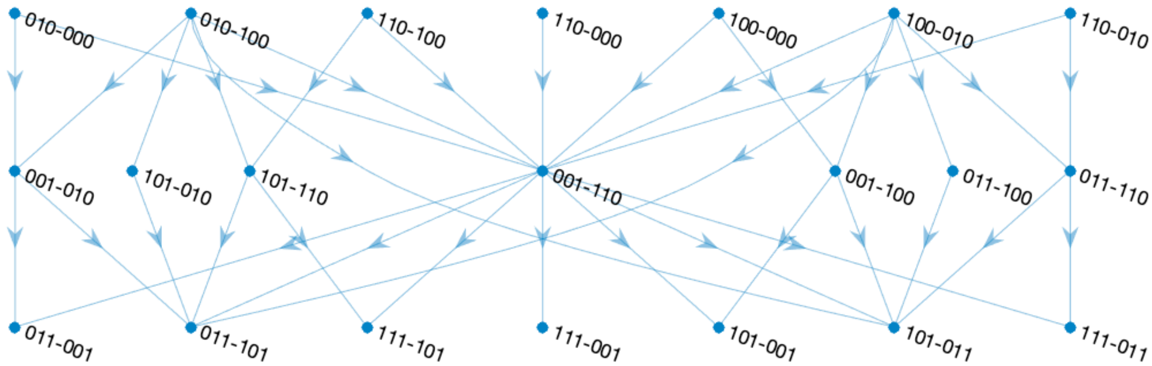

Figure 4: The precedence digraph with two sizes ($S = 2$) when left-lobe compatible pairs do not participate in exchange. We only denote left-lobe size of the donor types in this depiction, as their right-lobe size is uniquely determined by their left-lobe size. 16 pair types have no adjacent edges in the digraph, so those are not shown.

2. *there is a directed edge from type $X - Y$ to type $U - V$, denoted as $X - Y \longrightarrow U - V$,*[19] *if and only if*
$$X \leq V^\ell,\ U \not\leq Y^\ell \ \&\ U \leq Y^r, \quad \textit{and}$$
3. *D^τ is the resulting set of directed edges.*

We say that $X - Y$ **precedes** $U - V$, whenever $X - Y \longrightarrow U - V$. In a precedence digraph, type $X - Y$ precedes type $U - V$ if

- a donor of type V can donate his left lobe to a patient of type X, whereas
- a donor of type Y can donate his right lobe but not his left lobe to a patient of type U.

In this case, the two pairs fail to form a left-lobe-only exchange, but they can form an exchange once the pair of type $X - Y$ becomes available for right-lobe donation.

Figure A-16 in Appendix E depicts the precedence digraph for the case of two sizes $\mathbf{S} = \{0, 1\}$ such that
$$\text{for all } Y \in \mathbf{T^D}, \quad Y_{3\ell} = 0 \implies Y_{3r} = 1$$
when potentially all pair types are available for exchange. Figure 4 depicts the precedence digraph for the same case, but when only left-lobe incompatible pair types are available for exchange. Figure A-17 in Appendix E depicts the precedence digraphs for the case of three sizes $\mathbf{S} = \{0, 1, 2\}$ such that

$$\text{for all } Y \in \mathbf{T^D}, \quad [Y_{3\ell} = 0 \implies Y_{3r} = 1] \text{ and } [Y_{3\ell} = 1 \implies Y_{3r} = 2]$$

when only left-lobe incompatible pair types are available for exchange.

[19] This directed edge is also denoted as $(X - Y, U - V)$.

Note that the precedence digraphs in Figures 4, A-16, and A-17 are all acyclic. This observation is not specific to these examples. As stated by the next lemma, the precedence digraph for any liver-exchange pool is acyclic.

Lemma 1 *The precedence digraph* $(\mathbf{T} \times \mathbf{T^D}, D^\tau)$ *is acyclic.*

Proof of Lemma 1: Suppose for a contradiction that the precedence digraph has a cycle:

$$X^{(0)} - Y^{(0)} \longrightarrow X^{(1)} - Y^{(1)} \longrightarrow \ldots \longrightarrow X^{(n-1)} - Y^{(n-1)} \longrightarrow X^{(0)} - Y^{(0)}$$

where $n \geq 2$.

Note that for each $k \in \{0, 1, \ldots, n-1\}$ where all indexes are written in modulo n (i.e., $n \equiv 0$)

$$X^{(k)} - Y^{(k)} \longrightarrow X^{(k+1)} - Y^{(k+1)} \longrightarrow X^{(k+2)} - Y^{(k+2)}$$

implies that $X_3^{(k)} \leq Y_{3\ell}^{(k+1)}$. It also implies that $Y_{3\ell}^{(k+1)} < X_3^{(k+2)}$ since $Y^{(k+1)\,\ell} \not\geq X^{(k+2)}$ and $Y^{(k+1)\,r} \geq X^{(k+2)}$. Therefore, $X_3^{(k)} < X_3^{(k+2)}$. That is, a patient along the cycle has a smaller size than the patient two steps ahead in the cycle. This can be used to obtain a contradiction in two separate cases:

Case 1 "n is even": $X_3^{(0)} < X_3^{(2)} < \ldots < X_3^{(n-2)} < X_3^{(0)}$.

Case 2 "n is odd": $X_3^{(0)} < X_3^{(2)} < \ldots < X_3^{(n-1)} < X_3^{(1)} < X_3^{(3)} < \ldots < X_3^{(n-2)} < X_3^{(0)}$. ■

We extend the precedence digraph on the set of types of pairs $\mathbf{T} \times \mathbf{T^D}$ to the set of pairs $\mathcal{I}$. Let $(\mathbf{T} \times \mathbf{T^D}, D^\tau)$ be the precedence digraph of types of pairs. Construct a digraph $(\mathcal{I}, D)$ such that for any two types $X - Y,\ U - V \in \mathbf{T} \times \mathbf{T^D}$ and for any two pairs $i, j \in \mathcal{I}$ such that $\tau(i) = X - Y$ and $\tau(j) = U - V$:

$$(i, j) \in D \iff (X - Y, U - V) \in D^\tau.$$

We refer to $(\mathcal{I}, D)$ as the **precedence digraph** on $\mathcal{I}$.

A **topological order** Π of the precedence digraph $(\mathcal{I}, D)$ is a linear order over $\mathcal{I}$ such that for all $i, j \in \mathcal{I}$,

$$(i, j) \in D \implies i \,\Pi\, j.$$

Thus, whenever there is a feasible exchange $\{i, j\}$ in which pair i donates a right lobe (but cannot donate a left lobe) and pair j donates a left lobe, pair i is prioritized before pair j in a topological order of this digraph.

For a general digraph, a topological order may not exist. However, by Lemma 1, the precedence digraph on $\mathcal{I}$ is also acylic, and by Lemma 4 (in Appendix A), which states that every acyclic digraph has a topological order, there exists a topological order of the precedence digraph on $\mathcal{I}$. We illustrate this concept with a simple example:

Example 3 *In the problem given in Example 1, the precedence digraph over pairs* $(\mathcal{I}, D)$ *has the following directed edges:*

$$D = \Big\{(i_2, i_1),\ (i_3, i_2)\Big\}.$$

There is a unique topological order of this digraph:

$$\Pi' = i_3 - i_2 - i_1.$$

In order to maintain Pareto efficiency and incentive compatibility, our proposed mechanism uses a topological order to iteratively process pairs by (1) checking whether pairs can be matched while donating their left lobes, and (2) if they cannot but are willing, making them available as right-lobe donating pairs so that they can potentially be matched by donating their right lobes.

4.3 Transformations and Deletions

In Section 4.2 we have established the existence of a priority ordering, the topological order, that can potentially help us to overcome the challenges presented in Section 4.1. We next present some additional tools to facilitate the introduction of our proposed mechanism.

Fix a problem with the compatibility graph $G_c = (\mathcal{I}, E_c)$. For any set of matches $E' \subseteq E_c$, the graph $G' = (\mathcal{I}, E')$ is referred to as a **reduced compatibility graph**.

For any set of matches $E' \subseteq E_c$ with the reduced compatibility graph G', let $\mathbf{M}[G'] \subseteq \mathbf{M}_c$ denote the resulting set of matchings. That is,

$$M \in \mathbf{M}[G'] \iff M \in \mathbf{M}_c \text{ and } M \subseteq E'.$$

Also denote the **set of matches involving** i **in** E' as

$$E'(i) = \{\varepsilon \in E' : i \in \varepsilon\}.$$

For two reduced compatibility graphs $G' = (\mathcal{I}, E')$ and $G'' = (\mathcal{I}, E'')$ with $E'' \subseteq E'$, we refer to G'' as a **subgraph** of G'.

Given a preference profile $R' \in \mathbf{R}$, let $E_{IR}[R'] \subseteq E_c$ denote the set of **individually rational** matches, where, for any $i, j \in \mathcal{I}$,

$$\{i, j\} \in E_{IR}[R'] \iff \begin{cases} jR'_i\emptyset \text{ and if } i \in \mathcal{E}(i) \text{ then } jR'_i i \\ \text{and} \\ iR'_j\emptyset \text{ and if } j \in \mathcal{E}(j) \text{ then } iR'_j j \end{cases}$$

The reduced compatibility graph $G_{IR}[R'] = (\mathcal{I}, E_{IR}[R'])$ is referred to as the **individually rational (IR) compatibility graph for** R'.

