

**Economics 209A**  
**Theory and Application of Non-Cooperative Games**  
**(Fall 2011)**

**Strategic Games**

**Topics:** terminology and notations (OR 1.7), games and solutions (OR 1.1-1.3), rationality and bounded rationality (OR 1.4, 1.6), formalities (OR 2.1), best-response (OR 2.2), Nash equilibrium (OR 2.2), examples (OR 2.3), existence of Nash equilibrium (OR 2.4), mixed strategy Nash equilibrium (OR 3.1, 3.2), Strictly competitive games (OR 2.5), Bayesian equilibrium (OR 2.6), Bayesian equilibrium (OR 2.6), correlated equilibrium (OR 3.3), evolutionary equilibrium (OR 3.4), equivalence and symmetry, rationalizability (OR 4.1), dominance (OR 4.2, 4.3), trembling hand perfection (OR 12.5.1).

## Terminology and notations (OR 1.7)

**Sets** For  $x, y \in \mathbb{R}^n$ ,

$$x \geq y \iff x_i \geq y_i$$

for all  $i$ .

$$x > y \iff x_i \geq y_i \text{ and } x_j > y_j$$

for all  $i$  and some  $j$ .

$$x \gg y \iff x_i > y_i$$

for all  $i$ .

**Preferences**  $\succsim$  is a binary relation on some set of alternatives  $A \subseteq \mathbb{R}^n$ .  
From  $\succsim$  we derive two other relations on  $A$ :

– strict performance relation

$$a \succ b \iff a \succsim b \text{ and not } b \succsim a$$

– indifference relation

$$a \sim b \iff a \succsim b \text{ and } b \succsim a$$

**Utility representation**  $\succsim$  is said to be

- complete if  $\forall a, b \in A, a \succsim b$  or  $b \succsim a$  .
- transitive if  $\forall a, b, c \in A, a \succsim b$  and  $b \succsim c$  then  $a \succsim c$ .

$\succsim$  can be presented by a utility function only if it is complete and transitive (rational).

A function  $u : A \rightarrow \mathbb{R}$  is a utility function representing  $\succsim$  if  $\forall a, b \in A$

$$a \succsim b \iff u(a) \geq u(b).$$

$\succsim$  is said to be

- continuous (preferences cannot jump...) if for any sequence of pairs  $\{(a^k, b^k)\}_{k=1}^{\infty}$  with  $a^k \succsim b^k$ , and  $a^k \rightarrow a$  and  $b^k \rightarrow b$ ,  $a \succsim b$ .
- (strictly) quasi-concave if for any  $b \in A$  the upper counter set  $\{a \in A : a \succsim b\}$  is (strictly) convex.

These guarantee the existence of continuous well-behaved utility function representation.

**Profiles** Let  $N$  be a the set of players.

- $(x_i)_{i \in N}$  or simply  $(x_i)$  is a profile - a collection of values of some variable, one for each player.
- $(x_j)_{j \in N/\{i\}}$  or simply  $x_{-i}$  is the list of elements of the profile  $x = (x_j)_{j \in N}$  for all players except  $i$ .
- $(x_{-i}, x_i)$  is a list  $x_{-i}$  and an element  $x_i$ , which is the profile  $(x_i)_{i \in N}$ .

## Games and solutions (OR 1.1-1.3)

**A game** - a model of interactive (multi-person) decision-making. We distinguish between:

- Noncooperative and cooperative games - the units of analysis are individuals or (sub) groups.
- Strategic (normal) form games and extensive form games - players move simultaneously or precede one another.
- Games with perfect and imperfect information - players are perfectly or imperfectly informed about characteristics, events and actions.

**A solution** - a systematic description of outcomes in a family of games.

- Nash equilibrium.
- Subgame perfect equilibrium - extensive games with perfect information.
- Perfect Bayesian equilibrium - games with observable actions.
- Sequential equilibrium (and refinements) - extensive games with imperfect information.

Classic references: Von Neumann and Morgenstern (1944), Luca and Raiffa (1957) and Schelling (1960).

## Rational behavior and bounded rationality (OR 1.4, 1.6)

### A rational agent

- a set of actions  $A$ ,
- a set of consequences  $C$ ,
- a consequence function  $g : A \rightarrow C$ , and
- a preference relation  $\succsim$  on the set  $C$ .

