Efficient Resource Allocation on the Basis of Priorities*

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1 Introduction

Many institutions allocate resources by non-market mechanisms based on priorities. In this paper, we introduce a model of resource allocation on the basis of priorities and address the following questions: Is it possible to allocate resources on the basis of any priority structure? If yes, when can we do this Pareto efficiently, without creating incentives for manipulation and consistently across different groups of agents and resource levels?

We adopt a basic indivisible-objects model with a finite number of object types and a finite quota of goods of each type. Some interesting examples are the determination of access to education, allocation of graduate houses, offices or tasks. Agents are assumed to have strict preferences over object types and remaining unassigned. An assignment is an allocation of the objects to the agents such that every agent receives at most one object. A rule associates an assignment to each preference profile. We formalize a *priority structure* to be a collection of strict priority rankings of individuals indexed by the object types where $i \succ_a j$ would be read as "i has higher priority for object a than j". We assume that the priority structure is exogenously fixed and allow the preferences of the agents to vary. A rule is said to violate a priority of i for a if there is a preference profile under which i envies j who is assigned a and

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j has lower priority for a than i. A rule *adapts* to a priority structure if it never violates the specified priorities.

The interpretation of our model as a college admissions model gives an important initial result: for any priority structure, the deferred acceptance algorithm of Gale and Shapley (1962) yields the unique rule that adapts to the structure and is Pareto superior to any other rule that adapts to it. We therefore call this rule the *best rule* associated with the priority structure and restrict attention to best rules in the subsequent analysis. Balinski and Sönmez (1999) study a model of student placement where priorities are obtained from exam scores. Their model is essentially the same as ours and their fairness property coincides with our notion of adapting to a priority structure. They also note that the deferred acceptance algorithm yields the unique rule that is Pareto superior to any other adaptive rule, however do not investigate conditions that guarantee Pareto efficiency of the deferred acceptance outcome.

We will say that a priority structure is *acyclical* if it never gives rise to situations where an agent can block a potential settlement between any other two agents without affecting his own position. A rule is *group strategyproof* if it does not create any incentives for manipulation by groups of agents, it is *consistent* if its outcomes in problems involving different groups of agents and resource levels are coherent. Our main result shows that acyclicity of the priority structure is sufficient for Pareto efficiency, group strategyproofness and consistency of the associated best rule as well as necessary for each of these conditions separately. Since the best rule Pareto dominates any other rule that is adapted to the same structure, this in particular implies that a priority structure can be Pareto efficiently adapted if and only if it is acyclical. Finally, we show that a priority structure is acyclical if and only if the priority rankings for any pair of objects are similar in positions lower than the sum of their quotas, demonstrating the restrictiveness of the acyclicity condition.

In our model, priority rights are not always transferable. Consider a situation where there are three agents i, j, and k, one unit of each good type a and b with $i \succ_a j \succ_a k \succ_b i$, i strictly prefers getting b, j and k strictly prefer getting a and j prefers remaining unassigned to getting b. A mutually beneficial agreement between i and k would be to obtain the goods a and b respectively by exercising their priority rights and then to make an exchange so that finally i gets b and k gets a. However the final settlement would violate the priority of j for a, contradicting the allocation of resources on the basis of specified priorities. Here the priority structure is not acyclical, since j may block a potential settlement between i and k without affecting his own position.

In assignment models similar to ours, Abdulkadiroğlu and Sönmez (1998, 2002) and Pápai (2000)[1] introduce Pareto efficient and strategyproof classes of rules that generalize Gale's top trading cycles procedure first described by Shapley and Scarf (1974). These mechanisms are based on a collection of orderings of the agents indexed by the objects that specify a structure for hierarchical inheritance of transferable rights on the objects. Abdulkadiroğlu and Sönmez (2002) interprete the hierarchical inheritance orderings as priority orders where priorities are transferable. By making priorities transferable, they recover Pareto efficiency for any priority structure. On the other hand, the outcome of the top trading cycles algorithm applied to a priority structure need not be adapted to the specified priorities.

To see this, consider an example where there are three districts a, b, and c, one school in each district and three students 1, 2, and 3 residing in a, b, and c respectively. Student 1 is female, the others are male and the achievement test score of 1 is higher than that of 2 which is higher than that of 3. The central authority aims to assign the students to these three schools each of which has a quota of 1. Moreover, the authority desires to make the assignment on the basis of three criteria: Residency (students should be given priority in their own district), Test scores (students with higher test scores should be given priority), and Gender (Females should be given priority). Considering residency above test scores above gender yields the following priority structure:

School a	School b	School c
1	2	3
2	1	1
3	3	2

The above structure turns out to be cyclical, therefore there is no Pareto efficient rule that adapts to it. Suppose that one interprets the collection of orders above as hierarchical inheri-