In a general graph (i.e. not necessarily the compatibility graph of a liver-exchange pool) in which all pairs are indifferent between all exchanges, one has to recursively expand the set of simultaneously *matchable* pairs to find an efficient matching (e.g., as in the cardinality matching algorithm of Edmonds, 1965). While we allow for more general preferences in our model, we still rely on similar tools to design our mechanism.

A set of pairs $\mathcal{J} \subseteq \mathcal{I}$ is **matchable** in a reduced compatibility graph $G' = (\mathcal{I}, E')$, if there exists a matching $M \in \mathbf{M}[G']$ such that $M(j) \neq \emptyset$ for all $j \in \mathcal{J}$.[20]

Fix a pair $i \in \mathcal{I}$, a preference relation $R_i \in \mathbf{R}_i$, a reduced compatibility graph G', and a set of pairs $\mathcal{J} \subset \mathcal{I} \setminus \{i\}$ such that $\mathcal{J} \cup \{i\}$ is matchable in G'. When members of $\mathcal{J}$ are all committed to be matched, define the set of **achievable assignments** of pair i (while members of $\mathcal{J}$ are all matched) as

$$\mathcal{A}(i|\mathcal{J}, G') = \Big\{ j \in \mathcal{I} \ : \ \exists M \in \mathbf{M}[G'] \text{ such that } M(h) \neq \emptyset \ \ \forall h \in \mathcal{J} \text{ and } M(i) = j \Big\},$$

and the set of **best achievable assignments** of i (while members of $\mathcal{J}$ are all matched) as

$$\mathcal{B}(i|\mathcal{J}, G') \equiv \max_{R_i} \mathcal{A}(i|\mathcal{J}, G').$$

Whenever a pair j is an achievable assignment for a pair i, the corresponding match $\{i, j\}$ is referred to as an **achievable match**. Similarly, whenever a pair j is a best achievable assignment for a pair i, the corresponding match $\{i, j\}$ is referred to as a **best achievable match**.

Whether a set of pairs is matchable or not can be checked in polynomial time.[21] Moreover, sets $\mathcal{A}(\cdot)$ and $\mathcal{B}(\cdot)$ can also be constructed in polynomial time.

We obtain the outcome of our proposed mechanism through an iterative algorithm introduced in Section 4.4. At each step, the algorithm determines whether certain pairs are matchable in a reduced compatibility graph that we refer to as the **active graph** of the algorithm. Using this information and the information on right-lobe donation willingness of pairs, the active graph is revised as we proceed through the algorithm. At the initiation of our algorithm, the active graph is the reduced compatibility subgraph consisting of left-lobe-only individually rational matches. If necessary, right-lobe transplant possibilities are introduced to the active graph in subsequent steps.

To formalize the definition of the initial active graph, we define R^0 as an auxiliary pref-

[20] Our definition of matchability differs from the standard definition in graph theory, which requires the subgraph induced by $\mathcal{J}$ to have a perfect matching. See, for instance, Schrijver (2003, Vol I, p59).

[21] We provide a polynomial-time method for checking matchability in Appendix C.2.

erence profile obtained from preference profile R that deems only the assignments involving left-lobe donations by pairs as individually rational: For all $i \in \mathcal{I}$, if $R_i = R_i^{a/x}$ where its participation type is $a \in \{d, m\}$ and its willingness type is $x \in \{u, w\}$ then R_i^0 keeps the same participation type but assumes that the pair is unwilling to donate a right lobe. Formally,

$$\forall i \in \mathcal{I} \quad R_i = R_i^{a/x} \implies R_i^0 \equiv R_i^{a/u}.$$

At the initiation of the algorithm, the active reduced compatibility graph is $G_0 = (\mathcal{I}, E_0)$, where

$$E_0 \equiv E_{IR}[R^0] = \Big\{\{i,j\} \in E_{IR}[R] \ : \ t(i,j) = \ell \text{ and } t(j,i) = \ell\Big\}.$$

The active graph is modified through the following two operators as we run the algorithm:

1. **Deletions:** Left-lobe donation possibilities of pairs are fully utilized by sequentially processing them following a fixed topological order and committing to each processed pair one of its best achievable matches while donating a left lobe. Whenever a pair is committed to receive one of its best achievable matches in our algorithm, all less-preferred matches are *deleted* from the active reduced compatibility graph, and subsequently obtaining a new active subgraph.
   Let $G' = (\mathcal{I}, E')$ be the active reduced compatibility graph prior to processing of pair i, and let $\mathcal{J}$ be the set of pairs whose members are committed for an assignment up to this point. If pair i is deemed matchable in G' in addition to pairs in $\mathcal{J}$, i.e., $\mathcal{A}(i|\mathcal{J}, G') \neq \emptyset$, the **deletion** operator induces a new reduced compatibility graph $G'' = (\mathcal{I}, E'')$ in the algorithm, where

$$E'' \equiv \Big(E' \setminus E'(i)\Big) \cup \Big\{\{i,j\} \in E'(i) \ : \ j \in \mathcal{B}(i|\mathcal{J}, G')\Big\}.$$

   That is, E'' is obtained from E' by deleting all matches of i except its best achievable matches. Instead of fixing a pair's assignment, we will use deletion to determine the indifference class of assignments that this pair will eventually be matched. This will help us to ensure the efficiency of the mechanism.[22]
2. **Transformations:** As each pair is processed sequentially, the options for pairs further ahead in the topological order potentially shrink. If a pair i cannot be matched through a left-lobe donation, that means, not only its existing left-lobe donation possibilities

[22]Fixing a pair's assignment may lead to inefficiency in a priority-based algorithm when pairs can be indifferent among different assignments. Suppose that there are four pairs with types $\tau(i_1) = (0,0,1) - (1,1,0,1)$, $\tau(i_2) = (0,1,1) - (1,1,0,1)$, $\tau(i_3) = (1,1,0-0,1,1,1)$, $\tau(i_4) = (1,1,0) - (1,0,1,1)$, and that each pair is unwilling. The set of individually rational matches is $E_{IR} = \big\{\{i_1,i_3\},\{i_1,i_4\},\{i_2,i_3\}\big\}$ and it involves only left-lobe matches. Suppose i_1 and i_3 are indifferent between their IR assignments. The precedence digraph is empty so any priority order of pairs is a topological order. Suppose we process the pairs in the order $i_1 - i_2 - i_3 - i_4$. If we fixed initially i_1's assignment as i_3, then we would preclude both i_2 and i_4 to be matched later. Instead, we will fix the indifference class of possible assignments of i_1 as i_3 and i_4, and we will be able to match all four pairs through $M = \big\{\{i_1,i_4\},\{i_2,i_3\}\big\}$ when we process i_2 after i_1.

are exhausted, but also any potential left-lobe donation possibilities are also exhausted by Lemma 1.[23] At this point we **transform** pair i deeming it available for right-lobe donation, for the first time and until the termination of the algorithm, provided that it is willing.

Given a preference profile R, let $G' = (\mathcal{I}, E')$ be the active graph at this point in our algorithm when pairs in a set $\widetilde{\mathcal{J}} \subseteq \mathcal{I} \setminus \{i\}$ are already transformed to donate their right lobes. After we transform pair i, the active graph becomes $G'' = (\mathcal{I}, E'')$, where

$$E'' \equiv E' \cup E_{IR}[R_{\widetilde{\mathcal{J}}\cup\{i\}}, R^0_{-\widetilde{\mathcal{J}}\cup\{i\}}](i).$$

That is, we include all individually rational matches involving pair i donating right lobe when pairs in $\widetilde{\mathcal{J}}$ are also in right-lobe donation mode while the other pairs are in left-lobe donation mode.[24] Since $E'' \supseteq E'$, a transformation potentially enlarges the set of matches in the active graph.

4.4 Precedence-Induced Adaptive-Priority Mechanism

We are ready to present an iterative algorithm, which can be used to find the outcome of our proposed **Precedence-Induced Adaptive-Priority Mechanism**, referred to as $f^{\mathbf{P}}$. The mechanism we introduce is defined for a given pair (Π_ℓ, Π_r) of priority orders, where Π_ℓ is a topological order over pairs and Π_r is any priority order over pairs. While the priority order Π_r is completely flexible and it can be the same as the topological order Π_ℓ, it does not have to be. We refer to Π_ℓ as the **left-lobe matching topological order**, and to Π_r as the **right-lobe matching priority order**. For the rest of the section, we fix the left-lobe matching topological order Π_ℓ and the right-lobe matching priority order Π_r, and focus on the uniquely defined mechanism $f^{\mathbf{P}}$.[25]

For the rest of this section, also fix a liver-exchange problem $R \in \mathbf{R}$.

Before stating the formal definition of the algorithm, we next provide an overview of how it operates using the tools and concepts we have introduced so far. There are two main steps. In each step, we process pairs sequentially in substeps and determine whether we are to commit to match them to one of the best assignments they can achieve. We keep track of these through

- two sequences of committed pair sets, one for left-lobe-committed and one for right-lobe-

[23]Observe that, processing the pairs following a topological order is key for this last argument to hold.

[24]A pair in $\widetilde{\mathcal{J}}$ could potentially donate left lobe to pair i. However, as we process pairs in the topological order, there will be no pair in $j \in \mathcal{J}$ such that transformation of i induces a new individually rational match $\{i, j\}$ such that i donates right lobe, while j donates left lobe.

