University of California – Berkeley
Department of Economics
Game Theory in the Social Sciences
(ECON C110 | POLSCI C135)
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Lecture IV
Applications of Nash equilibrium:
The tragedy of the commons + Cournot's oligopoly

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The tragedy of the commons

William Forster Lloyd (1833)

- Cattle herders sharing a common parcel of land (the commons) on which they are each entitled to let their cows graze. If a herder put more than his allotted number of cattle on the common, overgrazing could result.
- Each additional animal has a positive effect for its herder, but the cost of the extra animal is shared by all other herders, causing a so-called "free-rider" problem. Today's commons include fish stocks, rivers, oceans, and the atmosphere.

Science AAAS

The Tragedy of the Commons

The population problem has no technical solution;
it requires a fundamental extension in morality.

Garrett Hardin

Garrett Hardin (1968)

– This social dilemma was populated by Hardin in his article "The Tragedy of the Commons," published in the journal *Science*. The essay derived its title from Lloyd (1833) on the over-grazing of common land.

– Hardin concluded that "...the commons, if justifiable at all, is justifiable only under conditions of low-population density. As the human population has increased, the commons has had to be abandoned in one aspect after another."

– "The only way we can preserve and nurture other and more precious freedoms is by relinquishing the freedom to breed, and that very soon. "Freedom is the recognition of necessity" – and it is the role of education to reveal to all the necessity of abandoning the freedom to breed. Only so, can we put an end to this aspect of the tragedy of the commons."

"Freedom to breed will bring ruin to all."

Let's put some game theoretic analysis (rigorous sense) behind this story:

- There are n players, each choosing how much to produce in a production activity that 'consumes' some of the clean air that surrounds our planet.

- There is K amount of clean air, and any consumption of clean air comes out of this common resource. Each player $i = 1, ..., n$ chooses his consumption of clean air for production $k_i \geq 0$ and the amount of clean air left is therefore

$$K - \sum_{i=1}^{n} k_i.$$

- The benefit of consuming an amount $k_i \geq 0$ of clean air gives player i a benefit equal to $\ln(k_i)$. Each player also enjoys consuming the reminder of the clean air, giving each a benefit

$$\ln\left(K - \sum_{i=1}^{n} k_i\right).$$

- Hence, the value for each player i from the action profile (outcome) $k = (k_1, ..., k_n)$ is give by

$$v_i(k_i, k_{-i}) = \ln(k_i) + \ln\left(K - \sum_{j=1}^{n} k_j\right).$$

– To get player i's best-response function, we write down the first-order condition of his payoff function:

$$\frac{\partial v_i(k_i, k_{-i})}{\partial k_i} = \frac{1}{k_i} - \frac{1}{K - \sum_{j=1}^{n} k_j} = 0$$

and thus

$$BR_i(k_{-i}) = \frac{K - \sum_{j \neq i} k_j}{2}.$$

The two-player Tragedy of the Commons

- To find the Nash equilibrium, there are n equations with n unknown that need to be solved. We first solve the equilibrium for two players. Letting $k_i(k_j)$ be the best response of player i, we have two best-response functions:

$$k_1(k_2) = \frac{K - k_2}{2} \quad \text{and} \quad k_2(k_1) = \frac{K - k_1}{2}.$$

- If we solve the two best-response functions simultaneously, we find the unique (pure-strategy) Nash equilibrium

$$k_1^{NE} = k_2^{NE} = \frac{K}{3}.$$

Can this two-player society do better? More specifically, is consuming $\frac{K}{3}$ clean air for each player too much (or too little)?

- The 'right way' to answer this question is using the Pareto principle (Vilfredo Pareto, 1848-1923) – can we find another action profile $k = (k_1, k_2)$ that will make both players better off than in the Nash equilibrium?

