

**Economics 201A**  
**Economic Theory**  
**(Fall 2009)**  
**Bargaining**

**Topics:** the axiomatic approach (OR 15) the strategic approach (OR 7).

## Nash (1953) bargaining

A bargaining situation is a tuple  $\langle N, A, D, (\succsim_i) \rangle$  where

- $N$  is a set of players or bargainers ( $N = \{1, 2\}$ ),
- $A$  is a set of agreements/outcomes,
- $D$  is a disagreement outcome, and
- $\succsim_i$  is a preference ordering over the set of lotteries over  $A \cup \{D\}$ .

The objects  $N$ ,  $A$ ,  $D$  and  $\succsim_i$  for  $i = \{1, 2\}$  define a bargaining situation.

$\succsim_1$  and  $\succsim_2$  satisfy the assumption of  $vNM$  so for each  $i$  there is a utility function  $u_i : A \cup \{D\} \rightarrow \mathbb{R}$ .

$\langle S, d \rangle$  is the primitive of Nash's bargaining problem where

- $S = (u_1(a), u_2(a))$  for  $a \in A$  the set of all utility pairs, and
- $d = (u_1(D), u_2(D))$ .

A bargaining problem is a pair  $\langle S, d \rangle$  where  $S \subset \mathbb{R}^2$  is compact and convex,  $d \in S$  and there exists  $s \in S$  such that  $s_i > d_i$  for  $i = 1, 2$ . The set of all bargaining problems  $\langle S, d \rangle$  is denoted by  $B$ .

A bargaining solution is a function  $f : B \rightarrow \mathbb{R}^2$  such that  $f$  assigns to each bargaining problem  $\langle S, d \rangle \in B$  a unique element in  $S$ .

## **Nash's axioms (OR 15.3)**

One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely - Nash 1953 -

Does not capture the details of a specific bargaining problem (e.g. alternating or simultaneous offers).

Rather, the approach consists of the following four axioms: invariance to equivalent utility representations, symmetry, independence of irrelevant alternatives, and (weak) Pareto efficiency.

## Invariance to equivalent utility representations (*INV*)

$\langle S', d' \rangle$  is obtained from  $\langle S, d \rangle$  by the transformations

$$s_i \mapsto \alpha_i s_i + \beta_i$$

for  $i = 1, 2$  if

$$d'_i = \alpha_i d_i + \beta_i$$

and

$$S' = \{(\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2) \in \mathbb{R}^2 : (s_1, s_2) \in S\}.$$

Note that if  $\alpha_i > 0$  for  $i = 1, 2$  then  $\langle S', d' \rangle$  is itself a bargaining problem.

If  $\langle S', d' \rangle$  is obtained from  $\langle S, d \rangle$  by the transformations

$$s_i \mapsto \alpha_i s_i + \beta_i$$

for  $i = 1, 2$  where  $\alpha_i > 0$  for each  $i$ , then

$$f_i(S', d') = \alpha_i f_i(S, d) + \beta_i$$

for  $i = 1, 2$ . Hence,  $\langle S', d' \rangle$  and  $\langle S, d \rangle$  represent the same situation.

*INV* requires that the utility outcome of the bargaining problem co-vary with representation of preferences.

The physical outcome predicted by the bargaining solution is the same for  $\langle S', d' \rangle$  and  $\langle S, d \rangle$ .

A corollary of *INV* is that we can restrict attention to  $\langle S, d \rangle$  such that

$$S \subset \mathbb{R}_+^2,$$

$$S \cap \mathbb{R}_{++}^2 \neq \emptyset, \text{ and}$$

$$d = (0, 0) \in S \text{ (reservation utilities).}$$

## Symmetry (*SYM*)

A bargaining problem  $\langle S, d \rangle$  is symmetric if  $d_1 = d_2$  and  $(s_1, s_2) \in S$  if and only if  $(s_2, s_1) \in S$ . If the bargaining problem  $\langle S, d \rangle$  is symmetric then

$$f_1(S, d) = f_2(S, d)$$

Nash does not describe differences between the players. All asymmetries (in the bargaining abilities) must be captured by  $\langle S, d \rangle$ .

Hence, if players are the same the bargaining solution must assign the same utility to each player.