[25]As it will be clear later, the algorithm for a given priority order profile, when executed for a given liver-exchange problem, will typically find an arbitrary matching from a subset of matchings. We interpret any matching from this set as an outcome of the induced mechanism $f^{\mathbf{P}}$, as all matchings in this outcome set are welfare equivalent.

committed pairs, such that each pair in each set is to be matched by the algorithm to one of its best achievable assignments either by donating left lobe or right lobe,

- a sequence of transformed willing pair sets, each of which includes pairs that could not be matched donating left lobe, but can potentially be matched donating right lobe, and
- a sequence of active reduced compatibility graphs each of which includes the matches that are deemed feasible so far.

Step 1 starts with the active graph that only consists of left-lobe-only individually rational matches (G_0 defined in Section 4.3). At this point, no pair is committed to be matched or transformed yet. We first use the left-lobe matching topological order Π_ℓ to process pairs sequentially. In each substep, we check whether the next pair in Π_ℓ is matchable (by donating its left lobe) in addition to the previously left-lobe-committed set of pairs in the active graph.

- If it is matchable, we commit to match it by donating its left lobe with one of its best achievable assignments in the active graph. To keep track of these assignments, we create a new active graph by deleting all matches of this pair except its best achievable matches from the latest active graph.
- If it is not matchable, this pair cannot be matched through left lobe donation without breaking at least one of the commitments to pairs who are processed earlier under the topological order Π_ℓ.[26] Thus, we check whether it is willing to donate a right lobe or not.
  - If it is not willing, this pair remains unmatched as all individually rational assignments of this pair are already committed to other pairs.
  - If it is willing, we add it to the set of transformed pairs and induce a new active graph by adding its all newly formed individually rational matches to the latest active graph.

So far, we committed pairs to be matched by donating their left lobes, if possible. Step 2 concerns only the commitment prospects of transformed pairs in Step 1. We start Step 2 with the reduced compatibility graph obtained at the end of Step 1 as its starting active graph. We also have the set of left-lobe committed pairs from Step 1. Using the right-lobe matching priority order Π_r, we process the transformed pairs sequentially. In each substep, we check whether the next transformed pair in Π_r is matchable in the active graph (by donating its right lobe) in addition to the previously left-lobe-committed and right-lobe-committed sets.

- If it is matchable, we commit to match it by donating a right lobe with one of its best achievable assignments in the active graph. To keep track of these assignments, we create a new active graph by deleting all matches of this pair except its best achievable matches from the latest active graph.
- If it is not matchable, this pair cannot be matched as all individually rational assignments of this pair are already committed to other pairs.

[26]This is true as we process pairs in a topological order induced by the precedence digraph: any future transformed pair will not induce new matches with this pair while it is donating left lobe.

When Step 2 terminates, we have a final active graph, a set of left-lobe-committed pairs, and a set of right-lobe-committed pairs. A matching of the active graph that matches all committed pairs is the outcome of our algorithm.

We are ready to formally present our algorithm:[27]

Precedence-Induced Adaptive-Priority-Matching Algorithm:

Step 1: Let $\mathcal{I} = \{i_1, \ldots, i_K\}$ be the enumeration of pairs with respect to the left-lobe matching topological order Π_ℓ. Define auxiliary preference profile $R^0 \in \mathbf{R}$ that is obtained from preference profile R and deems only the assignments involving left-lobe donations by pairs as individually rational: For all $i \in \mathcal{I}$, if $R_i = R_i^{a/x}$ where its participation type is $a \in \{d, m\}$ and its willingness type is $x \in \{u, w\}$ then R_i^0 keeps the same participation type but assumes the pair is unwilling to donate a right lobe:

$$\forall i \in \mathcal{I} \quad R_i = R_i^{a/x} \implies R_i^0 \equiv R_i^{a/u}.$$

We inductively construct

1. a sequence of reduced compatibility graphs

$$G_0 = (\mathcal{I}, E_0), \ldots, G_K = (\mathcal{I}, E_K)$$

   such that $E_k \subseteq E_{IR}[R]$ for all k, and
2. two sequences of enlarging pair sets

$$\mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \ldots \subseteq \mathcal{J}_K \subseteq \mathcal{I} \quad \text{and} \quad \widetilde{\mathcal{J}}_0 \subseteq \widetilde{\mathcal{J}}_1 \subseteq \ldots \subseteq \widetilde{\mathcal{J}}_K \subseteq \mathcal{I}$$

through substeps of Step 1. We refer to $\mathcal{J}_{k-1}$ as the set of left-lobe-committed pairs, $\widetilde{\mathcal{J}}_{k-1}$ as the set of transformed pairs, and G_{k-1} as the active graph at the beginning of Step 1.(k).

At the initiation, the active reduced compatibility graph is $G_0 = (\mathcal{I}, E_0)$ where

$$E_0 \equiv E_{IR}[R^0] = \Big\{\{i, j\} \in E_{IR}[R] \;:\; t(i, j) = \ell \text{ and } t(j, i) = \ell\Big\}.$$

That is, E_0 is restricted to individually rational matches in which each donor donates a left lobe. Also define

$$\mathcal{J}_0 \equiv \emptyset \quad \text{and} \quad \widetilde{\mathcal{J}}_0 \equiv \emptyset.$$

[27] Some of the readers may prefer to go through Example 4 at the end of this section, as well as the more detailed Example A-3 in Appendix D, prior to going through the formal definition of the algorithm.

Step 1's substeps proceed inductively:

Step 1.(k): Consider pair i_k, the k'th highest-priority pair under Π_ℓ. Sets $\mathcal{J}_{k-1}$, $\widetilde{\mathcal{J}}_{k-1}$, and the reduced compatibility graph $G_{k-1} = (\mathcal{I}, E_{k-1})$ are defined at Step 1.($k-1$).

- If $\mathcal{J}_{k-1} \cup \{i_k\}$ is matchable in G_{k-1}, then let

$$\mathcal{J}_k \equiv \mathcal{J}_{k-1} \cup \{i_k\}, \quad \widetilde{\mathcal{J}}_k \equiv \widetilde{\mathcal{J}}_{k-1}, \text{ and } G_k = (\mathcal{I}, E_k) \text{ where}$$

$$E_k \equiv \Big[E_{k-1} \setminus E_{k-1}(i_k)\Big] \cup \Big\{\{i_k, j\} \ : \ j \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_{k-1})\Big\}.$$

  That is, E_k is obtained from E_{k-1} by *deleting* all matches involving pair i_k except its best achievable matches (when all pairs in $\mathcal{J}_{k-1}$ can also be matched).

- Otherwise, let

$$\mathcal{J}_k \equiv \mathcal{J}_{k-1} \quad \text{and}$$

  * if i_k is not willing, then let

$$\widetilde{\mathcal{J}}_k \equiv \widetilde{\mathcal{J}}_{k-1} \text{ and } G_k = (\mathcal{I}, E_k) \equiv G_{k-1};$$

  * if i_k is willing, then let

$$\widetilde{\mathcal{J}}_k \equiv \widetilde{\mathcal{J}}_{k-1} \cup \{i_k\} \text{ and } G_k = (\mathcal{I}, E_k) \text{ where}$$

$$E_k \equiv E_{k-1} \cup E_{IR}[R_{\widetilde{\mathcal{J}}_k}, R^0_{-\widetilde{\mathcal{J}}_k}](i_k).$$

    That is, E_k is obtained by *transforming* pair i_k in addition to the previously transformed pairs while keeping other pairs as left-lobe donors and including newly formed individual rational matches to the active graph's set of edges.

Proceed with Step 1.($k+1$).

Step 1 terminates at substep K, where $K = |\mathcal{I}|$ is the number of pairs. This step determines:

1. the reduced compatibility graph $G_K = (\mathcal{I}, E_K)$, where one of its matchings is to be selected as the eventual outcome of the algorithm,
2. the finalized set of left-lobe-committed pairs $\mathcal{J}_K$, whose members will each be matched by donating a left lobe, and their eventual indifference class of assignments with one of which they will be matched,
3. the set of pairs $\mathcal{I} \setminus [\mathcal{J}_K \cup \widetilde{\mathcal{J}}_K]$, whose members –unwilling to donate a right lobe – definitely remain unmatched, and

4. the finalized set of transformed pairs $\widetilde{\mathcal{J}}_K$, whose members may be matched by donating right lobe or remain unmatched.

We proceed with Step 2 that determines which pairs in $\widetilde{\mathcal{J}}_K$ will be matched donating right lobe and their eventual indifference class of assignments (with one of which they will be matched), along with the final outcome of the algorithm.

Step 2: Inductively, we continue with the pairs in $\widetilde{\mathcal{J}}_K$, whose members can be potentially matched –albeit through right-lobe donation– in addition to members of $\mathcal{J}_K$ (who are already committed to be matched). Let $\widetilde{\mathcal{J}}_K = \{i_1^*, \ldots, i_N^*\}$ be the enumeration of those pairs with respect to the right-lobe matching priority order Π_r. We construct

1. a sequence of shrinking reduced compatibility graphs
$$G_0^* = (\mathcal{I}, E_0^*), \ldots, G_N^* = (\mathcal{I}, E_N^*)$$
such that $E_N^* \subseteq \ldots \subseteq E_0^*$, and
2. a sequence of enlarging pair sets
$$\mathcal{J}_0^* \subseteq \ldots \subseteq \mathcal{J}_N^*.$$
We refer to $\mathcal{J}_{n-1}^*$ as the set of right-lobe-committed pairs and G_{n-1}^* as the active graph at the beginning of Step 2.(n).