- To this end, the function we seek to maximize is the social welfare function w given by

$$w(v_1, v_2) = v_1 + v_2 = \sum_{i=1}^{2} \ln(k_i) + 2 \ln \left(K - \sum_{i=1}^{2} k_i \right).$$

– The first-order conditions for this problem are

$$\frac{\partial w(k_1, k_2)}{\partial k_1} = \frac{1}{k_1} - \frac{2}{K - k_1 - k_2} = 0$$

and

$$\frac{\partial w(k_1, k_2)}{\partial k_2} = \frac{1}{k_2} - \frac{2}{K - k_1 - k_2} = 0.$$

– Solving these two equations simultaneously result the unique Pareto optimal outcome

$$k_1^{PO} = k_2^{PO} = \frac{K}{4}.$$

The n-player Tragedy of the Commons

– In the n-player Tragedy of the Commons, the best response of each player $i = 1, ..., n$, $k_i(k_{-i})$, is given by

$$BR_i(k_{-i}) = \frac{K - \sum_{j \neq i} k_j}{2}.$$

– We consider a symmetric Nash equilibrium where each player i chooses the same level of consumption of clean air k^* (it is subtle to show that there cannot be asymmetric Nash equilibria).

– Because the best response must hold for each player i and they all choose the same level k^{SNE} then in the symmetric Nash equilibrium all best-response functions reduce to

$$k^{SNE} = \frac{K - \sum_{j \neq i} k^{SNE}}{2} = \frac{K - (n-1)k^{SNE}}{2}$$

or

$$k^{SNE} = \frac{K}{n+1}.$$

Hence, the sum of clean air consumed by the firms is $\frac{n}{n+1}K$, which increases with n as Hardin conjectured.

What is the socially optimal outcome with n players? And how does society size affect this outcome?

– With n players, the social welfare function w given by

$$\begin{aligned} w(v_1, ..., v_n) &= \textstyle\sum_{i=1}^n v_i \\ &= \textstyle\sum_{i=1}^n \ln(k_i) + n \ln\left(K - \sum_{i=1}^n k_i\right). \end{aligned}$$

And the n first-order conditions for the problem of maximizing this function are

$$\frac{\partial w(k_1, ..., k_n)}{\partial k_i} = \frac{1}{k_i} - \frac{n}{K - \sum_{j=1}^n k_j} = 0$$

for $i = 1, ..., n$.

– Just as for the analysis of the Nash equilibrium with n players, the solution here is also symmetric. Therefore, the Pareto optimal consumption of each player k^{PO} can be found using the following equation:

$$\frac{1}{k^{PO}} - \frac{n}{K - nk^{PO}} = 0$$

or

$$k^{PO} = \frac{K}{2n}$$

and thus the Pareto optimal consumption of air is equal $\frac{K}{2}$, for any society size n. for $i = 1, ..., n$.

Finally, we show there is no asymmetric equilibrium.

- To this end, assume there are two players, i and j, choosing two different $k_i \neq k_j$ in equilibrium.

- Because we assume a Nash equilibrium the best-response functions of i and j must hold simultaneously, that is

$$k_i = \frac{K - \bar{k} - k_j}{2} \quad \text{and} \quad k_j = \frac{K - \bar{k} - k_i}{2}$$

  where $\bar{k}$ be the sum of equilibrium choices of all other players except i and j.

– However, if we solve the best-response functions of players i and j simultaneously, we find that

$$k_i = k_j = \frac{K - \bar{k}}{3}$$

contracting the assumption we started with that $k_i \neq k_j$.

Oligopoly

Cournot's oligopoly model (1838)

– A single good is produced by two firms (the industry is a "duopoly").