Given any set  $B \subseteq A$  of actions, a rational agent chooses an action  $a^* \in B$  such that  $g(a^*) \succsim g(a)$  for all  $a \in B$ . When  $\succsim$  are specified by a utility function  $U : C \rightarrow \mathbb{R}$   $a^* \in \arg \max_{a \in B} U(g(a))$ .

With uncertainty (environment, events in the game, actions of other players and their reasoning), a rational agent is assumed to have in mind

- a state space  $\Omega$ ,
- a (subjective) probability measure over  $\Omega$ , and
- a consequence function  $g : A \times \Omega \rightarrow C$ ,

A rational agent is an (*vNM*) expected utility  $u(g(a, \omega))$  maximizer.

## Formalities (OR 2.1)

**A strategic game** A *finite* set  $N$  of players, and for each player  $i \in N$

- a non-empty set  $A_i$  of actions
- a preference relation  $\succsim_i$  on the set  $A = \times_{j \in N} A_j$  of possible outcomes.

We will denote a strategic game by

$$\langle N, (A_i), (\succsim_i) \rangle$$

or by

$$\langle N, (A_i), (u_i) \rangle$$

when  $\succsim_i$  can be represented by a utility function  $u_i : A \rightarrow \mathbb{R}$ .

A two-player finite strategic game can be described conveniently in a bi-matrix.

For example, a  $2 \times 2$  game

	<i>L</i>	<i>R</i>
<i>T</i>	$A_1, A_2$	$B_1, B_2$
<i>B</i>	$C_1, C_2$	$D_1, D_2$

## Best response (OR 2.2)

For any list of strategies  $a_{-i} \in A_{-i}$

$$B_i(a_{-i}) = \{a_i \in A_i : (a_{-i}, a_i) \succeq_i (a_{-i}, a'_i) \forall a'_i \in A_i\}$$

is the set of players  $i$ 's best actions given  $a_{-i}$ .

Strategy  $a_i$  is  $i$ 's best response to  $a_{-i}$  if it is the optimal choice when  $i$  conjectures that others will play  $a_{-i}$ .

## Nash equilibrium (OR 2.2)

Nash equilibrium ( $NE$ ) is a steady state of the play of a strategic game.

A  $NE$  of a strategic game  $\langle N, (A_i), (\succsim_i) \rangle$  is a profile  $a^* \in A$  of actions s.t.

$$(a_{-i}^*, a_i^*) \succsim_i (a_{-i}^*, a_i)$$

$\forall i \in N$  and  $\forall a_i \in A_i$ , or equivalently,

$$a_i^* \in B_i(a_{-i}^*)$$

$\forall i \in N$ .

In words, no player has a profitable deviation given the actions of the other players.

## Examples (OR 2.3)

	<i>L</i>	<i>R</i>
<i>T</i>	2, 1	0, 0
<i>B</i>	0, 0	1, 2

	<i>L</i>	<i>R</i>
<i>T</i>	2, 2	0, 0
<i>B</i>	0, 0	1, 1

	<i>L</i>	<i>R</i>
<i>T</i>	3, 3	0, 4
<i>B</i>	4, 0	1, 1

	<i>L</i>	<i>R</i>
<i>T</i>	3, 3	1, 4
<i>B</i>	4, 1	0, 0

## Existence of Nash equilibrium (OR 2.4)

Let the set-valued function  $B : A \rightarrow A$  defined by

$$B(a) = \times_{i \in N} B_i(a_{-i})$$

and rewrite the equilibrium condition

$$a_i^* \in B_i(a_{-i}^*) \quad \forall i \in N$$

in vector form as follows

$$a^* \in B(a^*)$$

Kakutani's fixed point theorem gives conditions on  $B$  under which  $\exists a^*$  such that  $a^* \in B(a^*)$ .

## Kakutani's fixed point theorem

Let  $X \subseteq \mathbb{R}^n$  be non-empty compact (closed and bounded) and convex set and  $f : X \rightarrow X$  be a set-valued function for which

- the set  $f(x)$  is non-empty and convex  $\forall x \in X$ .
- the graph of  $f$  is closed

$y \in f(x)$  for any  $\{x_n\}$  and  $\{y_n\}$  such that

$$y_n \in f(x_n) \forall n \text{ and } x_n \longrightarrow x \text{ and } y_n \longrightarrow y.$$

Then,  $\exists x^* \in X$  s.t.  $x^* \in f(x^*)$ .