At the initiation of Step 2, set
$$\mathcal{J}_0^* \equiv \emptyset \quad \text{and} \quad G_0^* = (\mathcal{I}, E_0^*) \equiv G_K.$$

Step 2.(n): Consider i_n^*, the n'th highest-priority pair in $\widetilde{\mathcal{J}}_K$ according to the right-lobe matching priority order Π_r. Pair set $\mathcal{J}_{n-1}^*$ and reduced compatibility graph $G_{n-1}^* = (\mathcal{I}, E_{n-1}^*)$ are constructed at Step 2.$(n-1)$.

- If $\mathcal{J}_K \cup \mathcal{J}_{n-1}^* \cup \{i_n^*\}$ is matchable in G_{n-1}^*, then let
$$\mathcal{J}_n^* \equiv \mathcal{J}_{n-1}^* \cup \{i_n^*\} \text{ and } G_n^* = (\mathcal{I}, E_n^*) \text{ where}$$
$$E_n^* \equiv \Big[E_{n-1}^* \setminus E_{n-1}^*(i_n^*)\Big] \cup \Big\{\{i_n^*, j\} \ : \ j \in \mathcal{B}(i_n^* | \mathcal{J}_K \cup \mathcal{J}_{n-1}^*, G_{n-1}^*)\Big\}.$$
That is, E_n^* is obtained from E_{n-1}^* by *deleting* the matches involving pair i_n^*, except its best achievable matches (when all pairs in $\mathcal{J}_K \cup \mathcal{J}_{n-1}^*$ are also matched).
- Otherwise, let
$$\mathcal{J}_n^* \equiv \mathcal{J}_{n-1}^* \quad \text{and} \quad G_n^* = (\mathcal{I}, E_n^*) \equiv G_{n-1}^*.$$

Proceed with Step 2.$(n+1)$.

When Step 2 terminates at substep $N = |\widetilde{\mathcal{J}}_K|$, the mechanism picks, as its outcome, a matching of G^*_N that matches all pairs in $\mathcal{J}_K$ and $\mathcal{J}^*_N$.

As we prove in Lemma 3 in Section 5, each pair is indifferent among all matchings of G^*_N that match all pairs in $\mathcal{J}_K$ and $\mathcal{J}^*_N$. Thus, the welfare achieved by the mechanism is uniquely defined although the matching it chooses is not uniquely defined.[28]

We illustrate how the algorithm works on the simple problem in Example 1.

Example 4 *Consider the liver exchange problem in Example 1. In Example 3, we showed that the unique topological order for its exchange pool is*

$$\Pi_\ell = i_3 - i_2 - i_1.$$

We fix this as the left-lobe matching topological order. And suppose right-lobe matching priority order is

$$\Pi_r = i_1 - i_2 - i_3.$$

We illustrate the steps of the algorithm for (Π_ℓ, Π_r)*:*

Step 1. *Each pair's auxiliary preferences are defined as follows: For each* $i \in \mathcal{I}$*, as* $R_i = R_i^{m/w}$*, we set*

$$R_i^0 \equiv R_i^{m/u}.$$

The initial active reduced compatibility has no edges in it as all matches in Figure 2 involves a right-lobe transplant: $E_0 \equiv E_{IR}[R^0] = \emptyset$ *and* $G_0 = (\mathcal{I}, E_0)$*. We also initialize* $\mathcal{J}_0 \equiv \emptyset$ *and* $\widetilde{\mathcal{J}}_0 \equiv \emptyset$*.*

Step 1.(1): $\mathcal{J}_0 \cup \{i_3\} = \{i_3\}$ *is not matchable in* G_0*, which is the empty graph. Thus,* $\mathcal{J}_1 \equiv \mathcal{J}_0 = \emptyset$*, the set of left-lobe-committed pairs does not change. However, pair* i_3 *is willing. Therefore, we transform it and obtain* $\widetilde{\mathcal{J}}_1 \equiv \widetilde{\mathcal{J}}_0 \cup \{i_3\} = \{i_3\}$*. We add to* E_0 *its only newly formed individually rational match in which* i_2 *donates a left lobe while* i_3 *donates a right lobe:*

$$E_1 \equiv E_0 \cup \Big\{\{i_3, i_2\}\Big\} = \Big\{\{i_3, i_2\}\Big\}$$

and $G_1 = (\mathcal{I}, E_1)$*.*

Step 1.(2): $\mathcal{J}_1 \cup \{i_2\} = \{i_2\}$ *is matchable in* G_1*:* $M = \Big\{\{i_2, i_3\}\Big\}$ *is the unique matching that matches* i_2*. Therefore, we commit it to be matched while donating a left lobe:* $\mathcal{J}_2 \equiv \mathcal{J}_1 \cup \{i_2\} = \{i_2\}$*. It is not transformed; therefore,* $\widetilde{\mathcal{J}}_2 = \widetilde{\mathcal{J}}_1$*. Pair* i_2 *has one achievable match in* G_1*. Therefore, we do not change the active graph:* $G_2 \equiv G_1$*.*

[28] We provide a polynomial-time method for how to find such a matching in Appendix C.2.

Step 1.(3): $\mathcal{J}_2 \cup \{i_1\} = \{i_2, i_1\}$ *is not matchable in* G_2*:* $M = \Big\{\{i_2, i_3\}\Big\}$ *is the unique one that matches the only pair in* $\mathcal{J}_2$*,* i_2*; but it leaves* i_1 *unmatched. Therefore,* $\mathcal{J}_3 \equiv \mathcal{J}_2 = \{i_2\}$*. Pair* i_1 *is willing and thus, it is transformed; therefore,* $\widetilde{\mathcal{J}}_3 \equiv \widetilde{\mathcal{J}}_2 \cup \{i_1\} = \{i_1, i_3\}$*. However, this transoformation does not induce new individually rational matches:* $G_3 \equiv G_3$*.*

Step 2. *We initialize the right-lobe-committed set to* $\mathcal{J}_0^* \equiv \emptyset$*, the active graph to* $G_0^* \equiv G_3 = G_1$*. The right-lobe matching priority order* Π_r *orders the transformed pairs as* $\Pi_r = i_1 - i_3$*.*

Step 2.(1): $\mathcal{J}_3 \cup \mathcal{J}_0^* \cup \{i_1\} = \{i_2, i_1\}$ *is not matchable in* G_0^* *for the same reason as in Step 1.(3). Pair* i_1 *will stay unmatched:* $\mathcal{J}_1^* \equiv \mathcal{J}_0^* = \emptyset$ *and* $G_1^* \equiv G_0^*$*.*

Step 2.(2): $\mathcal{J}_3 \cup \mathcal{J}_0^* \cup \{i_3\} = \{i_2, i_3\}$ *is matchable in* G_1^**:* $M = \Big\{\{i_2, i_3\}\Big\}$ *is the unique such matching. Thus, we commit to match pair* i_3 *while donating a right lobe:* $\mathcal{J}_2^* \equiv \mathcal{J}_1^* \cup \{i_3\} = \{i_3\}$*. As there is only one achievable match in* G_1^* *for* i_3*, we do not change active graph:* $G_2^* \equiv G_1^*$*.*

The outcome of the mechanism is a matching of G_2^* *that matches* $\mathcal{J}_3 \cup \mathcal{J}_2^* = \{i_2, i_3\}$*. There is one such matching:*

$$f^{\mathbf{P}}[R] \equiv \Big\{\{i_2, i_3\}\Big\}.$$

The algorithm prevents the possible manipulation by pair i_2 *under an arbitrary priority order as illustrated in Example 1, since it processes* i_2 *before* i_1 *using the concept of topological order of the precedence digraph. Thus,* i_2 *is never transformed, and hence, it does not need to misrepresent its willing preferences.*

We also illustrate how the algorithm works and how the detailed constructions in substeps are used with a more involved example in Appendix D.

5 Results

The main result of our paper is as follows:

Theorem 1 *A precedence-induced adaptive-priority mechanism is individually rational, Pareto efficient, and incentive compatible.*

To prove Theorem 1, we rely on some additional notation and preliminary results presented below.

We first state the properties of the reduced compatibility graphs, the resulting matchings, and the sets of pairs constructed through the steps of our algorithm. We define the following sets of matchings, which can be interpreted as the outcomes of Step 1.(k) for any

$k \in \{1, \ldots, K\}$ and Step 2.(n) for any $n \in \{1, \ldots, N\}$ of the mechanism. We analyze the properties of these sets of matchings in Lemmas 2 and 3 below.

Define

$$\mathbf{M}_k \equiv \{M \in \mathbf{M}[G_k] \ : \ M(j) \neq \emptyset \ \ \forall j \in \mathcal{J}_k\}$$

as the subset of matchings of G_k, the active graph at the end of Step 1.(k), such that each matching in this set matches each pair in $\mathcal{J}_k$, which is the set of committed pairs up to the end of Step 1.(k).