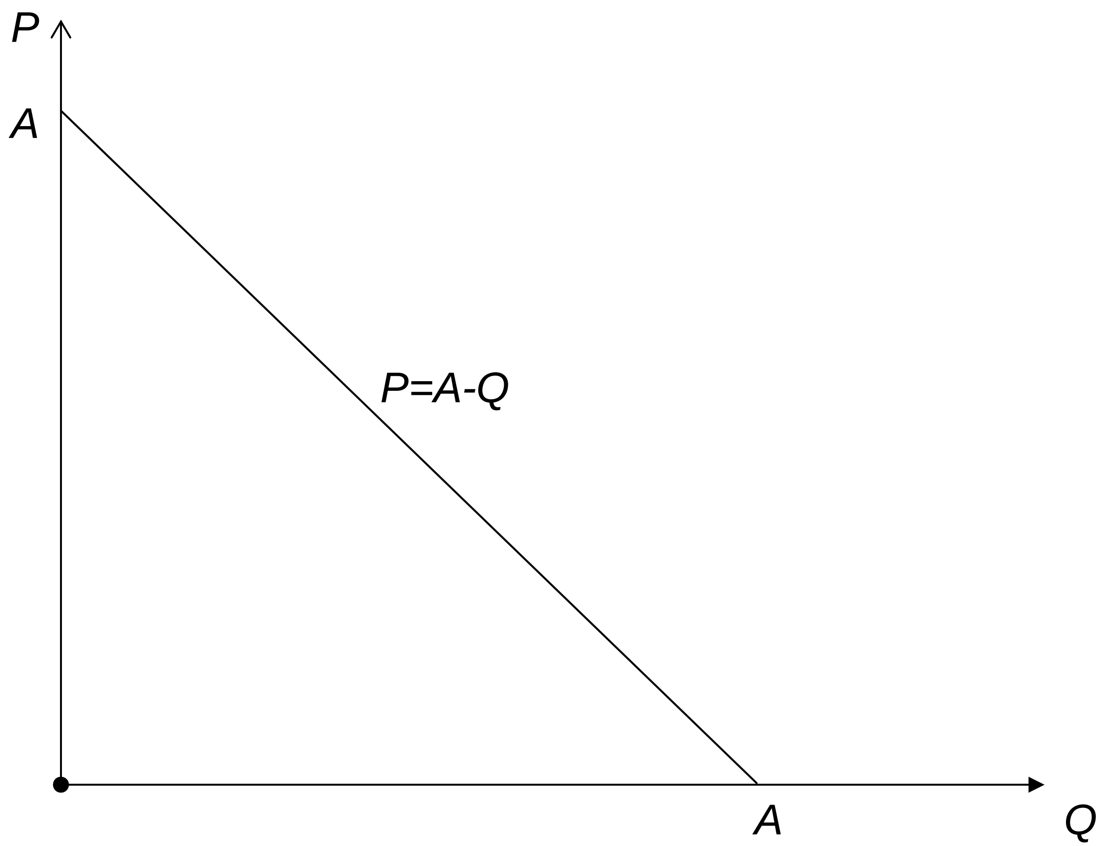
– The cost for firm $i = 1, 2$ for producing q_i units of the good is given by $c_i q_i$ ("unit cost" is constant equal to $c_i > 0$).

– If the firms' total output is $Q = q_1 + q_2$ then the market price is

$$P = A - Q$$

if $A \geq Q$ and zero otherwise (linear inverse demand function). We also assume that $A > c$.

The inverse demand function



To find the Nash equilibria of the Cournot's game, we can use the procedures based on the firms' best response functions.