## **Independence of irrelevant alternatives (*IIA*)**

If  $\langle S, d \rangle$  and  $\langle T, d \rangle$  are bargaining problems with  $S \subset T$  and  $f(T, d) \in S$  then

$$f(S, d) = f(T, d)$$

If  $T$  is available and players agree on  $s \in S \subset T$  then they agree on the same  $s$  if only  $S$  is available.

*IIA* excludes situations in which the fact that a certain agreement is available influences the outcome.

## Weak Pareto efficiency (*WPO*)

If  $\langle S, d \rangle$  is a bargaining problem where  $s \in S$  and  $t \in S$ , and  $t_i > s_i$  for  $i = 1, 2$  then  $f(S, d) \neq s$ .

In words, players never agree on an outcome  $s$  when there is an outcome  $t$  in which both are better off.

Hence, players never disagree since by assumption there is an outcome  $s$  such that  $s_i > d_i$  for each  $i$ .

*SYM* and *WPO*

restrict the solution on single bargaining problems.

*INV* and *IIA*

requires the solution to exhibit some consistency across bargaining problems.

Nash 1953: there is precisely one bargaining solution, denoted by  $f^N(S, d)$ , satisfying *SYM*, *WPO*, *INV* and *IIA*.

## Nash's solution (OR 15.4)

The unique bargaining solution  $f^N : B \rightarrow \mathbb{R}^2$  satisfying *SYM*, *WPO*, *INV* and *IIA* is given by

$$f^N(S, d) = \arg \max_{(d_1, d_2) \leq (s_1, s_2) \in S} (s_1 - d_1)(s_2 - d_2)$$

and since we normalize  $(d_1, d_2) = (0, 0)$

$$f^N(S, 0) = \arg \max_{(s_1, s_2) \in S} s_1 s_2$$

The solution is the utility pair that maximizes the product of the players' utilities.

## Proof

Pick a compact and convex set  $S \subset \mathbb{R}_+^2$  where  $S \cap \mathbb{R}_{++}^2 \neq \emptyset$ .

Step 1:  $f^N$  is well defined.

- Existence: the set  $S$  is compact and the function  $f = s_1 s_2$  is continuous.
- Uniqueness:  $f$  is strictly quasi-concave on  $S$  and the set  $S$  is convex.

Step 2:  $f^N$  is the only solution that satisfies *SYM*, *WPO*, *INV* and *IIA*.

Suppose there is another solution  $f$  that satisfies *SYM*, *WPO*, *INV* and *IIA*.

Let

$$S' = \left\{ \left( \frac{s_1}{f_1^N(S)}, \frac{s_2}{f_2^N(S)} \right) : (s_1, s_2) \in S \right\}$$

and note that  $s'_1 s'_2 \leq 1$  for any  $s' \in S'$ , and thus  $f^N(S', 0) = (1, 1)$ .

Since  $S'$  is bounded we can construct a set  $T$  that is symmetric about the  $45^\circ$  line and contains  $S'$

$$T = \{(a, b) : a + b \leq 2\}$$

By *WPO* and *SYM* we have  $f(T, 0) = (1, 1)$ , and by *IIA* we have  $f(S', 0) = f(T, 0) = (1, 1)$ .

By *INV* we have that  $f(S', 0) = f^N(S', 0)$  if and only if  $f(S, 0) = f^N(S, 0)$  which completes the proof.

## Is any axiom superfluous?

INV

The bargaining solution given by the maximizer of

$$g(s_1, s_2) = \sqrt{s_1} + \sqrt{s_2}$$

over  $\langle S, 0 \rangle$  where  $S := \text{co}\{(0, 0), (1, 0), (0, 2)\}$ .

This solution satisfies *WPO*, *SYM* and *IIA* (maximizer of an increasing function). The maximizer of  $g$  for this problem is  $(1/3, 4/3)$  while  $f^N = (1/2, 1)$ .

## SYM

The family of solutions  $\{f^\alpha\}_{\alpha \in (0,1)}$  over  $\langle S, 0 \rangle$  where

$$f^\alpha(S, d) = \arg \max_{(d_1, d_2) \leq (s_1, s_2) \in S} (s_1 - d_1)^\alpha (s_2 - d_2)^{1-\alpha}$$

is called the asymmetric Nash solution.

