Advanced Microeconomics
(Economics 104)
Fall 2008
Maxminimization and strictly competitive games

A two-player strategic game $\langle \{1,2\}, (A_i), (\succsim_i) \rangle$ is *strictly competitive* if preferences are diametrically opposes. That is, for any $a, a' \in A$,

$$a \succsim_1 a' \text{ if and only if } a' \succsim_2 a.$$

When $\succsim_i$is represented by a utility function u_i then for any $a \in A$ we have

$$u_1(a) = -u_2(a).$$

Thus, a strictly competitive game is sometimes called *zero-sum*.

An interesting character of a zero-sum game is that a strategy profile is a NE if and only if the action of each player is a $\max\min$ strategy.

This is an important result and it helps us understand the decision-making basis for NE.

Maxminimization (O 11.1-11.2, OR 2.5)

Consider a strategic game $\langle N, (A_i), (u_i) \rangle$ (vNM preference).

A max min mixed strategy of player i is a mixed strategy that solves the problem

$$\max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})$$

where $U_i(\alpha)$ is player i's expected payoff to the profile of mixed strategies α.

Equivalently, α_i^* is a max min for player i if and only if

$$\min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i^*, \alpha_{-i}) = \max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})$$

In words, player i chooses a mixed strategy that is best for him under the assumption that whatever he does, all other players will choose their actions to hurt him as much as possible.

For example, in the BoS player 1's max min strategy is $(1/3, 2/3)$ while player 2's is $(1/3, 2/3)$ (you should verify this).

Note that a player's payoff in a mixed strategy NE is at least her max min payoff.

To see this suppose that α^* is a mixed strategy NE. Then, for any player i and for all α_i

$$\begin{aligned} U_i(\alpha^*) &\geq U_i(\alpha_i, \alpha_{-i}^*) \\ &\geq \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \\ &\geq \max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \end{aligned}$$

and the last step follows since the above holds for all α_i.

Two $\min\max$ **propositions (O 11.3-11.4, OR 2.5)**

We next prove two $\min\max$ propositions.

Proposition 1 In any strategic game $G = \langle N, (A_i), (u_i) \rangle$,

$$\max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i} \in \Delta A_{-i}} \max_{\alpha_i \in \Delta A_i} U_i(\alpha_i, \alpha_{-i})$$

Proof.

For every α'_i and α'_{-i}

$$\min_{\alpha_{-i}} U_i(\alpha'_i, \alpha_{-i}) \leq U_i(\alpha'_i, \alpha'_{-i})$$

and thus

$$\min_{\alpha_{-i}} U_i(\alpha'_i, \alpha_{-i}) \leq \max_{\alpha_i} U_i(\alpha_i, \alpha'_{-i})$$

However, since the above holds for every α'_i and α'_{-i} it must hold for the "best" and "worst" such choices

$$\max_{\alpha_i} \min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i}} \max_{\alpha_i} U_i(\alpha_i, \alpha_{-i})$$

More precisely, the above result follows from the following Lemma (you can skip that part).

Lemma Let X_1 and X_2 be arbitrary sets then for any function $f : X \times X \to \mathbb{R}$

$$\inf_{x_2}(\sup_{x_1} f(x_1, x_2)) \geq \sup_{x_1}(\inf_{x_2} f(x_1, x_2))$$

Proof. Fix $\varepsilon > 0$. For each $x_1 \in X_1$ define

$$f_1(x_1) \equiv \inf_{x_2} f(x_1, x_2)$$

and for each $x_2 \in X_2$ define

$$f_2(x_2) \equiv \sup_{x_1} f(x_1, x_2)$$

Choose x_1' and x_2' such that

$$\sup_{x_1} f_1(x_1) < f_1(x_1') + \varepsilon$$

and

$$\inf_{x_2} f_2(x_2) > f_2(x_2') - \varepsilon$$

Then,

$$\sup_{x_1}(\inf_{x_2} f(x_1, x_2)) \equiv \sup_{x_1} f_1(x_1) < f_1(x_1') + \varepsilon \leq f(x_1', x_2') + \varepsilon$$

and

$$\inf_{x_2}(\sup_{x_1} f(x_1, x_2)) \equiv \inf_{x_2} f_2(x_2) > f_2(x_2') - \varepsilon \geq f(x_1', x_2') - \varepsilon$$

By combining the two inequalities

$$\inf_{x_2}(\sup_{x_1} f(x_1, x_2)) > \sup_{x_1}(\inf_{x_2} f(x_1, x_2)) + 2\epsilon$$

and letting $\epsilon \to 0$ gives the desired result.