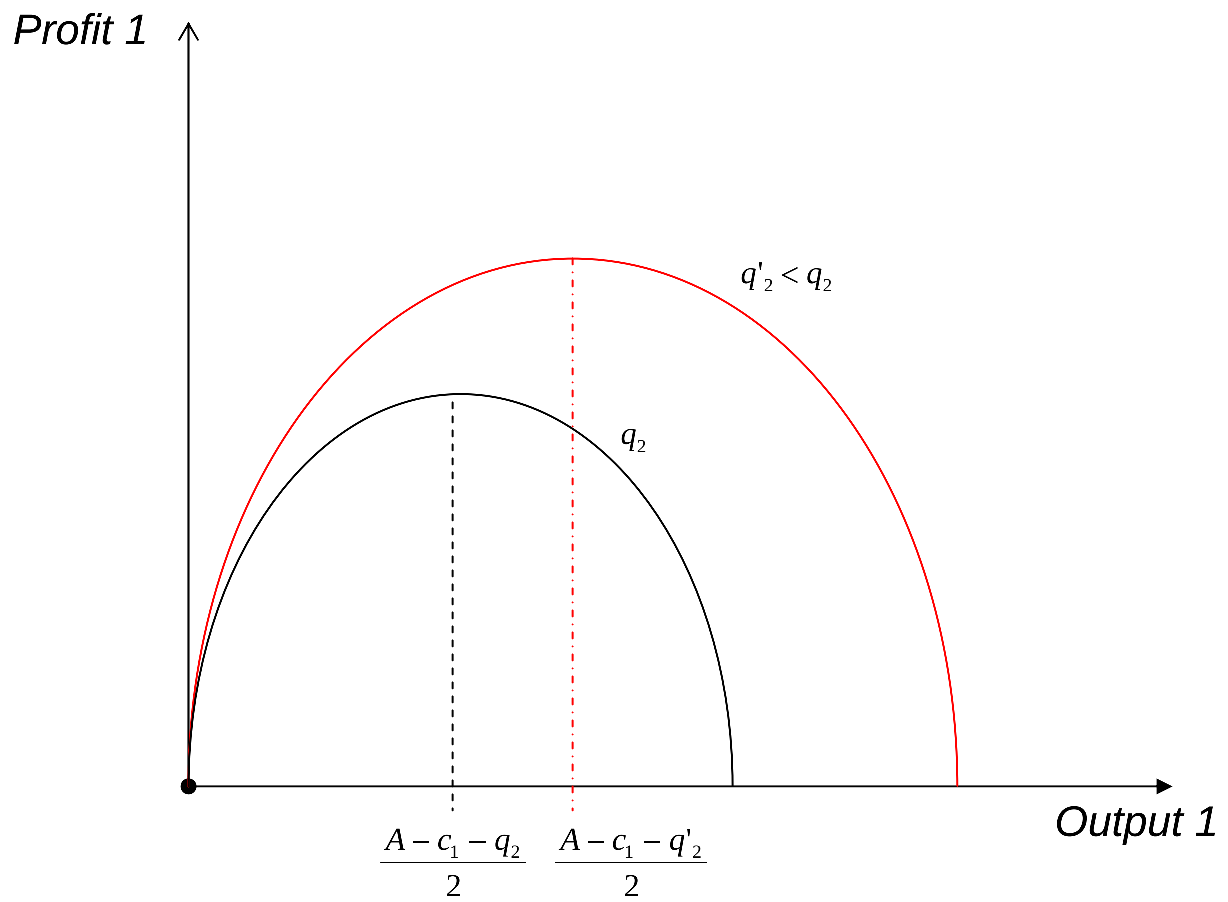
But first we need the firms payoffs (profits):

$$\begin{aligned}\pi_1 &= Pq_1 - c_1 q_1 \\ &= (A-Q)q_1 - c_1 q_1 \\ &= (A - q_1 - q_2)q_1 - c_1 q_1 \\ &= (A - q_1 - q_2 - c_1)q_1\end{aligned}$$

and similarly,

$$\pi_2 = (A - q_1 - q_2 - c_2)q_2$$

Firm 1's profit as a function of its output
(given firm 2's output)



To find firm 1's best response to any given output q_2 of firm 2, we need to study firm 1's profit as a function of its output q_1 for given values of q_2.

Using calculus, we set the derivative of firm 1's profit with respect to q_1 equal to zero and solve for q_1:

$$q_1 = \frac{1}{2}(A - q_2 - c_1).$$

We conclude that the best response of firm 1 to the output q_2 of firm 2 depends on the values of q_2 and c_1.