Any  $f^\alpha$  satisfies *INV*, *IIA* and *WPO* by the same arguments used for  $f^N$ .

For  $\langle S, 0 \rangle$  where  $S := \text{co}\{(0, 0), (1, 0), (0, 1)\}$  we have  $f^\alpha(S, 0) = (\alpha, 1 - \alpha)$  which is different from  $f^N$  for any  $\alpha \neq 1/2$ .

## WPO

Consider the solution  $f^d$  given by  $f^d(S, d) = d$  which is different from  $f^N$ .  $f^d$  satisfies *INV*, *SYM* and *IIA*.

*WPO* in the Nash solution can be replaced with strict individual rationality (*SIR*)  
 $f(S, d) \gg d$ .

## An application - risk aversion

Dividing a dollar: the role of risk aversion: Suppose that

$$A = \{(a_1, a_2) \in \mathbb{R}_+^2 : a_1 + a_2 \leq 1\}$$

(all possible divisions),  $D = (0, 0)$  and for all  $a, b \in A$   $a \succsim_i b$  if and only if  $a_i \geq b_i$ .

Player  $i$ 's preferences over  $A \cup D$  can be represented by  $u_i : [0, 1] \rightarrow \mathbb{R}$  where each  $u_i$  is concave and (WLOG)  $u_i(0) = 0$ .

Then,

$$S = \{(s_1, s_2) \in \mathbb{R}_+^2 : (s_1, s_2) = (u_1(a_1), u_2(a_2))\}$$

for some  $(a_1, a_2) \in A$  is compact and convex and

$$d = (u_1(0), u_2(0)) = (0, 0) \in S.$$

First, note that when  $u_1(a) = u_2(a)$  for all  $a \in (0, 1]$  then  $\langle S, d \rangle$  is symmetric so by *SYM* and *WPO* the Nash solution is  $(u(1/2), u(1/2))$ .

Now, suppose that  $v_1 = u_1$  and  $v_2 = h \circ u_2$  where  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is increasing and concave and  $h(0) = 0$  (player 2 is more risk averse).

Let  $\langle S', d' \rangle$  be bargaining problem when the preferences of the players are represented by  $v_1$  and  $v_2$ .

Let  $z_u$  be the solution of

$$\max_{0 \leq z \leq 1} u_1(z)u_2(1 - z),$$

and  $z_v$  the corresponding solution when  $u_i = v_i$  for  $i = 1, 2$ .

Then,

$$f^N(S, d) = (u_1(z_u), u_2(1 - z_u)) \text{ and } f^N(S', d') = (v_1(z_v), v_2(1 - z_v)).$$

If  $u_i$  for  $i = 1, 2$  and  $h$  are differentiable then  $z_u$  and  $z_v$  are, in respect, the solutions of

$$\frac{u'_1(z)}{u_1(z)} = \frac{u'_2(1 - z)}{u_2(1 - z)}, \quad (1)$$

and

$$\frac{u'_1(z)}{u_1(z)} = \frac{h'(u_2(1 - z))u'_2(1 - z)}{h(u_2(1 - z))}. \quad (2)$$

Since  $h$  is increasing and concave and  $h(0) = 0$  we have

$$h'(t) \leq \frac{h(t)}{t}$$

for all  $t$ , so the RHS of (1) is at least as the RHS of (2) and thus  $z_u \leq z_v$ . Thus, if player 2 becomes more risk-averse, then  $f_1^N$  increases and  $f_2^N$  decreases.

If player 2's marginal utility declines more rapidly than that of player 1, then player 1's share exceeds 1/2.

## The strategic approach (OR 7.1, 7.2)

The players bargain over a pie of size 1.

An agreement is a pair  $(x_1, x_2)$  where  $x_i$  is player  $i$ 's share of the pie. The set of possible agreements is

$$X = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 = 1\}$$

Player  $i$  prefers  $x \in X$  to  $y \in X$  if and only if  $x_i > y_i$ .

## The bargaining protocol

The players can take actions only at times in the (infinite) set  $T = \{0, 1, 2, \dots\}$ . In each  $t \in T$  player  $i$ , proposes an agreement  $x \in X$  and  $j \neq i$  either accepts ( $Y$ ) or rejects ( $N$ ).