Define

$$\mathbf{M}^*_n \equiv \{M \in \mathbf{M}[G^*_n] \ : \ M(j) \neq \emptyset \ \ \forall j \in \mathcal{J}_K \cup \mathcal{J}^*_n\}$$

as the subset of matchings of G^*_n, the active graph of the algorithm at the end of Step 2.(n), such that each matching in this set matches each pair in $\mathcal{J}_K \cup \mathcal{J}^*_n$, which is the set of committed pairs up to the end of Step 2.(n). Thus, any matching in $\mathbf{M}^*_N$ can be the outcome of the precedence-induced adaptive-priority mechanism.

We state the properties of matchings in $\mathbf{M}_K$ below in Lemma 2.

Lemma 2 (Properties of Constructs in Step 1) *Consider the sequences of reduced compatibility graphs $\{G_k\}_{k=1}^K$ and pair sets $\{\mathcal{J}_k, \widetilde{\mathcal{J}}_k\}_{k=1}^K$ constructed through the substeps of Step 1 of the precedence-induced adaptive-priority-matching algorithm.*

1. *Set of pairs $\mathcal{J}_K$, i.e., the set of pairs that are committed to be matched in Step 1, is indeed matchable in G_K, the active graph at the end of Step 1.*
2. *$\mathbf{M}_K \neq \emptyset$, and, for any matching $M \in \mathbf{M}_K$,*
   (a) *for all $j \in \mathcal{I} \setminus \widetilde{\mathcal{J}}_K$ and all $M' \in \mathbf{M}_K$, $M(j) \ I_j \ M'(j)$; that is, any pair that is not transformed in Step 1 is indifferent between any two matchings in $\mathbf{M}_K$,*
   (b) *for all $j \in \mathcal{J}_K$, $M(j) \in \mathcal{E}^\ell(j)$; that is, any pair that is committed to be matched in Step 1, is matched by donating its left lobe in M,*
   (c) *for all $j \in \widetilde{\mathcal{J}}_K$, $M(j) \notin \mathcal{E}^\ell(j)$; that is, any pair that is transformed in Step 1 is not matched by donating its left lobe in M,*
   (d) *for all $j \in \mathcal{I} \setminus [\mathcal{J}_K \cup \widetilde{\mathcal{J}}_K]$, $M(j) = \emptyset$; that is any pair that is neither committed to be matched nor transformed in Step 1 remains unmatched in M,*
   (e) *for all $i_k \in \mathcal{J}_K$, $M(i_k) \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_K)$; moreover, $M(i_k) \ I_{i_k} \ j$ for all $j \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_k)$; that is, each committed pair i_k is matched in M to one of its best available assignments in G_K when all pairs committed prior to i_k are also matched;*
   moreover, in M, i_k's assignment is indifferent to one of its best achievable assignments in G_k, the active graph at the end of Step 1.(k), when all pairs committed prior to i_k are also matched.[29]

We state the properties of matchings in $\mathbf{M}^*_N$ below in Lemma 3.

[29]The proofs of this lemma and other results in this section are in Appendix B.

Lemma 3 (Properties of Constructs in Step 2) *Consider the sequences of reduced compatibility graphs $\{G_n^*\}_{n=1}^N$ and pair sets $\{\mathcal{J}_n^*\}_{n=1}^N$ constructed through the substeps of Step 2 of the precedence-induced adaptive-priority-matching algorithm.*

1. $\mathbf{M}_N^* \subseteq \mathbf{M}_K$ *and* $\mathcal{J}_N^* \subseteq \widetilde{\mathcal{J}}_K$.
2. *Set of pairs $\mathcal{J}_K \cup \mathcal{J}_N^*$, the set of pairs committed in Steps 1 and 2, is matchable in G_N^*, the final active graph of the algorithm.*
3. $\mathbf{M}_N^* \neq \emptyset$, *and, for any matching $M \in \mathbf{M}_N^*$,*
   (a) *for all $j \in \mathcal{I}$ and all $M' \in \mathbf{M}_N^*$, $M(j)\ I_j\ M'(j)$; that is any pair is indifferent between any two matchings of set $\mathbf{M}_N^*$,*
   (b) *for all $j \in \mathcal{J}_K$, $M(j) \in \mathcal{E}^\ell(j)$; that is, all pairs committed in Step 1 are matched by donating left lobe in M,*
   (c) *for all $j \in \mathcal{J}_N^*$, $M(j) \in \mathcal{E}^r(j)$; that is, all pairs committed in Step 2 are matched by donating right lobe in M,*
   (d) *for all $j \in \mathcal{I} \setminus [\mathcal{J}_K \cup \mathcal{J}_N^*]$, $M(j) = \emptyset$; that is, all non-committed pairs remain unmatched in M,*
   (e) *for all $i_k \in \mathcal{J}_K$, $M(i_k) \in \mathcal{B}(i_k|\mathcal{J}_{k-1}, G_N^*)$; that is, each committed pair i_k in Step 1 is matched in M to one of its best available assignments in G_N^* when all pairs committed prior to i_k are also matched, and*
   (f) *for all $i_n^* \in \mathcal{J}_N^*$, $M(i_n^*) \in \mathcal{B}(i_n^*|\mathcal{J}_K \cup \mathcal{J}_{n-1}^*, G_N^*)$; moreover, $M(i_n^*) \in \mathcal{B}(i_n^*|\mathcal{J}_K \cup \mathcal{J}_{n-1}^*, G_n^*)$; that is, each committed pair i_n^* in Step 2 is matched in M to one of its best available assignments in G_N^* and also in G_n^* when all pairs committed prior to i_n^* in Steps 1 and 2 are also matched.*

We state the following corollary to Lemma 3 regarding our mechanism:

Corollary 1 *Fix a liver-exchange problem, a left-lobe matching topological order, and a right-lobe matching priority order. For any given pair, the pair receives the same welfare under all possible outcome matchings of the precedence-induced adaptive priority mechanism.*

Parts of these two lemmas will also be used in proving our main result, Theorem 1 given at the beginning of the section.

5.1 Transplant Maximization and Incentive Compatibility

It is well known that when all patients are indifferent among compatible grafts and right-lobe donation is not allowed, every Pareto efficient matching maximizes the number of transplants (see Korte and Vygen, 2011, Roth, Sönmez, and Ünver, 2005, and Sönmez and Ünver, 2014).

When right-lobe donation becomes feasible, this equivalence no longer holds. Moreover, although our proposed precedence-induced adaptive-priority mechanism is Pareto efficient, it may not maximize the number of transplants even when there are only two sizes and all patients are indifferent among all compatible grafts.

The example below shows that one has to sacrifice incentive compatibility in order to maximize the number of transplants, even when patients are indifferent among all compatible grafts. Indeed the same example also shows that, one has to sacrifice incentive compatibility in order to maximize the number of the safer left-lobe transplants as well.

Example 5 *Suppose there are two sizes, small and large, denoted as* $\mathbf{S} = \{0, 1\}$*. Consider a liver-exchange pool with* $\mathcal{I} = \{i_1, i_2, i_3, i_4\}$*. The pair types are given as follows:*

$$\begin{aligned} \tau_P(i_1) &= (1,0,1) & \tau_D(i_1) &= (0,1,1,1), \\ \tau_P(i_2) = \tau_P(i_4) &= (0,1,1) & \tau_D(i_2) = \tau_D(i_4) &= (1,0,0,1) \\ \tau_P(i_3) &= (1,0,0) & \tau_D(i_3) &= (0,1,1,1). \end{aligned}$$

Suppose all pairs are indifferent among all compatible grafts under the received-graft preference profile $\succsim$*. Suppose also that, pairs* i_2 *and* i_4 *are both willing. The individually rational compatibility graph of this problem is given in Figure 5.*

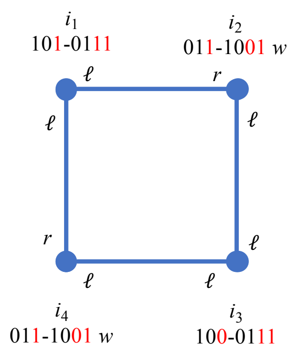

Figure 5: The individually rational compatibility graph for Example 5. The willing types are denoted by letter w following their types. The left- and right-lobe donations are denoted by letters ℓ and r. There are four individually rational exchanges.

Any left-lobe-donation-maximizing or total-transplant-maximizing matching (two of which can be obtained by swapping i_2 *and* i_4 *with each other) generates two exchanges. Consider these two matchings:*

$$M = \Big\{\{i_1, i_2\}, \{i_3, i_4\}\Big\} \quad \& \quad M' = \Big\{\{i_1, i_4\}, \{i_2, i_3\}\Big\}.$$

Observe that $t(i_2, i_1) = t(i_4, i_1) = r$ *while* $t(i_2, i_3) = t(i_4, i_3) = \ell$*. Any (probabilistic) mechanism that chooses a matching with the maximum number of transplants or the maximum* *number of left-lobe transplants chooses at least one of these two matchings in its support. Without loss of generality, suppose M is that matching. Then i_2 has an incentive to announce its right-lobe donation willingness type as unwilling by revealing $R'_{i_2} = R^{m/u}_{i_2}$, as the mechanism will choose M', which is the unique left-lobe-donation- and total-transplant-maximizing matching in this case, with probability 1. Hence, there is no incentive-compatible mechanism that maximizes the total number of transplants or left-lobe transplants.*

Example 5 also serves as a proof for the following impossibility result:

Proposition 1 *There is no individually rational and incentive-compatible mechanism that maximizes the number of transplants or the number of left-lobe transplants even when all patients are indifferent among compatible grafts.*

Establishing such an impossibility is straightforward when received-graft preferences admit strict preferences, thus we skip it.