[1] Although she does not aim to provide a formal treatment of priorities, Pápai (2000) also hints at priorities. Her interpretation is different since she allows the priority order of lower priority agents to depend on the preferences of higher priority agents. In our model, the priority structure is exogenous to all agents.

tance orderings. Then the induced top trading cycle outcome (where the endowments of 1, 2, and 3 are a, b, and c respectively) for the profile where bP_1aP_1c, cP_2aP_2b and bP_3aP_3c would be:

$$\begin{pmatrix} 1 & 2 & 3 \\ a & c & b \end{pmatrix}.$$

Note that 3 is assigned to b while 1 both prefers to attend b rather than a and has higher priority for b, i.e., the above assignment violates a priority specified in the structure via which it was derived. Consider the market interpretation of the above outcome: initially 1, 2, and 3 *own* a, b, and c respectively. Then 2 and 3 trade c and b which yields the final Pareto efficient assignment. However in our model, agent 2's priority for b is not transferable: he can attend b if he wishes because he has top priority for it, but he does not have the right to make an exchange with 3 when agent 1 wants to attend b and is ranked above 3 in the priority list for b.

2 The Model

Let N denote the set of agents and A the set of indivisible good types. From time to time, we will refer to N as the set of students and to A as the set of schools. Let $q = (q_a)_{a \in A}$ where $q_a \geq 1$ denotes the number of available goods of type a. A *preference profile* is a vector of linear orders (complete, transitive and antisymmetric relations) $R = (R_i)_{i \in N}$ where R_i denotes the preference of agent i over $A \cup \{i\}$.[2] Being assigned to oneself is interpreted as not being assigned to any school. A school a is *acceptable* to i if aR_ii. For any subset of agents $N' \subset N$, let $R_{N'} = (R_i)_{i \in N'}$ and for any agent i and any $x, y \in A \cup \{i\}$, let xP_iy if and only if xR_iy and not yR_ix. An *assignment* is a function $\mu \colon N \to A \cup N$ satisfying: (i) $\forall i \in N : \mu(i) \in A \cup \{i\}$ and (ii) $\forall a \in A : |\mu^{-1}(a)| \leq q_a$.

A *rule* f is a function that associates an assignment to every preference profile. A *priority structure* is a profile of linear orders over agents $\succeq = (\succeq_a)_{a \in A}$ where for each $a \in A$, $\succeq_a$ ranks agents with respect to their priority for a. For any $a \in A$ and $i, j \in N$, let $i \succ_a j$ if and only if $i \succeq_a j$ and not $j \succeq_a i$. For every $a \in A$ and $i \in N$, let $U_a(i) = \{j \in N | j \succ_a i\}$. A rule

[2]Ehlers (2002) proves a maximal domain result showing that one cannot go much beyond strict preferences if one insists on Pareto efficiency and a weak form of group strategyproofness.

f is *Pareto efficient* if for any preference profile R there does not exist an assignment μ such that $\mu(i) R_i f_i(R)$ for every $i \in N$ and $\mu(j) P_j f_j(R)$ for some $j \in N$. A rule f is *strategyproof* if no single agent can be better-off by misrepresenting his preferences, i.e., for any R, i and R'_i we have that $f_i(R) R_i f_i(R_{N\setminus\{i\}}, R'_i)$. It is *group strategyproof* if no subset of agents can gain by jointly misrepresenting their preferences, i.e., if there do not exist $\emptyset \neq N' \subset N$, R and $R'_{N'}$ such that $f_i(R_{N\setminus N'}, R'_{N'}) R_i f_i(R)$ for every $i \in N'$ and $f_j(R_{N\setminus N'}, R'_{N'}) P_j f_j(R)$ for some $j \in N'$. Nonbossiness requires that no agent can maintain his assignment and cause a change in others' assignments by reporting different preferences. Formally, f is *nonbossy* if for any $i \in N$, R and R'_i, $f(R'_i, R_{N\setminus\{i\}}) = f(R)$ whenever $f_i(R'_i, R_{N\setminus\{i\}}) = f_i(R)$. Given a priority structure $\succeq$ and a preference profile R, the assignment μ *violates the priority of* i *for* a, if there is a student j such that j receives a under μ whereas i both prefers a to what he receives under μ and has higher priority for a, i.e.: $\mu(j) = a$, $a P_i \mu(i)$ and $i \succ_a j$. Given R, the assignment μ *adapts* to $\succeq$ if it does not violate any priorities. A rule f *adapts* to the priority structure $\succeq$ if for any preference profile R, the assignment $\mu = f(R)$ adapts to $\succeq$.

Definition 1 Let $\succeq$ be a priority structure and q a vector of quotas. A *cycle* is constituted of distinct $a, b \in A$ and $i, j, k \in N$ such that the following are satisfied:

(C) Cycle condition: $i \succ_a j \succ_a k \succ_b i$,

(S) Scarcity condition: There exist (possibly empty) disjoint sets of agents $N_a, N_b \subset N \setminus \{i, j, k\}$ such that $N_a \subset U_a(j)$, $N_b \subset U_b(i)$, $|N_a| = q_a - 1$ and $|N_b| = q_b - 1$.