Because firm 2's cost function is $c_2 \neq c_1$, its best response function is given by

$$q_2 = \frac{1}{2}(A - q_1 - c_2).$$

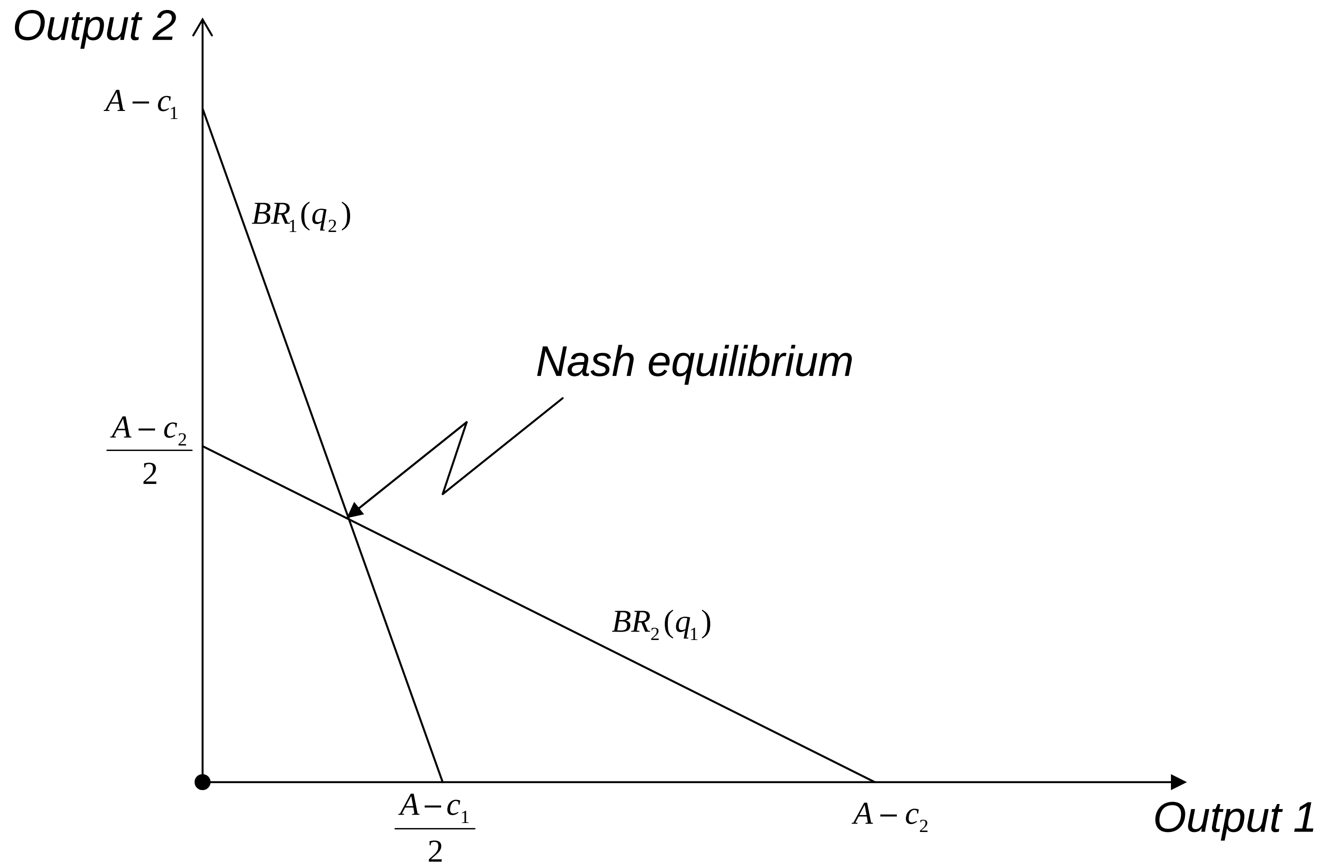
A Nash equilibrium of the Cournot's game is a pair (q_1^*, q_2^*) of outputs such that q_1^* is a best response to q_2^* and q_2^* is a best response to q_1^*.