## Necessity of conditions in Kakutani's theorem

- $X$  is compact

$$X = \mathbb{R}^1 \text{ and } f(x) = x + 1$$

- $X$  is convex

$$X = \{x \in \mathbb{R}^2 : \|x\| = 1\} \text{ and } f \text{ is } 90^\circ \text{ clock-wise rotation.}$$

–  $f(x)$  is convex for any  $x \in X$

$X = [0, 1]$  and

$$f(x) = \begin{cases} \{1\} & \text{if } x < \frac{1}{2}, \\ \{0, 1\} & \text{if } x = \frac{1}{2}, \\ \{0\} & \text{if } x > \frac{1}{2}. \end{cases}$$

–  $f$  has a closed graph

$X = [0, 1]$  and

$$f(x) = \begin{cases} 1 & \text{if } x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

A strategic game  $\langle N, (A_i), (\succsim_i) \rangle$  has a  $NE$  if for all  $i \in N$

- $A_i$  is non-empty, compact and convex.
- $\succsim_i$  is continuous and quasi-concave on  $A_i$ .

$B$  has a fixed point by Kakutani:

- $B_i(a_{-i}) \neq \emptyset$  ( $A_i$  is compact and  $\succsim_i$  is continuous).
- $B_i(a_{-i})$  is convex ( $\succsim_i$  is quasi-concave on  $A_i$ ).
- $B$  has a closed graph ( $\succsim_i$  is continuous).

## Randomization (OR 3.1)

Recall that a strategic game is a triple  $\langle N, (A_i), (\succsim_i) \rangle$  where

- $N$  is a finite set of players, and for each player  $i \in N$
- a non-empty set  $A_i$  of actions
- a preference relation  $\succsim_i$  on the set  $A = \times_{j \in N} A_j$  of possible outcomes.

or a triple  $\langle N, (A_i), (u_i) \rangle$  when  $\succsim_i$  can be represented by a utility function  $u_i : A \rightarrow \mathbb{R}$ .

Suppose that,

- each player  $i$  can randomize among all her strategies so choices are not deterministic, and
- player  $i$ 's preferences over lotteries on  $A$  can be represented by  $vNM$  expected utility function.

Then, we need to add these specifications to the primitives of the model of strategic game  $\langle N, (A_i), (\succsim_i) \rangle$ .

A mixed strategy of player  $i$  is  $\alpha_i \in \Delta(A_i)$  where  $\Delta(A_i)$  is the set of all probability distributions over  $A_i$ .

- A profile  $(\alpha_i)_{i \in N}$  of mixed strategies induces a probability distribution over the set  $A$ .
- Assuming independence, the probability of an action profile (outcome)  $a$  is then

$$\prod_{i \in N} \alpha_i(a_i).$$

A *vNM* utility function

$$U_i : \times_{j \in N} \Delta(A_j) \rightarrow \mathbb{R}$$

represents player  $i$ 's preferences over the set of lotteries over  $A$ .

For any mixed strategy profile  $\alpha = (\alpha_j)_{j \in N} \in \times_{j \in N} \Delta(A_j)$

$$U_i(\alpha) = \sum_{a \in A} \left( \prod_{j \in N} \alpha_j(a_j) \right) u_i(a)$$

which is linear in  $\alpha$ .

The mixed extension of a the strategic game  $\langle N, (A_i), (u_i) \rangle$  is the strategic game  $\langle N, (\Delta(A_i)), (U_i) \rangle$ .

## Existence of mixed strategy Nash equilibrium

Every finite (action sets) strategic game has a mixed strategy  $NE$ .

- The set of player  $i$ 's mixed strategies  $\Delta(A_i)$

$$\{(p_k)_{k=1}^{m_i} : \sum_{k=1}^{m_i} p_k = 1 \text{ and } p_k \geq 0 \forall k\}$$

where  $m_i$  is the number of  $a_i \in A_i$  (pure strategies) is non empty, convex and compact.

- $vNM$  expected utility is linear probabilities so  $U_i$  is quasi-concave and continuous.

Therefore, the mixed extension has a  $NE$  by Kakutani.

## Two results on mixed strategy Nash equilibrium

Let  $G = \langle N, (A_i), (u_i) \rangle$  be a strategic game and  $G' = \langle N, (\Delta(A_i)), (U_i) \rangle$  be its mixed extension.