A priority structure is *acyclical* if it has no cycles.

Note that acyclicity is a joint property of the priority structure and the vector of quotas, although the latter will often be suppressed. The scarcity condition (S) requires that there are enough people with higher priority for a and b such that there may be instants when i, j and k would compete for admission in either a or b. For example, if the quota of every school is $|N|$ the condition (S) is never satisfied: there is no scarcity of resources, it is possible to assign each agent to his favorite school and this assignment will be Pareto efficient. On the other hand, if all the quotas are one then (S) is always satisfied: resources are fully scarce and cycles are characterized by condition (C). Given a priority structure $\succeq$ and two vectors of quotas q and q' with $q'_a \leq q_a$ for each a, any cycle of $(\succeq, q)$ is also a cycle of $(\succeq, q')$. Therefore, acyclicity becomes more restrictive as resources become more scarce.

2.1 *The Deferred Acceptance Algorithm and the Best Rule*

Our model is closely related to the college admissions model of Gale and Shapley (1962). The college admissions model is a two sided matching problem where there are two finite sets of students and colleges. Every student has a preference over colleges and remaining unassigned. Similarly, each college has a preference over students, it also has a fixed quota determining the maximum number of students that it can admit. Our model can be interpreted as a college admissions model where the priority orders are seen as the *fixed* preferences of colleges over students. However one must be careful with this interpretation, since there are fundamental differences between priority orders and preferences. First, preferences are allowed to vary whereas priorities are fixed: members of A are object types that are associated with *fixed* priority orderings and only the preferences of the agents in N are subject to change. Secondly, preferences evoke strategic considerations whereas priorities do not, therefore in our model members of A are not strategic. Finally, preferences constitute the criteria for Pareto efficiency whereas priorities are irrelevant from a welfare perspective. As a consequence, Pareto efficiency is harder to achieve in our model: given $(\succeq, R)$, any Pareto efficient assignment when $\succeq$ is interpreted as a priority structure is also a Pareto efficient assignment when $\succeq$ is interpreted as a preference profile, but not vice versa.[3]

In the following, consider N and A as the two sides of a college admissions model where the college a is perceived to have the *fixed* "preference" $\succeq_a$ on the set of applicants and R is the preference profile of the members of N as usual. Then Gale and Shapley (1962) define a *stable* assignment μ to be one such that the following conditions are satisfied:

1. *Individual Rationality:* $\forall i \in N$, $\mu(i) R_i i$.
2. *Pairwise stability:*
    2.1. There do not exist i, j and a such that $a = \mu(j)$, $aP_i\mu(i)$ and $i \succ_a j$.

[3] The implication holds independently of the way in which $\succeq$ is extended to a preference profile over subsets of students but it is crucial that students have strict preferences over schools.

2.2. There do not exist i and a such that $aP_i\mu(i)$ and $q_a > |\mu^{-1}(a)|$.[4]

Note that Condition 2.1 says that given R, μ adapts to $\succeq$. Therefore, any stable assignment μ is adapted to $\succeq$. However, adaptability is a weaker property than stability. Thus for any given $\succeq$ and R, the set of adapted assignments are larger than stable ones. Given a priority structure and a preference profile, there always exists a stable assignment. This assignment is found via the *deferred acceptance algorithm* of Gale and Shapley (1962):

> At the first step, every student applies to his favorite acceptable school. For each school a, q_a applicants who have highest priority for a (all applicants if there are fewer than q_a) are placed on the waiting list of a, and the others are rejected.
>
> At the rth step, those applicants who were rejected at step $r-1$ apply to their next best acceptable schools. For each school a, the highest priority q_a students among the new applicants and those in the waiting list are placed on the new waiting list and the rest are rejected.

The algorithm terminates when every student is either on a waiting list or has been rejected by every school that is acceptable to him. After this procedure ends, schools admit students on their waiting lists which yields the desired assignment. Gale and Shapley (1962) show that the algorithm described above yields the unique stable assignment that is Pareto superior to any other stable assignment from the viewpoint of the students. Although the set of adapted assignments is larger than the stable ones, it also turns out that the assignment induced by the deferred acceptance algorithm is Pareto superior to any assignment that is adapted to the structure. The following Proposition provides a summary of the above results. It is identical to Theorem 2 in Balinski and Sönmez (1999).