From the figure below, we see that there is exactly one such pair of outputs

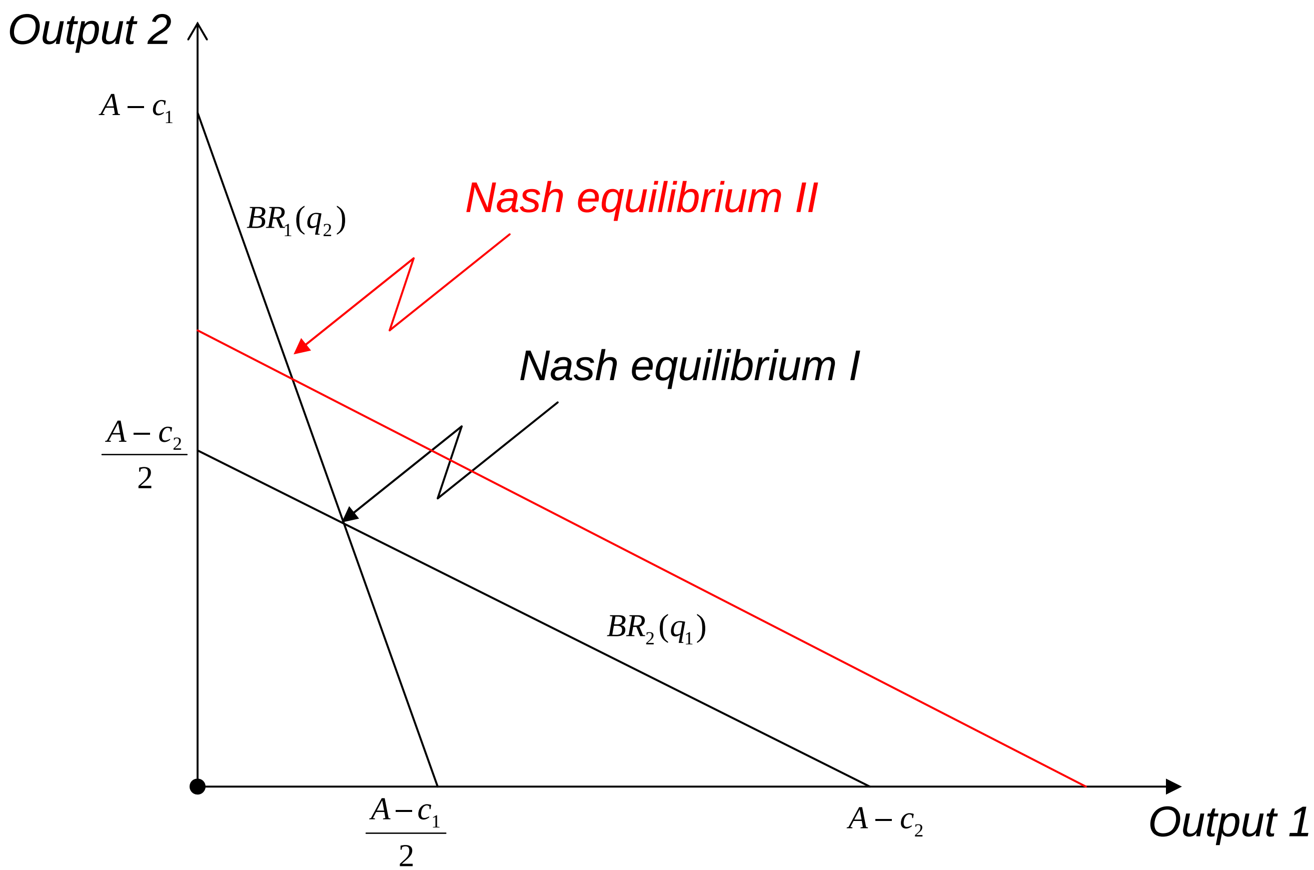
$$q_1^* = \frac{A+c_2-2c_1}{3} \quad \text{and} \quad q_2^* = \frac{A+c_1-2c_2}{3}$$

which is the solution to the two equations above.

The best response functions in the Cournot's duopoly game



Nash equilibrium comparative statics
(a decrease in the cost of firm 2)



A question: what happens when consumers are willing to pay more (*A* increases)?

In summary, this simple Cournot's duopoly game has a unique Nash equilibrium.

Two economically important properties of the Nash equilibrium are (to economic regulatory agencies):

[1] The relation between the firms' equilibrium profits and the profit they could make if they act collusively.

[2] The relation between the equilibrium profits and the number of firms.