It is instructive to find outcome of the precedence-adjusted priority mechanism for Example 5, with particular emphasis on how it prevents manipulation:

Example 6 *The precedence digraph over the pairs for the problem in Example 5 is $(\mathcal{I}, D)$ with directed edge set*

$$D = \Big\{(i_2, i_1), (i_4, i_1)\Big\}.$$

Hence, in all topological orders of this digraph, i_2 and i_4 are ordered before i_1, but otherwise the ordering is arbitrary. Recall that i_2 and i_4 are of the same pair type.

Suppose we fix a topological order Π_ℓ that orders i_2 before i_4 and i_4 before i_1 as the left-lobe matching topological order and an arbitrary right-lobe matching priority order Π_r. The outcome of our mechanism is

$$f^{\mathbf{P}}[R] = \Big\{\{i_2, i_3\}, \{i_4, i_1\}\Big\},$$

as i_2 is ordered before i_4 it will get the unique left-lobe matching opportunity with i_3, and as i_4 is willing, it will be matched with i_1 by donating right lobe.

On the other hand, if i_4 announced its right-lobe donation willingness type as unwilling, i.e.,

$$R'_{i_4} = R^{m/u}_i$$

then the outcome of the mechanism would be

$$f^{\mathbf{P}}[R'_{i_4}, R_{-i_4}] = \Big\{\{i_2, i_3\}\Big\}.$$

Thus, i_4 would remain unmatched. Instead of finding the unique maximum matching for the *problem* (R'_{i_4}, R_{-i_4})

$$M = \Big\{\{i_1, i_2\}, \{i_3, i_4\}\Big\},$$

our mechanism matches two pairs Pareto efficiently. This ensures our mechanism is incentive compatible for this problem.

6 Simulations

In this section, we report the results of computer simulations to determine the potential welfare gains from liver exchange. We use South Korean aggregate statistics in our simulations, since this country leads the world both in living-donor liver transplants and in liver exchange.

Calibration Statistics for Simulations from South Korean Population				
	Live-Donation Recipients	**Live Donors**	**Height (cm)**	
Female	1492 (34.55%)	1149 (26.61%)	*Mean:* 157.40	*Std Dev:* 5.99
Male	2826 (64.45%)	3169 (73.39%)	*Mean:* 170.70	*Std Dev:* 6.40
Total	4318 (100.0%)	4318 (100.0%)		
Blood-Type Distribution				
O	**A**	**B**	**AB**	**Total**
37%	33%	21%	9%	100%

Table 1: Calibration statistics from South Korea for liver-exchange simulations. Blood-type distribution is obtained from `http://bloodtypes.jigsy.com/East_Asia-bloodtypes` on 04/10/2016. Mean and standard deviation for South Korean adult height distribution are obtained from the Korean Agency for Technology and Standards (KATS) website `http://sizekorea.kats.go.kr` on 04/10/2016. The transplant data is obtained from the Korean Network for Organ Sharing (KONOS) 2014 Annual Report, retrieved from `http://www.konos.go.kr/konosis/index.jsp` on 04/10/2016 and contains the years 2010–2014.

Table 1 summarizes the calibration parameters used in our simulations. Each patient is assumed to be paired with a donor. Blood type, gender, and height characteristics for patients and their donors are determined independently and randomly.[30]

A donor and patient are deemed left-lobe compatible if they are blood-type compatible and the donor's left lobe volume is at least 40% of the total liver volume of the patient. A donor and patient are deemed right-lobe-only compatible if they are blood-type compatible,

[30]We use the following weight determination formula as a function of height (also see Ergin et al. 2017): $w = a\, h^b$, where w is weight in kilograms, h is height in meters, and constants a and b are set as $a = 26.58, b = 1.92$ for males and $a = 32.79, b = 1.45$ for females (Diverse Populations Collaborative Group, 2005). The body surface area (BSA in m^2) of an individual is determined through the Mostellar formula given in Um et al. (2015) as $BSA = \frac{\sqrt{h\, w}}{6}$, and the liver volume ($l_v$ in ml) of Korean adults is determined through the estimated formula in Um et al. (2015) as $l_v = 893.485\, BSA - 439.169$. Each patient and donor have a height drawn independently from the truncated normal distribution using the mean and std. dev. reported in this table with the support [mean - 3 std. dev., mean + 3 std. dev.]. We assume that the left lobe of each donor is 35% of all his liver, as this is reported as the mean of the left-lobe volume in Korea (Um et al. 2015).

the donor's right-lobe volume is at least 40% of the total liver volume of the patient, although the donor's left lobe volume is less than 40% of the total liver volume of the patient.

We generate $K = 50$, 100, and 250 patient-donor pairs in three sets of simulations. Since we do not have empirical statistics on the willingness of donors for right-lobe donation, we consider 6 scenarios for each population size in which on average $0, 20, 40, 60, 80$, and 100% of all pairs are willing. For each willingness rate, we randomly determine each pair's willingness.

We make the following two assumptions for preferences of pairs, as we do not have a better measure of more nuanced preferences over liver exchanges:

1. All pairs are *direct-transplant biased*. This implies all left-lobe compatible pairs prefer a direct transplant to any type of exchange and all right-lobe compatible pairs prefer a direct transplant to any exchange in which they donate right lobe. Incompatible pairs are not affected by this assumption.
2. All pairs are indifferent among all compatible received-grafts in their received-graph preferences $\succsim$.

These two assumptions are standard in the kidney exchange literature, as well as most of its real-life applications. In most kidney exchange applications, pairs have direct-transplant bias as they may not want to wait for an exchange. Moreover, transplant doctors mostly care about the compatibility of the received graft as the first-order coarse received-graph preference relation. Thus, we expect these also to be the case in liver exchange, especially in the short run until more data becomes available regarding exchanges.

We consider the following four treatments:

1. *No exchange.* Left-lobe-compatible pairs and right-lobe-only-compatible willing pairs participate in direct transplants.
2. *RSÜ priority mechanism for left-lobe exchanges.* Left-lobe-compatible pairs participate exclusively in direct transplants. Restricting the compatibility graph to left-lobe-only exchanges in the remaining problem, the outcome of the Roth, Sönmez, and Ünver (2005) (RSÜ) priority mechanism, mainly introduced for kidney exchange, is determined for an arbitrary priority order. Right-lobe-only-compatible willing pairs participate in direct transplants only if they cannot be matched through left-lobe only exchanges.
3. *Proposed Pareto-efficient, individually rational, and incentive-compatible mechanism.* An outcome of our precedence-induced adaptive-priority mechanism is determined for arbitrary topological and priority orders.
4. *A maximum individually rational matching under full information.* Assuming that the willingness profile is known, we find a maximum individually rational matching as follows: We first deem each left-lobe compatible pair only compatible with themselves. For willing right-lobe-only compatible pairs, in addition to their left-lobe transplant options, we only make direct (right-lobe) transplant a feasible option. We transform all willing in-

compatible pairs at the initiation, deeming them available for right-lobe transplantation right away. Then, we find a maximum matching of the induced compatibility graph.[31]

The second treatment depicts a baseline scenario for measuring the benefits from liver exchange using off-the-shelf methods introduced for kidney exchange. Thus, it utilizes exchanges only for left-lobe transplants. Although this procedure is individually rational and incentive compatible as a mechanism, it is not Pareto efficient.

The fourth treatment depicts a hypothetical situation assuming willingness profiles of the pairs are known. We use this as a benchmark when the goal is to maximize number of transplants. This procedure is not incentive compatible as a mechanism.

The results of the simulations are given in Table 2 and Figure 6.[32] About 12.5% of all pairs are left-lobe compatible and their patients receive a direct left-lobe transplant. About 45.5% of all pairs are right-lobe-only compatible, and up to this percentage of the patients receive a direct right-lobe transplant as a linear, increasing function of the willingness rate (see the no exchange treatment in the figure). Therefore, in the absence of liver exchange, 12.5% to 58.0% of patients with living donors receive a direct transplant as a linear, increasing function of the willingness rate. Our mechanism, on the other hand, matches from 18% to 78% of all pairs, in a seemingly concave, increasing function of the willingness rate for $K = 100$ (see proposed PE&IR&IC treatment in the figure).[33] Thus, for a population size of $K = 100$, the percentage-wise increase in the number of transplants due to exchange is in the range of 44% to 34%, higher for the lower values of the willingness rate.[34] Our proposed mechanism not only increases the number of living-donor liver transplants, but also increases the reliance on the lower-risk left-lobe liver transplantation in the spirit of the central tenet of the hippocratic oath "first do no harm." For example, when all pairs are willing, the share of left-lobe transplants increases from 21.5% to 31.1%. In general for any willingness rate, the rate of increase in left-lobe transplants is higher than the rate of increase in right-lobe

[31] We implement the Sönmez and Ünver (2014) priority mechanism in this case to find a maximum matching from an arbitrary priority order. This mechanism is maximum when the compatibility graph includes two-way exchanges and direct transplants.