If  $x$  is accepted ( $Y$ ) then the bargaining ends and  $x$  is implemented. If  $x$  is rejected ( $N$ ) then the play passes to period  $t + 1$  in which  $j$  proposes an agreement.

At all times players have perfect information. Every path in which all offers are rejected is denoted as disagreement ( $D$ ). The only asymmetry is that player 1 is the first to make an offer.

## Preferences

Time preferences (toward agreements at different points in time) are the driving force of the model.

A bargaining game of alternating offers is

- an extensive game of perfect information with the structure given above, and
- player  $i$ 's preference ordering  $\succsim_i$  over  $(X \times T) \cup \{D\}$  is complete and transitive.

Preferences over  $X \times T$  are represented by  $\delta_i^t u_i(x_i)$  for any  $0 < \delta_i < 1$  where  $u_i$  is increasing and concave.

## Assumptions on preferences

A1 Disagreement is the worst outcome

For any  $(x, t) \in X \times T$ ,

$$(x, t) \succsim_i D$$

for each  $i$ .

A2 Pie is desirable

– For any  $t \in T$ ,  $x \in X$  and  $y \in X$

$$(x, t) \succ_i (y, t) \text{ if and only if } x_i > y_i.$$

### A3 Time is valuable

For any  $t \in T$ ,  $s \in T$  and  $x \in X$

$$(x, t) \succsim_i (x, s) \text{ if } t < s$$

and with strict preferences if  $x_i > 0$ .

### A4 Preference ordering is continuous

Let  $\{(x_n, t)\}_{n=1}^{\infty}$  and  $\{(y_n, s)\}_{n=1}^{\infty}$  be members of  $X \times T$  for which

$$\lim_{n \rightarrow \infty} x_n = x \text{ and } \lim_{n \rightarrow \infty} y_n = y.$$

Then,  $(x, t) \succsim_i (y, s)$  whenever  $(x_n, t) \succsim_i (y_n, s)$  for all  $n$ .

**A2-A4** imply that for any outcome  $(x, t)$  either there is a unique  $y \in X$  such that

$$(y, 0) \sim_i (x, t)$$

or

$$(y, 0) \succ_i (x, t)$$

for every  $y \in X$ .

Note  $\succsim_i$  satisfies **A2-A4** iff it can be represented by a continuous function

$$U_i : [0, 1] \times T \rightarrow \mathbb{R}$$

that is increasing (decreasing) in the first (second) argument.

## A5 Stationarity

For any  $t \in T$ ,  $x \in X$  and  $y \in X$

$(x, t) \succsim_i (y, t + 1)$  if and only if  $(x, 0) \succsim_i (y, 1)$ .

If  $\succsim_i$  satisfies **A2-A5** then for every  $\delta \in (0, 1)$  there exists a continuous increasing function  $u_i : [0, 1] \rightarrow \mathbb{R}$  (not necessarily concave) such that

$$U_i(x_i, t) = \delta_i^t u_i(x_i).$$

## Present value

Define  $v_i : [0, 1] \times T \rightarrow [0, 1]$  for  $i = 1, 2$  as follows

$$v_i(x_i, t) = \begin{cases} y_i & \text{if } (y, 0) \sim_i (x, t) \\ 0 & \text{if } (y, 0) \succsim_i (x, t) \text{ for all } y \in X. \end{cases}$$

We call  $v_i(x_i, t)$  player  $i$ 's present value of  $(x, t)$  and note that

$$(y, t) \succsim_i (x, s) \text{ whenever } v_i(y_i, t) > v_i(x_i, s).$$

If  $\succsim_i$  satisfies **A2-A4**, then for any  $t \in T$   $v_i(\cdot, t)$  is continuous, non decreasing and increasing whenever  $v_i(x_i, t) > 0$ .

Further,  $v_i(x_i, t) \leq x_i$  for every  $(x, t) \in X \times T$  and with strict whenever  $x_i > 0$  and  $t \geq 1$ .

With **A5**, we also have that

$$v_i(v_i(x_i, 1), 1) = v_i(x_i, 2)$$

for any  $x \in X$ .