Proposition 1 *For any priority structure and preference profile, the assignment given by the deferred acceptance algorithm is adapted to the structure and is Pareto superior to any other assignment that is adapted to the structure.*

[4]Different from Gale and Shapley (1962), in 1. and 2.2. we ignore individual rationality conditions for members of A. This simplifying assumption is W.L.O.G. in our analysis. Even if we allow for priority structures where certain students are not acceptable to certain schools, rules that adapt to such structures would not be Pareto efficient. Balinski and Sönmez (1999) call this assumption *non-wastefulness.*

We will call the rule that associates to each profile the deferred acceptance outcome, the *best rule* induced by $\succeq$ and denote it by $f^{\succeq}$. Note that $f^{\succeq}$ is (in particular) constrained Pareto efficient within the class all rules that adapt to $\succeq$. However, in general, it need not be Pareto efficient since it may be Pareto dominated by rules that are not adapted to $\succeq$. Dubins and Freedman (1981) show that under a best rule, agents by themselves do not have any incentives to misrepresent their preferences. However there may be cases when a group of agents can improve their positions by jointly misrepresenting their preferences.

Given a pair $(\succeq, q)$, a subset of agents $\emptyset \neq N' \subset N$, a subset of resources $q' = (q'_a)_{a\in A}$ where $q'_a \leq q_a$ for each a and a preference profile R, there is a unique deferred acceptance outcome induced by the restriction $\succeq |_{N'} = (\succeq_a |_{N'})_{a\in A}$ of the original structure, for the smaller economy $(N', q', R_{N'})$. Let us call the map that associates the deferred acceptance with any such smaller economy $(N', q', R_{N'})$, the *extended best rule* associated with the structure $\succeq$ and denote it by $\tilde{f}^{\succeq}$. Given an economy $\mathcal{E} = (N, q, R)$, an assignment μ for $\mathcal{E}$ and a subset of agents $\emptyset \neq N' \subset N$, the *reduced problem* $r^{\mu}_{N'}(\mathcal{E})$ *of* $\mathcal{E}$ *with respect to* μ *and* N' is the smaller problem consisting of agents N' and remaining resources after agents in $N \setminus N'$ have left with their assignments under μ, i.e., $r^{\mu}_{N'}(\mathcal{E}) = (N', q', R_{N'})$ where $q'_a = q_a - |\mu^{-1}(a) \setminus N'|$ for each $a \in A$. Consistency requires that once an assignment is determined and a group of agents receive their physical assignments before the others, the rule should not change the assignments of the remaining agents in the reduced problem involving the remaining agents and resources. Formally, a priority structure $\succeq$ induces a *consistent* extended best rule if for any economy $\mathcal{E} = (N, q, R)$, one has $\mu|_{N'} = \tilde{f}^{\succeq}(r^{\mu}_{N'}(\mathcal{E}))$ where $\mu = \tilde{f}^{\succeq}(\mathcal{E})$. Consistent rules are coherent in their outcomes for problems involving different groups of agents and robust to non-simultaneous physical assignment of the objects.[5]

3 The Results

Theorem 1 *For any pair $(\succeq, q)$, the following are equivalent:*

(i) $f^{\succeq}$ is Pareto efficient,

(ii) $f^{\succeq}$ is group strategyproof,

[5]See Ergin (2000) for a discussion of consistency in the indivisible-object assignment setting and Thomson (1990) for a survey of the consistency principle in general in allocation problems.

(iii) $\tilde{f}^{\succeq}$ *is consistent,*

(iv) $\succeq$ *is acyclical.*

Since the best rule associated with a structure $\succeq$ is Pareto superior to all other rules that adapt to $\succeq$, the "$(i) \iff (iv)$" part above can also be read as: There exists a Pareto efficient rule that adapts to a priority structure if and only if the structure is acyclical. As the goods become more scarce, the acyclicity condition becomes more restrictive and therefore fewer priority structures can be Pareto efficiently adapted. The "$(ii) \iff (iv)$" part reinforces the above result by confirming that the best rule associated with an acyclical structure is immune to strategic behavior on the part of the agents. If we wish to achieve the weaker concept of constrained Pareto efficiency while keeping group incentives in line, then the result implies that we are restricted to acyclical structures.

Given a linear order $\succeq'$ on N and $m \leq |N|$, let $L(\succeq', m) = \{i : |\{j : i \succeq' j\}| \leq m\}$ be the set of agents who share the lowest m ranks of $\succeq'$.[6] Next, we provide a characterization of acyclicity showing that the priority rankings for any pair of objects should be similar in positions lower than the sum of their quotas.

