Advanced Microeconomics
(Economics 104)
Fall 2011
Extensive games with perfect information

Topics

- Formalities.
- Reduced strategic form.
- Backward induction and subgame perfection.

The need for refinements of Nash equilibrium

The concept of NE is unsatisfactory since it

- ignores the sequential structure of the decision problems, and
- in sequential decision problems not all NE are self-enforcing.

The following refinements have been proposed:

- subgame perfect, perfect, sequential, perfect sequential, proper
- persistent, justifiable, neologism proof, stable, intuitive, divine, undefeated and explicable.

All the refinements represent attempts to formulize the same two or three intuitive ideas (Kohlberg 1990).

Formalities (O 5.1-5.2, OR 6.1)

Definition

An extensive game with perfect information $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ consists of

- A set N of players.
- A finite or infinite set H of sequences (histories), each component an action taken by a player.
- A player function $P : H \backslash Z \to N$ s.t. $P(h)$ being the player who takes an action after history h.
- A preference relation $\succsim_i$on Z for each player $i \in N$ where,

  The empty sequence $\varnothing$ is a member of H.
  If $(a^k)_{k=1}^K \in H$ then $(a^k)_{k=1}^L \in H$ for any $L < K$.
  If $(a^k)_{k=1}^\infty$satisfies $(a^k)_{k=1}^L \in H$ for any L then $(a^k)_{k=1}^\infty \in H$.

And,

- A set of terminal histories $Z \subseteq H$ s.t. $(a^k)_{k=1}^K \in Z$ if it is infinite, or $\nexists a^{K+1}$ s.t. $(a^k)_{k=1}^{K+1} \in H$.
- If h is a history of length k then (h, a) is a history of length $k+1$ consists of h followed by a.

If the longest history is finite then the game has a *finite horizon*.

Strategies and outcomes

A strategy s_i of player i is a plan that specifies the action taken for every $h \in H \backslash Z$ for which $P(h) = i$.

For any $s = (s_i)_{i \in N}$, the outcome $O(s)$ of s is $h \in Z$ that results when each player $i \in N$ follows s_i.

Nash equilibrium (O 5.3)

A NE of $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ is a strategy profile s^* s.t. for any $i \in N$

$$O(s^*) \succsim_i O(s_i, s^*_{-i}) \ \forall s_i$$

Note that

– strategies are once-in-a-lifetime decisions made before the game starts.

– non-self-enforcing outcome (Selten 96.2).

The (reduced) strategic form

Consider an extensive game $\Gamma = \langle N, H, P, (\succsim_i) \rangle$

The strategic form of Γ is a game $\langle N, (S_i), (\succsim_i') \rangle$ in which for each $i \in N$

- S_i is player i's strategy set in Γ.
- $\succsim_i'$ is defined by

$$s \succsim_i' s' \Leftrightarrow O(s) \succsim_i' O(s') \ \forall s, s' \in \times_{i \in N} S_i$$

The reduced strategic form of Γ is a game $\langle N, (S_i'), (\succsim_i'') \rangle$ in which for each $i \in N$

- S_i' contains one member of equivalent strategies in S_i, i.e., $s_i \in S_i$ and $s_i' \in S_i$ are equivalent if

$$(s_{-i}, s_i) \sim_j' (s_{-i}, s_i') \forall j \in N$$

- $\succsim_i''$defined over $\times_{j \in N} S_j'$ and induced by $\succsim_i'$.

Subgame perfection (O 5.4 OR 6.2)

Selten (1965, 1975) and Kreps and Wilson (1982) proposed a condition for differentiating the self-enforcing equilibria.

A subgame of Γ that follows the history h is the game $\Gamma(h)$

$$\langle N, H|_h, P|_h, (\succsim_i|_h)\rangle$$

where for each $h' \in H|_h$

$$(h, h') \in H, P|_h(h') = P(h, h')$$

and

$$h' \succsim_i|_h h'' \Leftrightarrow (h, h') \succsim_i (h, h'')$$

s^* is a subgame perfect equilibrium (SPE) of Γ if

$$O_h(s_i^*|_h, s_{-i}^*|_h) \succsim_i|_h O_h(s_i|_h, s_{-i}^*|_h)$$

for each $i \in N$ and $h \in H \backslash Z$ for which $P(h) = i$ and for any $s_i|_h$.

The equilibrium of the full game must induce on equilibrium on every subgame.

Backward induction

An algorithm for calculating the set of SPE (Zermelo 1912)

- make payoff-maximizing choices at nodes which are one move from the end
- given those, make payoff-maximizing choices at nodes which are two move from the end,
- and so on.

SPE eliminates NE in which players' threats are not credible (non-self-enforcing).

Kuhn's theorems
Consider a finite extensive game with perfect information Γ

(Kuhn's theorem) Γ has a SPE.

- The proof is by backwards induction.
- Kuhn makes no claim about uniqueness.

Γ has a unique SPE if there is no $i \in N$ and $z, z' \in Z$ such that $z \sim_i z'$.

Γ is dominance solvable if

$$z \sim_i z' \ \exists i \in N \Rightarrow z \sim_j z' \ \forall j \in N$$

where $z, z' \in Z$.

But, elimination of weakly dominated strategies in G may eliminate the SPE in Γ (OR 6.6.1).