## Delay

### A6 Increasing loss to delay

$x_i - v_i(x_i, \mathbf{1})$  is an increasing function of  $x_i$ .

If  $u_i$  is differentiable then under **A6** in any representation  $\delta_i^t u_i(x_i)$  of  $\succsim_i$

$$\delta_i u_i'(x_i) < u_i'(v_i(x_i, \mathbf{1}))$$

whenever  $v_i(x_i, \mathbf{1}) > 0$ .

This assumption is weaker than concavity of  $u_i$  which implies

$$u_i'(x_i) < u_i'(v_i(x_i, \mathbf{1})).$$

## The single crossing property of present values

If  $\succsim_i$  for each  $i$  satisfies **A2-A6**, then there exist a unique pair  $(x^*, y^*) \in X \times X$  such that

$$y_1^* = v_1(x_1^*, 1) \text{ and } x_2^* = v_2(y_2^*, 1).$$

– For every  $x \in X$ , let  $\psi(x)$  be the agreement for which

$$\psi_1(x) = v_1(x_1, 1)$$

and define  $H : X \rightarrow \mathbb{R}$  by

$$H(x) = x_2 - v_2(\psi_2(x), 1).$$

- The pair of agreements  $x$  and  $y = \psi(x)$  satisfies also  $x_2 = v_2(\psi_2(x), 1)$  iff  $H(x) = 0$ .
- Note that  $H(0, 1) \geq 0$  and  $H(1, 0) \leq 0$ ,  $H$  is a continuous function, and

$$H(x) = [v_1(x_1, 1) - x_1] + [1 - v_1(x_1, 1) - v_2(1 - v_1(x_1, 1), 1)].$$

- Since  $v_1(x_1, 1)$  is non decreasing in  $x_1$ , and both terms are decreasing in  $x_1$ ,  $H$  has a unique zero by **A6**.

## Examples

[1] For every  $(x, t) \in X \times T$

$$U_i(x_i, t) = \delta_i^t x_i$$

where  $\delta_i \in (0, 1)$ , and  $U_i(D) = 0$ .

[2] For every  $(x, t) \in X \times T$

$$U_i(x_i, t) = x_i - c_i t$$

where  $c_i > 0$ , and  $U_i(D) = -\infty$  (constant cost of delay).

Although **A6** is violated, when  $c_1 \neq c_2$  there is a unique pair  $(x, y) \in X \times X$  such that  $y_1 = v_1(x_1, 1)$  and  $x_2 = v_2(y_2, 1)$ .

## Strategies

Let  $X^t$  be the set of all sequences  $\{x^0, \dots, x^{t-1}\}$  of members of  $X$ .

A strategy of player 1 (2) is a sequence of functions

$$\sigma = \{\sigma^t\}_{t=0}^{\infty}$$

such that  $\sigma^t : X^t \rightarrow X$  if  $t$  is even (odd), and  $\sigma^t : X^{t+1} \rightarrow \{Y, N\}$  if  $t$  is odd (even).

The way of representing a player's strategy is closely related to the notion of automation.

## Nash equilibrium

For any  $\bar{x} \in X$ , the outcome  $(\bar{x}, 0)$  is a *NE* when players' preference satisfy **A1-A6**.

To see this, consider the stationary strategy profile

Player 1	proposes	$\bar{x}$
	accepts	$x_1 \geq \bar{x}_1$
Player 2	proposes	$\bar{x}$
	accepts	$x_2 \geq \bar{x}_2$

This is an example for a pair of one-state automata.

The set of outcomes generated in the Nash equilibrium includes also delays (agreements in period 1 or later).

## Subgame perfect equilibrium (OR 7.3)

Any bargaining game of alternating offers in which players' preferences satisfy **A1-A6** has a unique *SPE* which is the solution of the following equations

$$y_1^* = v_1(x_1^*, 1) \text{ and } x_2^* = v_2(y_2^*, 1).$$

Note that if  $y_1^* > 0$  and  $x_2^* > 0$  then

$$(y_1^*, 0) \sim_1 (x_1^*, 1) \text{ and } (x_2^*, 0) \sim_2 (y_2^*, 1).$$

The equilibrium strategy profile is given by

Player 1	proposes	$x^*$
	accepts	$y_1 \geq y_1^*$
Player 2	proposes	$y^*$
	accepts	$x_2 \geq x_2^*$

The unique outcome is that player 1 proposes  $x^*$  in period 0 and player 2 accepts.