Theorem 2 *Let a pair $(\succeq, q)$ be given. Then $\succeq$ is acyclical if and only if for any two $a, b \in A$, if $i \in N$ ranks lower than $(q_a + q_b)$st from the top in N with respect to $\succeq_a$ or $\succeq_b$, then the ranks of i in N with respect to $\succeq_a$ and $\succeq_b$ differ at most by one, i.e.:*

$$(*) \quad \forall a, b \in A, \forall i \in L\left(\succeq_a, |N| - q_a - q_b\right) \cup L\left(\succeq_b, |N| - q_a - q_b\right): \quad ||U_a(i)| - |U_b(i)|| \leq 1.$$

Let $\succeq$ be acyclical and for any pair of objects a and b, let us say that members of $L\left(\succeq_a, |N| - q_a - q_b\right) \cup L\left(\succeq_b, |N| - q_a - q_b\right)$ belong to the *lower class* and the remaining agents belong to the *upper class* for a and b. Let $p_{a,b}$ denote the number of lower class agents for a and b. By Theorem 2, $|N| - q_a - q_b \leq p_{a,b} \leq |N| - q_a - q_b + 1$ and the lower class agents share the bottom $p_{a,b}$ positions with respect to *both* priority orders. Acyclicity brings no restrictions as to the relative positions of upper class members as long as they are ranked in the top $|N| - p_{a,b}$ positions in both priority orders. However, it restricts the ranks of a lower class member across the two priority orders to differ at most by one.

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[6]Note that if $m \leq 0$, then by definition $L(\succeq', m) = \emptyset$.

Appendix

1 *Proof of Theorem 1*

The Theorem is proved in two parts: (iv)$\Longrightarrow$(i)$\Longrightarrow$ (iii)$\Longrightarrow$(iv) and (iii)$\Longrightarrow$(ii)$\Longrightarrow$(iv). Given a priority structure $\succeq$, a *generalized cycle* is constituted of distinct $a_0, a_1, \ldots, a_{n-1} \in A$ and $j, i_0, i_1, \ldots, i_{n-1} \in N$ with $n \geq 2$ such that the following are satisfied:

(C) $i_0 \succ_{a_0} j \succ_{a_0} i_{n-1} \succ_{a_{n-1}} i_{n-2} \succ_{a_{n-2}} \ldots i_2 \succ_{a_2} i_1 \succ_{a_1} i_0$,

(S) There exist disjoint sets of agents $N_{a_0}, \ldots, N_{a_{n-1}} \subset N \setminus \{j, i_0, i_1, \ldots, i_{n-1}\}$ such that $N_{a_0} \subset U_{a_0}(j)$, $N_{a_1} \subset U_{a_1}(i_0)$, $N_{a_2} \subset U_{a_2}(i_1)$, $\ldots$ $N_{a_{n-2}} \subset U_{a_{n-1}}(i_{n-3})$, $N_{a_{n-1}} \subset U_{a_{n-1}}(i_{n-2})$ and $|N_{a_l}| = q_{a_l} - 1$ for $l = 0, 1, \ldots, n-1$.

(iv)$\Longrightarrow$(i): *Step 1* If $f^{\succeq}(R)$ is not Pareto efficient then $\succeq$ has a generalized cycle.

Suppose that μ' Pareto dominates $\mu = f^{\succeq}(R)$. The first part is to show that there exist agents $i_0, i_1, \ldots, i_{n-1}, i_n = i_o \in N$ with $n \geq 2$ such that each agent envies the next under μ. Let $N' = \{i \in N | \mu'(i) P_i \mu(i)\}$, then $N' \neq \emptyset$. Since no agent is worse-off under μ', those that are not strictly better-off should be indifferent between μ and μ'. Therefore, by our assumption of strict preferences they are assigned to the same school under both assignments, i.e., $N \setminus N' = \{i \in N | \mu'(i) = \mu(i)\}$. For any $i \in N'$, $\mu'(i) \in A$ since $\mu'(i) P_i \mu(i) R_i i$, by the individual rationality of the best rule outcome .

Let $i \in N'$, since $\mu'(i) P_i \mu(i)$, i has been rejected by $\mu'(i)$ at a step in the algorithm leading to μ. At that step the waiting list of $\mu'(i)$ was full, therefore at the end of the algorithm the school $\mu'(i)$ has full quota, i.e., $|\mu^{-1}(\mu'(i))| = q_{\mu'(i)}$. Moreover, there are people in N' who were assigned to $\mu'(i)$ under μ. Because otherwise the set of $q_{\mu'(i)}$ people who were assigned to $\mu'(i)$ under μ would be a subset of $N \setminus N'$, therefore they would be assigned to $\mu(i)$ also under μ'. But then when we also include $i \in N'$, there are at least $q_{\mu'(i)} + 1$ people assigned to $\mu'(i)$ under μ', which leads to a contradiction.