[1] If  $a \in NE(G)$  then  $a \in NE(G')$ .

[2]  $\alpha \in NE(G')$  if and only if

$$U_i(\alpha_{-i}, a_i) \geq U_i(\alpha_{-i}, a'_i)$$

for all  $a'_i$  and all  $\alpha_i(a_i) > 0$ .

[1] If  $a \in NE(G)$  then

$$u_i(a_{-i}, a_i) \geq u_i(a_{-i}, a'_i) \quad \forall i \in N \text{ and } \forall a'_i \in A_i.$$

Then, by the linearity of  $U_i$  in  $\alpha_i$

$$U_i(a_{-i}, a_i) \geq U_i(a_{-i}, \alpha_i) \quad \forall i \in N \text{ and } \forall \alpha_i \in \Delta(A_i)$$

and thus  $a \in NE(G')$ .

[2] Let  $\alpha \in NE(G')$

Suppose that  $\exists a_i \in A_i$  such that  $\alpha_i(a_i) > 0$  and

$$U_i(\alpha_{-i}, a'_i) \geq U_i(\alpha_{-i}, a_i) \text{ for some } a'_i \neq a_i.$$

Then, player  $i$  can increase her payoff by transferring probability from  $a_i$  to  $a'_i$  so  $\alpha$  is not a  $NE$ .

This implies that  $U_i(\alpha_{-i}, a_i) = U_i(\alpha_{-i}, a'_i)$  for all  $a_i, a'_i$  in the support of  $\alpha$ .

## Interpretation of mixed strategy Nash equilibrium (OR 3.2)

Since she is indifferent among all strategies in the support, why should a player choose her  $NE$  mixed strategy?

- [1] Mixed strategies as objects of choice
- [2] Mixed strategy  $NE$  as a steady state
- [3] Mixed strategies as pure strategies in an extended game
- [4] Mixed strategies as pure strategies in a perturbed game (Harsanyi 1973).

## Strictly competitive game (OR 2.5)

A strategic game  $\langle \{1, 2\}, (A_i), (\succsim_i) \rangle$  is strictly competitive if for any  $a \in A$  and  $b \in A$  we have  $a \succsim_1 b$  if and only if  $b \succsim_2 a$ .

	<i>L</i>	<i>R</i>
<i>T</i>	$A, -A$	$B, -B$
<i>B</i>	$C, -C$	$D, -D$

If  $(x^*, y^*)$  is a *NE* of a strictly competitive game then

$$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y).$$

## Maxminimization

A max min mixed strategy of player  $i$  is a mixed strategy that solves the problem

$$\max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})$$

A player's payoff in  $\alpha^* \in NE(G)$  is at least her max min payoff:

$$\begin{aligned} U_i(\alpha^*) &\geq U_i(\alpha_i, \alpha_{-i}^*) \\ &\geq \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \\ &\geq \max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \end{aligned}$$

and the last step follows since the above holds for all  $\alpha_i \in \Delta(A_i)$ .

## Two min-max results

$$[1] \quad \max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i} \in \Delta A_{-i}} \max_{\alpha_i \in \Delta A_i} U_i(\alpha_i, \alpha_{-i})$$

For every  $\alpha'_i$

$$\min_{\alpha_{-i}} U_i(\alpha'_i, \alpha_{-i}) \leq U_i(\alpha'_i, \alpha'_{-i})$$

and thus

$$\min_{\alpha_{-i}} U_i(\alpha'_i, \alpha_{-i}) \leq \max_{\alpha_i} U_i(\alpha_i, \alpha'_{-i})$$

However, since the above holds for every  $\alpha'_i$  and  $\alpha'_{-i}$  it must hold for the “best” and “worst” such choices

$$\max_{\alpha_i} \min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i}} \max_{\alpha_i} U_i(\alpha_i, \alpha_{-i}).$$

[2] In a zero-sum game

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha^*)$$

$\Leftarrow$  Since  $\alpha^* \in NE(G)$

$$U_1(\alpha^*) = \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2^*) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)$$

and since  $U_1 = -U_2$  at the same time

$$U_1(\alpha^*) = \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1^*, \alpha_2) \leq \max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2)$$

Hence,

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)$$

which together with [1] gives the desired conclusion.