Step 1  $(x^*, y^*)$  is a *SPE*

Player 1:

- proposing  $x^*$  at  $t^*$  leads to an outcome  $(x^*, t^*)$ . Any other strategy generates either

$$(x, t) \text{ where } x_1 \leq x_1^* \text{ and } t \geq t^*$$

or

$$(y^*, t) \text{ where } t \geq t^* + 1$$

or  $D$ .

- Since  $x_1^* > y_1^*$  it follows from **A1-A3** that  $(x^*, t^*)$  is a best response.

Player 2:

- accepting  $x^*$  at  $t^*$  leads to an outcome  $(x^*, t^*)$ . Any other strategy generates either

$$(y, t) \text{ where } y_2 \leq y_2^* \text{ and } t \geq t^* + 1$$

or

$$(x^*, t) \text{ where } t \geq t^*$$

or  $D$ .

– By **A1-A3** and **A5**

$$(x^*, t^*) \succsim_2 (y^*, t^* + 1)$$

and thus accepting  $x^*$  at  $t^*$ , which leads to the outcome  $(x^*, t^*)$ , is a best response.

Note that similar arguments apply to a subgame starting with an offer of player 2.

Step 2  $(x^*, y^*)$  is the unique  $SPE$

Let  $G_i$  be a subgame starting with an offer of player  $i$  and define

$$M_i = \sup\{v_i(x_i, t) : (x, t) \in SPE(G_i)\},$$

and

$$m_i = \inf\{v_i(x_i, t) : (x, t) \in SPE(G_i)\}.$$

It suffices to show that

$$M_1 = m_1 = x_1^* \text{ and } M_2 = m_2 = y_2^*.$$

First, note that in any *SPE* the first offer is accepted because

$$v_1(y_1^*, 1) \leq y_1^* < x_1^*.$$

Thus, after a rejection, the present value for player 1 is less than  $x_1^*$ .

Then, it remains to show that

$$m_2 \geq 1 - v_1(M_1, 1) \tag{3}$$

and

$$M_1 \leq 1 - v_2(m_2, 1). \tag{4}$$

3 implies that the pair  $(M_1, 1 - m_2)$  lies below the line

$$y_1 = v_1(x_1, 1)$$

and 4 implies that the pair  $(M_1, 1 - m_2)$  lies to the left the line

$$x_2 = v_2(y_2, 1).$$

Thus,

$$M_1 = x_1^* \text{ and } m_2 = y_2^*,$$

and with the role of the players reversed, the same argument show that

$$M_2 = y_2^* \text{ and } m_1 = x_1^*.$$

With constant discount rates the equilibrium condition implies that

$$y_1^* = \delta_1 x_1^* \text{ and } x_2^* = \delta_2 y_2^*$$

so that

$$x^* = \left( \frac{1 - \delta_2}{1 - \delta_1 \delta_2}, \frac{\delta_2(1 - \delta_1)}{1 - \delta_1 \delta_2} \right) \text{ and } y^* = \left( \frac{\delta_1(1 - \delta_2)}{1 - \delta_1 \delta_2}, \frac{1 - \delta_1}{1 - \delta_1 \delta_2} \right).$$

Thus, if  $\delta_1 = \delta_2 = \delta$  ( $v_1 = v_2$ ) then

$$x^* = \left( \frac{1}{1 + \delta}, \frac{\delta}{1 + \delta} \right) \text{ and } y^* = \left( \frac{\delta}{1 + \delta}, \frac{1}{1 + \delta} \right)$$

so player 1 obtains more than half of the pie.

But, shrinking the length of a period by considering a sequence of games indexed by  $\Delta$  in which  $u_i = \delta_i^{\Delta t} x_i$  we have

$$\lim_{\Delta \rightarrow 0} x^*(\Delta) = \lim_{\Delta \rightarrow 0} y^*(\Delta) = \left( \frac{\log \delta_2}{\log \delta_1 + \log \delta_2}, \frac{\log \delta_1}{\log \delta_1 + \log \delta_2} \right).$$