We can now define the correspondence $\pi\colon N' \longrightarrow N'$ by $\pi(i) = \mu^{-1}(\mu'(i)) \cap N'$. The correspondence π is non-empty valued by the above arguments. By construction, we can choose a branch $\overline{\pi}$ of π such that for any $i, j \in N'$ with $\mu'(i) = \mu'(j)$ we have that $\overline{\pi}(i) =$

$\overline{\pi}(j)$. Note that $\overline{\pi}(i) \neq i$ for any $i \in N'$, therefore there is $n \geq 2$ and n distinct agents $i_1, \ldots, i_{n-1}, i_n = i_0 \in N'$ with $i_r = \overline{\pi}(i_{r-1})$, for $r = 1, 2, \ldots, n$. Set $a_r = \mu(i_r)$, then $a_r = \mu\left(\overline{\pi}(i_{r-1})\right) = \mu'(i_{r-1})$ for $r = 1, 2, \ldots, n$. Note also that since $i_1, \ldots, i_{n-1}, i_n = i_0$ are distinct, the schools $a_1, \ldots, a_n = a_0$ are also distinct by the particular choice of branch $\overline{\pi}$. Moreover since $a_r = \mu(i_r) = \mu'(i_{r-1}) P_{i_{r-1}} \mu(i_{r-1})$, and μ adapts to the structure $\succeq$ for the profile R, we have that $i_r \succ_{a_r} i_{r-1}$ for $r = 1, 2, \ldots, n$. We conclude the first part of the proof by noting that $i_0 \succ_{a_0} i_{n-1} \succ_{a_{n-1}} i_{n-2} \succ_{a_{n-2}} \ldots i_2 \succ_{a_2} i_1 \succ_{a_1} i_0$, where $a_0, a_1, \ldots, a_{n-1} \in A$, $i_0, i_1, \ldots, i_{n-1} \in N$ are distinct and $n \geq 2$.

For the second part of the proof, consider the deferred acceptance algorithm that leads to μ. Let r be the latest step in the algorithm when someone in $\{i_1, \ldots, i_{n-1}\}$ applies to (and is accepted by) the school that he is assigned to under μ. W.L.O.G., suppose that i_0 applies to (and is accepted by) $\mu(i_0)$ at step r. Note that after step r, all agents in $\{i_1, \ldots, i_{n-1}\}$ never get rejected again, since they are in the waiting list of their final allocation.

Let $l \in \{1, 2, \ldots, n\}$. Since $a_l P_{i_{l-1}} a_{l-1}$, agent i_{l-1} was rejected by a_l at an earlier step than when he applied to a_{l-1}. The latest step at which i_{l-1} could have applied to $a_{l-1} = \mu(i_{l-1})$ is r, so he was rejected by a_l at a step $r' < r$. Therefore, after the end of step r', (in particular right after step $r-1$) the waiting list of a_l is full. At the end of step $r-1$, the waiting list of a_0 is full and does not include any $i_l \in \{i_1, \ldots, i_{n-1}\}$, otherwise i_l would apply to a_l at a step later than r, a contradiction. Thus, there is $j \in N$ distinct from $i_0, i_1, i_2, \ldots, i_{n-1}$ such that he is rejected by a_0 at step r when i_0 applies to (and is accepted in) the waiting list of a_0 and he is accepted to the waiting list of a_0 at or after the step when i_{n-1} is rejected by a_0. Note that $i_0 \succ_{a_0} j \succ_{a_0} i_{n-1}$. For any $l \in \{0, 1, 2, \ldots, n-1\}$, let N_{a_l} be the set of agents in the waiting list of a_l other than i_l at the end of step r. It is now straightforward to see that condition (S) in the definition of generalized cycle is also satisfied.

Step 2 If $\succeq$ has a generalized cycle, then it also has a cycle.

Suppose that $\succeq$ has a generalized cycle and let the size of its shortest generalized cycle be n. We will show that $n = 2$ which will prove step 2, since a cycle is a generalized cycle of size 2. Suppose that $a_0, a_1, \ldots, a_{n-1} \in A$; $j, i_0, i_1, \ldots, i_{n-1} \in N$ and $N_{a_0}, \ldots, N_{a_{n-1}} \subset N \setminus \{j, i_0, \ldots, i_{n-1}\}$ form a shortest generalized cycle of size $n > 2$.

If $i_0 \succ_{a_2} i_2$, then $i_0 \succ_{a_2} i_2 \succ_{a_2} i_1 \succ_{a_1} i_0$ and $N_{a_2}, N_{a_1} \subset N \setminus \{i_2, i_0, i_1\}$ are disjoint sets satisfying $N_{a_2} \subset U_{a_2}(i_2)$, $N_{a_1} \subset U_{a_1}(i_0)$, $|N_{a_2}| = q_{a_2} - 1$ and $|N_{a_1}| = q_{a_1} - 1$, a contradiction to $\succeq$ having no cycles. If $i_2 \succ_{a_2} i_0$, then $i_0 \succ_{a_0} j \succ_{a_0} i_{n-1} \succ_{a_{n-1}} \dots i_3 \succ_{a_3} i_2 \succ_{a_2} i_0$, and $N_{a_0}, N_{a_2}, \dots, N_{a_{n-1}} \subset N \setminus \{j, i_0, i_2, \dots, i_{n-1}\}$ are disjoint sets satisfying $N_{a_0} \subset U_{a_0}(j)$, $N_{a_2} \subset U_{a_2}(i_0)$, $N_{a_3} \subset U_{a_3}(i_2)$, $\dots$ $N_{a_{n-2}} \subset U_{a_{n-2}}(i_{n-3})$, $N_{a_{n-1}} \subset U_{a_{n-1}}(i_{n-2})$ and $|N_{a_l}| = q_{a_l} - 1$ for $l = 0, 2, 3, \dots, n-1$. So, there is a generalized cycle of size $n-1$, a contradiction with the choice of the initial generalized cycle.