Interchangeability in zero-sum games

Before proving the second min max proposition, we prove a result about the interchangeability of NE in zero-sum games.

If (α_1, α_2) and (α_1', α_2') are NE in a zero-sum game, then so are (α_1, α_2') and (α_1', α_2).

- Let (α_1, α_2) and (α_1', α_2') be NE in a zero-sum game.
- Since (α_1, α_2) is an equilibrium

  $$U_1(\alpha_1, \alpha_2) \geq U_1(\alpha_1', \alpha_2)$$

  and since (α_1', α_2') is an equilibrium

  $$U_2(\alpha_1', \alpha_2') \geq U_2(\alpha_1', \alpha_2)$$

  and because $U_1 = -U_2$ (zero-sum game)

  $$U_1(\alpha_1', \alpha_2') \leq U_1(\alpha_1', \alpha_2)$$

  Therefore,

  $$U_1(\alpha_1, \alpha_2) \geq U_1(\alpha_1', \alpha_2) \geq U_1(\alpha_1', \alpha_2') \tag{1}$$

  and similar analysis gives that

  $$U_1(\alpha_1, \alpha_2) \leq U_1(\alpha_1, \alpha_2') \leq U_1(\alpha_1', \alpha_2') \tag{2}$$

  (1) and (2) yield

  $$U_1(\alpha_1, \alpha_2) = U_1(\alpha_1', \alpha_2) = U_1(\alpha_1, \alpha_2') = U_1(\alpha_1', \alpha_2')$$

- Since (α_1, α_2) is an equilibrium

  $$U_2(\alpha_1, \alpha_2'') \leq U_2(\alpha_1, \alpha_2) = U_2(\alpha_1, \alpha_2')$$

  for any $\alpha_2'' \in \Delta A_2$, and since (α_1', α_2') is an equilibrium

  $$U_1(\alpha_1'', \alpha_2') \leq U_1(\alpha_1', \alpha_2') = U_1(\alpha_1, \alpha_2')$$

  for any $\alpha_1'' \in \Delta A_1$. Therefore, (α_1, α_2') is an equilibrium and similarly also (α_1, α_2').
- Note that equilibrium strategies do not in general have this property (consider, for example, a coordination game).

Proposition 2 In a two-player aero-sum game,

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha_1^*, \alpha_2^*)$$

where (α_1^*, α_2^*) is a mixed strategy NE.

Proof.

$\Leftarrow$ Suppose that (α_1^*, α_2^*) is a NE. Then, by definition of an equilibrium

$$\begin{aligned} U_1(\alpha_1^*, \alpha_2^*) &= \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2^*) \\ &\geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) \end{aligned}$$

and since $U_1 = -U_2$ at the same time

$$\begin{aligned} U_1(\alpha_1^*, \alpha_2^*) &= \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1^*, \alpha_2) \\ &\leq \max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) \end{aligned}$$

Hence,

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)$$

which together with Proposition 1 gives the desired conclusion.

$\Rightarrow$ Suppose that

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)$$

and let $\alpha_1^{\max}$ be player 1's $\max\min$ strategy and $\alpha_2^{\min}$ be player 2's $\min\max$ strategy. Then,

$$\begin{aligned} \max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) &= \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1^{\max}, \alpha_2) \\ &\leq U_1(\alpha_1^{\max}, \alpha_2) \; \forall \alpha_2 \in \Delta A_2 \end{aligned}$$

and

$$\begin{aligned} \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) &= \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2^{\min}) \\ &\geq U_1(\alpha_1, \alpha_2^{\min}) \; \forall \alpha_1 \in \Delta A_1 \end{aligned}$$

But

$$\begin{aligned} \max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) &= \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) \\ &= U_1(\alpha_1^{\max}, \alpha_2^{\min}) \end{aligned}$$

implies that

$$U_1(\alpha_1, \alpha_2^{\min}) \leq U_1(\alpha_1^{\max}, \alpha_2^{\min}) \leq U_1(\alpha_1^{\max}, \alpha_2)$$

$\forall \alpha_2 \in \Delta A_2$ and $\forall \alpha_1 \in \Delta A_1$. Hence, $(\alpha_1^{\max}, \alpha_2^{\min})$ is an equilibrium.