[32] We caution the readers that we do not consider the possible genetic relationship between a paired donor and patient and assume that their blood types and sizes are independently distributed. Also pairs consisting of spouses may have positive correlation for their sizes although their blood types are unrelated. This independence assumption works in favor of our simulated gains from exchange. Also we only consider adult patients. Living-donor transplants from parents to their children are in non-negligible numbers in countries such as the US. The left-lobe compatibility instances within such pairs should be more frequent than within baseline pairs. The exclusion of such pairs works in favor of our simulated gains from exchange as well. On the other hand, we used the recipient percentages to determine the gender of donors and patients. Females are in general smaller than males. This data has selection bias as probably we observe more size-compatible pairs than the underlying entry population. This effect works against our exchange simulations.

[33] This concavity is caused by the initial fast increase in the scope of exchange when right-lobe donation becomes feasible for lower willingness fractions. For example, for w-fraction equal to 0, the exchange's contribution is rather low only an additional 10% of pairs are matched over no exchange.

[34] The increase in the number of transplants (rather than the percentage-wise increase compared to no exchange scenario) is higher for higher willingness rates.

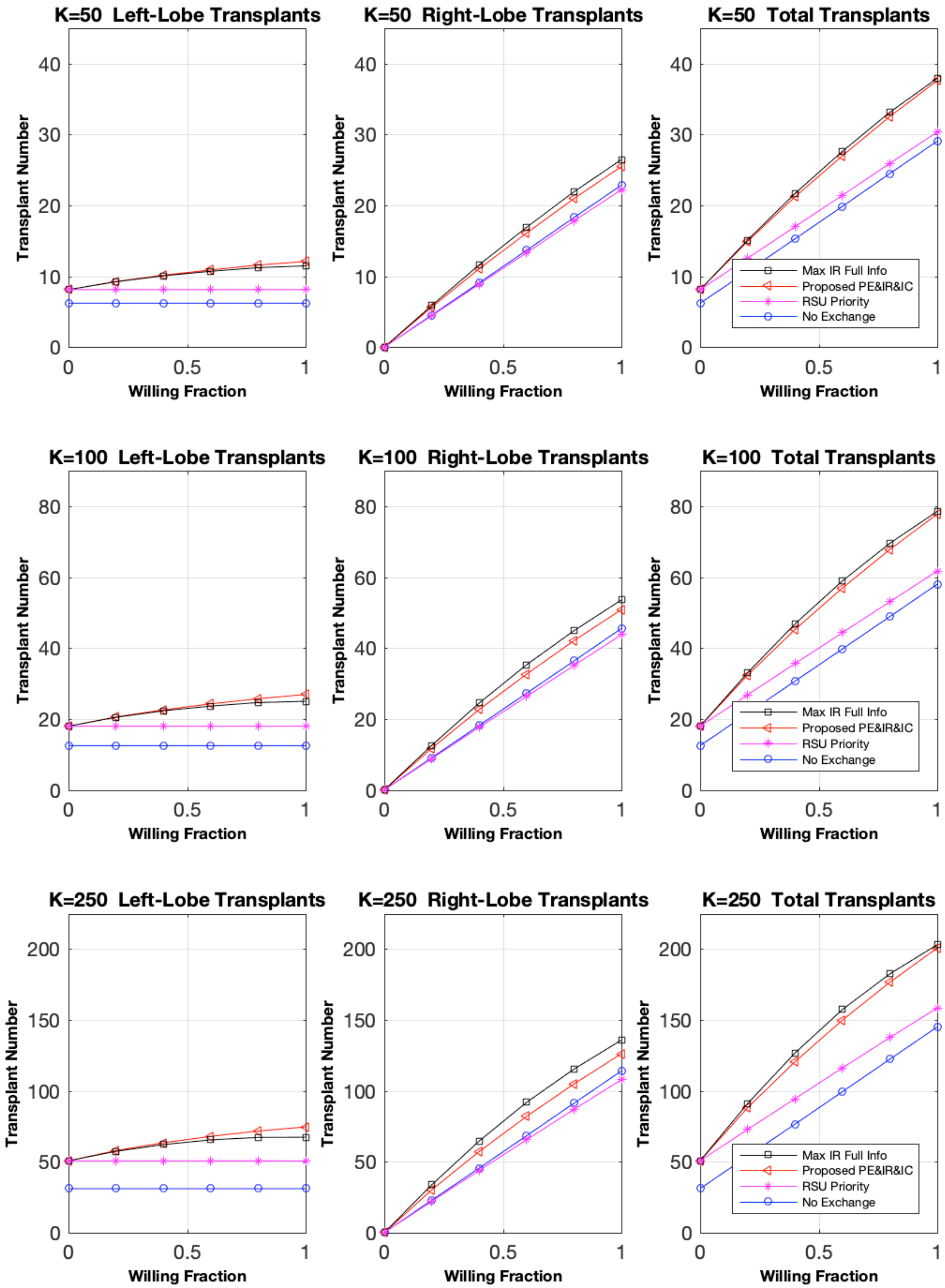

Figure 6: Simulation averages

		No Exchange			RSÜ Priority Mechanism			Proposed PE&IR&IC Mechanism			A Maximum IR Matching under Full Information		
Pop. Size K	w Fraction	Transplants Left L.	Right L.	Total	Transplants Left L.	Right L.	Total	Transplants Left L.	Right L.	Total	Transplants Left L.	Right L.	Total
	0	6.204	0	6.204	8.122	0	8.122	8.122	0	8.122	8.122	0	8.122
		(2.280)	(0.000)	(2.280)	(2.831)	(0.000)	(2.831)	(2.831)	(0.000)	(2.831)	(2.831)	(0.000)	(2.831)
	0.2	6.204	4.529	10.733	8.12	4.439	12.559	9.272	5.649	14.921	9.232	5.885	15.117
		(2.280)	(2.064)	(2.912)	(2.826)	(2.050)	(3.305)	(2.869)	(2.270)	(3.609)	(2.855)	(2.313)	(3.694)
	0.4	6.204	9.081	15.285	8.12	8.881	17.001	10.164	11.059	21.223	10.074	11.616	21.69
50		(2.280)	(2.810)	(3.330)	(2.826)	(2.775)	(3.644)	(2.920)	(2.899)	(3.966)	(2.895)	(2.993)	(4.139)
	0.6	6.204	13.665	19.869	8.122	13.318	21.44	10.904	16.103	27.007	10.715	16.922	27.637
		(2.280)	(3.179)	(3.491)	(2.831)	(3.150)	(3.756)	(2.923)	(3.211)	(3.945)	(2.887)	(3.296)	(4.096)
	0.8	6.204	18.301	24.505	8.122	17.808	25.93	11.589	20.963	32.552	11.224	21.932	33.156
		(2.280)	(3.496)	(3.581)	(2.831)	(3.489)	(3.766)	(2.931)	(3.476)	(3.825)	(2.868)	(3.542)	(3.924)
	1	6.204	22.87	29.074	8.12	22.239	30.359	12.101	25.507	37.608	11.481	26.465	37.946
		(2.280)	(3.624)	(3.507)	(2.832)	(3.671)	(3.607)	(2.940)	(3.614)	(3.558)	(2.833)	(3.695)	(3.630)
	0	12.497	0	12.497	17.995	0	17.995	17.995	0	17.995	17.995	0	17.995
		(3.367)	(0.000)	(3.367)	(4.526)	(0.000)	(4.526)	(4.526)	(0.000)	(4.526)	(4.526)	(0.000)	(4.526)
	0.2	12.497	9.097	21.594	17.995	8.839	26.834	20.593	11.754	32.347	20.484	12.56	33.044
		(3.367)	(2.931)	(4.255)	(4.526)	(2.909)	(5.107)	(4.586)	(3.263)	(5.544)	(4.582)	(3.401)	(5.752)
	0.4	12.497	18.196	30.693	17.991	17.641	35.632	22.594	22.671	45.265	22.329	24.5	46.829
100		(3.367)	(3.928)	(4.605)	(4.520)	(3.865)	(5.298)	(4.548)	(4.079)	(5.725)	(4.514)	(4.310)	(6.082)
	0.6	12.497	27.277	39.774	17.989	26.378	44.367	24.292	32.679	56.971	23.667	35.318	58.985
		(3.367)	(4.442)	(4.814)	(4.524)	(4.412)	(5.336)	(4.545)	(4.479)	(5.595)	(4.380)	(4.761)	(5.944)
	0.8	12.497	36.402	48.899	17.989	35.106	53.095	25.738	42.111	67.849	24.658	44.937	69.595
		(3.367)	(4.884)	(5.032)	(4.524)	(4.935)	(5.323)	(4.526)	(4.895)	(5.389)	(4.318)	(4.999)	(5.572)
	1	12.497	45.561	58.058	17.971	43.806	61.777	26.945	50.897	77.842	25.006	53.629	78.635
		(3.367)	(5.186)	(5.062)	(4.513)	(5.306)	(5.234)	(4.514)	(5.138)	(5.243)	(4.229)	(5.196)	(5.255)
	0	31.031	0	31.031	50.683	0	50.683	50.683	0	50.683	50.683	0	50.683
		(5.236)	(0.000)	(5.236)	(7.681)	(0.000)	(7.681)	(7.681)	(0.000)	(7.681)	(7.681)	(0.000)	(7.681)
	0.2	31.031	22.895	53.926	50.683	22.109	72.792	57.889	30.228	88.117	57.354	33.686	91.04
		(5.236)	(4.746)	(6.572)	(7.681)	(4.692)	(8.329)	(7.820)	(5.175)	(9.060)	(7.661)	(5.597)	(9.579)
	0.4	31.031	45.5	76.531	50.679	43.81	94.489	63.368	57.189	120.557	62.138	64.615	126.753
250		(5.236)	(6.355)	(7.263)	(7.677)	(6.280)	(8.592)	(7.819)	(6.488)	(9.052)	(7.501)	(6.963)	(9.827)
	0.6	31.031	68.387	99.418	50.659	65.528	116.187	67.925	81.993	149.918	65.457	91.913	157.37
		(5.236)	(7.287)	(7.639)	(7.673)	(7.329)	(8.502)	(7.797)	(7.272)	(8.508)	(7.411)	(7.730)	(9.107)
	0.8	31.031	91.294	122.325	50.643	86.973	137.616	71.718	104.914	176.632	67.178	115.387	182.565
		(5.236)	(7.870)	(7.777)	(7.668)	(8.097)	(8.275)	(7.765)	(7.838)	(8.262)	(7.139)	(8.105)	(8.449)
	1	31.031	114.084	145.115	50.613	107.917	158.53	74.598	126.228	200.826	67.291	135.859	203.15
		(5.236)	(8.290)	(7.744)	(7.654)	(8.660)	(7.879)	(7.677)	(8.480)	(7.759)	(6.888)	(8.391)	(7.726)