(i)$\Longrightarrow$(iii): Assume that $\tilde{f}^{\succeq}$ is not consistent. Then, there is R and $\emptyset \neq N' \subset N$ such that $\mathcal{E} = (N, q, R)$, $\mu = \tilde{f}^{\succeq}(\mathcal{E})$, $\mu' = \tilde{f}^{\succeq}(r^{\mu}_{N'}(\mathcal{E}))$ and $\mu|_{N'} \neq \mu'$. Since μ adapts to $\succeq$ in the original economy $\mathcal{E}$, the restricted assignment $\mu|_{N'}$ adapts to $\succeq|_{N'} = (\succeq_a |_{N'})_{a \in A}$ in the reduced economy $r^{\mu}_{N'}(\mathcal{E})$. So by Proposition 1, μ' Pareto dominates $\mu|_{N'}$ in the reduced economy $r^{\mu}_{N'}(\mathcal{E})$. But then the assignment ν for the original economy $\mathcal{E}$ defined by:

$$\nu(i) = \begin{cases} \mu'(i) & \text{if } i \in N', \\ \mu(i) & \text{otherwise.} \end{cases}$$

Pareto dominates μ, therefore the best rule associated with $\succeq$ is not Pareto efficient.

(iii)$\Longrightarrow$(iv): Let N, A and q and $\succeq$ be given. Assume that $\succeq$ has a cycle with a, b, i, j, k, N_a and N_b. Consider the preference profile R where agents in N_a and N_b respectively rank a and b as their top choice and the preferences of i, j and k are such that $bP_iaP_iiP_i\dots$, $aP_jjP_j\dots$ and $aP_kbP_kkP_k\dots$. Finally, let agents outside $N_a \cup N_b \cup \{i, j, k\}$ prefer not to be assigned to any school. Then, the deferred acceptance outcome μ for R is such that $\forall l \in N_a \cup \{i\}: \quad \mu(l) = a$ and $\forall l \in N_b \cup \{k\}: \quad \mu(l) = b$. Let $\mathcal{E} = (N, q, R)$, then the reduced problem $r^{\mu}_{\{i,k\}}(\mathcal{E}) = \left(\{i, k\}, q', R_{\{i,k\}}\right)$ is such that the preferences of i and k are as in above, $q'_a = 1$, $q'_b = 1$ and $q'_x = q_x$ for any $x \in A \setminus \{a, b\}$. The deferred acceptance outcome μ' of the reduced problem $r^{\mu}_{\{i,k\}}(\mathcal{E})$ is such that $\mu'(i) = b$ and $\mu'(k) = a$. Since $\mu' \neq \mu|_{N'}$, $\tilde{f}^{\succeq}$ is not consistent.

(iii)$\Longrightarrow$(ii): An argument exactly similar to that of Lemma 1 in Pápai (2000) shows that strategyproofness and nonbossiness imply group strategyproofness in our model.[7] By Dubins and Freedman (1981), $f^{\succeq}$ is strategyproof. We will next show that if $\tilde{f}^{\succeq}$ is consistent then it is also nonbossy, completing the proof of this part. Assume that $\tilde{f}^{\succeq}$ is consistent. Let i, R

[7] See Barberà and Jackson (1995) for a related result.

and R_i' be given and set $\mathcal{E} = (N, q, R)$, $\mathcal{E}' = \left(N, q, (R_i', R_{N\setminus\{i\}})\right)$, $\mu = \tilde{f}^{\succeq}(\mathcal{E})$ and $\nu = \tilde{f}^{\succeq}(\mathcal{E}')$. Assume that $\mu(i) = \nu(i)$, then the two reduced problems $r^{\mu}_{N\setminus\{i\}}(\mathcal{E})$ and $r^{\nu}_{N\setminus\{i\}}(\mathcal{E}')$ are the same. Moreover by consistency of $\tilde{f}^{\succeq}$, $\mu|_{N\setminus\{i\}} = \tilde{f}^{\succeq}\left(r^{\mu}_{N\setminus\{i\}}(\mathcal{E})\right)$ and $\nu|_{N\setminus\{i\}} = \tilde{f}^{\succeq}\left(r^{\nu}_{N\setminus\{i\}}(\mathcal{E}')\right)$, therefore $\mu|_{N\setminus\{i\}} = \nu|_{N\setminus\{i\}}$. Since $\mu(i) = \nu(i)$ and $\mu|_{N\setminus\{i\}} = \nu|_{N\setminus\{i\}}$, we conclude that $\mu = \nu$.