Table 2: Simulation results for population sizes $K = 50$, 100, 250 and willingness (w) rates 0, 0.2, 0.4, 0.6, 0.8, 1. Standard deviations of the populations for the total number of transplants are reported below the averages in parentheses for 1000 simulations. For the standard errors of the averages all these standard errors need to be divided by $\sqrt{1000} \approx 31.62$.

transplants.[35]

The baseline off-the-shelf RSÜ priority treatment matches more patients than no exchange, and the difference slightly decreases as willingness rate increases. However, when compared to our mechanism, it results in substantially fewer transplants whenever right-lobe transplantation is a viable option, i.e., w-fraction is positive. The percentage-wise increase

[35] Even under the most conservative predictions, our simulations also show that potential gains from liver exchange has not been fully realized in South Korea, especially those due to size incompatibility. ASAM Medical Center, the leading living-donor liver transplantation center in the world, reports that between 2003 and 2011 only 26 patients were transplanted through exchange, which is only 1.2% of all living-donor transplants conducted in the center (see Jung et al., 2014). Moreover, they note that only 4 of these patients participated in exchange for size incompatibility reasons, while the rest participated in exchange because of blood-type incompatibility.

in the number of transplants due to the availability of our mechanism instead of the RSÜ priority mechanism is in the range of 20.5% to 28.4% for $K = 100$ when right-lobe transplant option is viable, and more than 26% when w-fraction is 0.4 or more.

When compared to the maximum IR matching treatment, our mechanism does fairly well in terms of the numbers of transplants, despite the favorable treatment received by the former mechanism due to the full information assumption on willingness to donate a right lobe. Unlike our proposed mechanism, this mechanism is not incentive compatible, and hence it's outcome is best interpreted as a hypothetical maximum. Indeed, the number of left-lobe transplants are higher under our proposed mechanism than this hypothetical maximum.[36] Since the total number of transplants has to be weakly higher under the hypothetical maximum, our proposed mechanism yields fewer right-lobe transplants. As the worst case, when $K = 250$, the total number of transplants change between 100% (when w-fraction is 0) to 95.1% (when w-fraction is 0.4) and then back to 98.9% (when w-fraction is 1) of those of the hypothetical maximum IR matching. These ratios, while always less than 100% by definition, they are more favorable for our proposed mechanism when the population size is smaller with $K = 50$ and $K = 100$.

While the total number of transplants is higher under the maximum IR matching treatment than our proposed mechanism, the total number of transplants does not necessarily represent the best metric for social welfare. The *double equipoise* is a widely accepted concept in evaluating the balance between donor risk and recipient benefit in living donor liver transplantation, and according to this theory it should be performed only if the donor risk is justified by the acceptable outcome for the recipient. Based on this theory, Roll et al. (2013) propose the metric of *recipient lives saved at 5 years per donor death* to evaluate various liver transplantation policies. For a population size of $K = 100$ and willingness rate of 100%, the expected number of left lobe transplants under our proposed mechanism is more than 2 units higher than under the maximum IR matching treatment, whereas the expected number of right lobe transplants is less than 3 units lower. Therefore, given the five-fold donor-mortality risk under the right-lobe transplantation, our proposed mechanism performs significantly better than the maximum IR matching treatment based on this performance metric proposed in liver transplantation literature.

7 Conclusion

We introduced a liver-exchange model where the donor of each pair can donate either the smaller and safer-to-donate left liver lobe or the larger and riskier-to-donate right liver lobe. While liver exchange is inspired by the increasingly widespread kidney exchange, analytically it is a more challenging problem due to its dual-donation possibility. On the one hand, right-

[36] We choose one arbitrary full-information priority matching in this graph, and we do not aim to minimize the number of right-lobe transplants among all possible priority matchings of the graph, as there may be multiple priority matchings that match the same set of pairs in the induced compatibility graph.

lobe donation expands the set of feasible exchanges, increasing the number of patients who can receive a transplant. On the other hand, it is a considerably higher-risk procedure for the donor, thereby possibly discouraging some of the donors from this option. And since some donors will be willing to donate their left lobes but not their right lobes, the liver-exchange problem harbors a novel incentive compatibility consideration that is not present in kidney exchange. Exploiting the acyclicity of a certain directed graph among pairs which can participate in exchange both through left-lobe donation and right-lobe donation, we introduced a novel exchange mechanism that is Pareto efficient and incentive compatible. The welfare gains from adopting our mechanism are considerable, and depending on the ratio of donors who are willing to donate a right lobe, it increases the number of living-donor liver transplants by 34–44%.

Recently Mishra et al. (2018) advocated for organized liver exchange in the US, emphasizing the choice of a matching algorithm as one of the most difficult issues to be resolved. We believe our proposed mechanism is a viable solution for this important problem.

Appendix A Mathematical Preliminaries

In this section, we will state some definitions and a result from graph theory that will be used in subsequent proofs.

A tuple $G = (\mathcal{V}, E)$ is a **graph** if $\mathcal{V}$ is a nonempty set such that $\emptyset \notin \mathcal{V}$ and $E \subseteq \Big\{\{x, y\} : x, y \in \mathcal{V}\Big\}$. The elements of $\mathcal{V}$ are called **vertices**. The elements of E are called **edges**.

Note that in the definition of a graph, we are allowing for loops, i.e., edges $\{x, y\}$ such that $x = y$.[37]

A **matching** in a graph $G = (\mathcal{V}, E)$ is a subset $M \subseteq E$ of pairwise disjoint edges, i.e., $\varepsilon, \varepsilon' \in M$ such that $\varepsilon \cap \varepsilon' \neq \emptyset \implies \varepsilon = \varepsilon'$. Given a matching M in G, we will abuse notation and also define the function $M : \mathcal{V} \to \mathcal{V} \cup \{\emptyset\}$ by:

$$M(x) = \begin{cases} y & \text{if there exists } y \in \mathcal{V} \text{ such that } \{x, y\} \in M \\ \emptyset & \text{otherwise} \end{cases}$$

for all $x \in \mathcal{V}$. We call $M(x)$ the **assignment of** x **in** M. We will say that a subset $\mathcal{W} \subseteq \mathcal{V}$ is **matchable in** G, if there is a matching M in G such that $M(x) \neq \emptyset$ for all $x \in \mathcal{W}$.

In a graph, the vertices corresponding to each edge $\varepsilon = \{x, y\}$ are unordered. We will also need the notion of a *directed graph* where the order of the vertices does matter.

[37]In some texts, a *simple undirected graph with loops* is what we call a graph here. See for example Korte and Vygen (2011, p13-14).

A tuple $G = (\mathcal{V}, E)$ is a **directed graph (digraph)** if $\mathcal{V}$ is a nonempty set and $E \subseteq \{(x,y) \in \mathcal{V} \times \mathcal{V} : x \neq y\}$. When the digraph is understood, we will also use $x \rightarrow y$ to denote $(x,y) \in E$.

Note that as opposed to our definition of an undirected graph, in the definition of a digraph, we are ruling out loops, i.e., directed edges (x,y) such that $x = y$.[38]

Given a digraph $G = (\mathcal{V}, E)$, a **topological order on** G is a linear order Π on $\mathcal{V}$ such that: $x \rightarrow y$ implies $x \Pi y$, for all $x, y \in \mathcal{V}$.

A digraph $G = (\mathcal{V}, E)$ is **acyclic** if there does not exist an integer $n \geq 2$ and $v_1, \ldots, v_n \in \mathcal{V}$ such that: $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n \rightarrow v_1$.

The following lemma is a standard result in graph theory.[39]

Lemma 4 *Given a digraph $G = (\mathcal{V}, E)$, there exists a topological order on G if and only if G is acyclic.*

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[38] In some texts, a *simple directed graph without loops* is what we call a digraph here. See again Korte and Vygen (2011, p13-14).

[39] For example, see Proposition 2.9 in Korte and Vygen (2011, p20).

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