(ii)$\Longrightarrow$(iv): Assume that $\succeq$ has a cycle. Let R be as defined in part "(iii)$\Longrightarrow$(iv)", $N' = \{i, j, k\}$, $R'_{-j} = R_{-j}$ and let R_j' rank j at the top. Then, we have:

$$f^{\succeq}(R_{N\setminus N'}, R_{N'}) = \begin{pmatrix} i & j & k \\ a & j & b \end{pmatrix} \text{ and } f^{\succeq}(R_{N\setminus N'}, R'_{N'}) = \begin{pmatrix} i & j & k \\ b & j & a \end{pmatrix},$$

contradicting the group strategyproofness of $f^{\succeq}$ under the true preferences R. Q.E.D.

2 *Proof of Theorem 2*

Let $\succeq$ be acyclical and suppose that there exist $a, b \in A$ and $i \in L(\succeq_a, |N| - q_a - q_b) \cup L(\succeq_b, |N| - q_a - q_b)$ such that $||U_a(i)| - |U_b(i)|| > 1$. W.L.O.G. let $|U_b(i)| \geq |U_a(i)| + 2$, then $|U_b(i)| \geq q_a + q_b$. Let k be the bottom and j the second from bottom agent in $U_b(i) \cup \{i\}$ with respect to $\succeq_a$ (there exist two such agents since $|U_b(i) \cup \{i\}| \geq 3$). Then, $(U_b(i) \cup \{i\}) \setminus \{j, k\} \subset U_a(j)$, i.e. $|U_a(j)| \geq |(U_b(i) \cup \{i\}) \setminus \{j, k\}| = |U_b(i)| - 1$, so $|U_a(j)| > |U_a(i)|$ and $|U_a(j)| \geq q_a + q_b - 1 \geq q_a$. Thus j (and therefore also k) ranks lower than i from the top in N with respect to $\succeq_a$, i.e., $i \succ_a j$ and in particular i, j and k are distinct. So we have $i \succ_a j \succ_a k \succ_b i$, $|U_a(j)| \geq q_a$ and $|U_b(i)| \geq q_a + q_b$. Therefore $\succeq$ has a cycle, a contradiction.

For the converse, let $\succeq$ satisfy $(*)$ and suppose that it has a cycle with a, b, i, j, k, N_a and N_b. Suppose that both i and k are in the top $q_a + q_b$ positions with respect to $\succeq_a$ and $\succeq_b$. Then $j \succ_a k$ and $N_a \subset U_a(j)$ imply that $\{j\} \cup N_a$ are in the top $q_a + q_b$ positions with respect to $\succeq_a$. Also $N_b \subset U_b(i)$ implies that elements of N_b are in the top $q_a + q_b - 1$ positions with respect to $\succeq_b$. Moreover since by $(*)$, $L(\succeq_a, |N| - q_a - q_b) \subset L(\succeq_b, |N| - q_a - q_b + 1)$, by taking complements we deduce that the elements in the top $q_a + q_b - 1$ positions with respect to $\succeq_b$ are in the top $q_a + q_b$ positions with respect to $\succeq_a$. In particular, elements of N_b are in the top $q_a + q_b$ positions with respect to $\succeq_a$. Thus, elements of $\{i, j, k\} \cup N_a \cup N_b$ are in the top $q_a + q_b$ positions with respect to $\succeq_a$, a contradiction to $|\{i, j, k\} \cup N_a \cup N_b| = q_a + q_b + 1$. So i or k is in $L(\succeq_a, |N| - q_a - q_b) \cup L(\succeq_b, |N| - q_a - q_b)$. Suppose that $k \notin L(\succeq_a, |N| - q_a - q_b) \cup$

$L(\succeq_b, |N| - q_a - q_b)$, then $i \in L(\succeq_a, |N| - q_a - q_b) \cup L(\succeq_b, |N| - q_a - q_b)$. By applying $(*)$, $|U_a(i)| \geq q_a + q_b - 1$ and $i \succ_a k$, so $k \in L(\succeq_a, |N| - q_a - q_b)$, a contradiction. Thus $k \in L(\succeq_a, |N| - q_a - q_b) \cup L(\succeq_b, |N| - q_a - q_b)$. But then $i \succ_a j \succ_a k \succ_b i$ and $(*)$ imply that $|U_a(i)| \leq |U_a(k)| - 2 \leq |U_b(k)| - 1 \leq |U_b(i)| - 2$, a contradiction. Q.E.D.

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