$\Rightarrow$  Let  $\alpha_1^{\max}$  be player 1's max min strategy and  $\alpha_2^{\min}$  be player 2's min max strategy. Then,

$$\begin{aligned} \max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) &= \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1^{\max}, \alpha_2) \\ &\leq U_1(\alpha_1^{\max}, \alpha_2) \quad \forall \alpha_2 \in \Delta A_2 \end{aligned}$$

and

$$\begin{aligned} \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) &= \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2^{\min}) \\ &\geq U_1(\alpha_1, \alpha_2^{\min}) \quad \forall \alpha_1 \in \Delta A_1 \end{aligned}$$

But

$$\begin{aligned}\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) &= \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) \\ &= U_1(\alpha_1^{\max}, \alpha_2^{\min})\end{aligned}$$

implies that

$$U_1(\alpha_1, \alpha_2^{\min}) \leq U_1(\alpha_1^{\max}, \alpha_2^{\min}) \leq U_1(\alpha_1^{\max}, \alpha_2)$$

$\forall \alpha_2 \in \Delta A_2$  and  $\forall \alpha_1 \in \Delta A_1$ .

Hence,  $(\alpha_1^{\max}, \alpha_2^{\min})$  is an equilibrium.

## Interchangeability

If  $\alpha$  and  $\alpha'$  are *NE* in a zero-sum game, then so are  $(\alpha_1, \alpha'_2)$  and  $(\alpha'_1, \alpha_2)$ .

– Since  $\alpha$  and  $\alpha'$  are equilibria

$U_1(\alpha_1, \alpha_2) \geq U_1(\alpha'_1, \alpha_2)$  and  $U_2(\alpha'_1, \alpha'_2) \geq U_2(\alpha'_1, \alpha_2)$ ,  
and because  $U_1 = -U_2$

$$U_1(\alpha'_1, \alpha'_2) \leq U_1(\alpha'_1, \alpha_2).$$

Therefore,

$$U_1(\alpha_1, \alpha_2) \geq U_1(\alpha'_1, \alpha_2) \geq U_1(\alpha'_1, \alpha'_2). \quad (1)$$

and similar analysis gives that

$$U_1(\alpha_1, \alpha_2) \leq U_1(\alpha_1, \alpha'_2) \leq U_1(\alpha'_1, \alpha'_2). \quad (2)$$

– (1) and (2) yield

$$U_1(\alpha_1, \alpha_2) = U_1(\alpha'_1, \alpha_2) = U_1(\alpha_1, \alpha'_2) = U_1(\alpha'_1, \alpha'_2)$$

– Since  $\alpha$  is an equilibrium

$$U_2(\alpha_1, \alpha''_2) \leq U_2(\alpha_1, \alpha_2) = U_2(\alpha_1, \alpha'_2)$$

for any  $\alpha''_2 \in \Delta A_2$ , and since  $\alpha'$  is an equilibrium

$$U_1(\alpha''_1, \alpha'_2) \leq U_1(\alpha'_1, \alpha'_2) = U_1(\alpha_1, \alpha'_2)$$

for any  $\alpha''_1 \in \Delta A_1$ . Therefore,  $(\alpha_1, \alpha'_2)$  is an equilibrium and similarly also  $(\alpha_1, \alpha'_2)$ .

## Bayesian equilibrium (OR 2.6)

A Bayesian game consists of a finite set  $N$  of players, a finite set  $\Omega$  of decision-relevant states (characteristics of players), and for each player  $i \in N$

- a set  $A_i$  of actions
- a finite set  $T_i$  of types and a signal function  $\tau_i : \Omega \rightarrow T_i$
- a probability measure  $p_i$  on  $\Omega$  (prior belief) for which  $p_i(\tau_i^{-1}(t_i)) > 0$  for all  $t_i \in T_i$ .
- a preference relation  $\succsim_i$  on the set of probability measure over  $A \times \Omega$ .

$a^* \in \times_{(i,t_i)} A_i$  is a Bayes-Nash equilibrium of a Bayesian game

$$\langle N, \Omega, (A_i), (T_i), (\tau_i), (p_i), (\succeq_i) \rangle$$

if it is a *NE* in which the set of players is the set of all pairs  $(i, t_i)$  for all  $i \in N$  and  $t_i \in T_i$ , and for each player  $(i, t_i)$

$$a^* \succeq_{(i,t_i)} b^* \Leftrightarrow L_i(a^*, t_i) \succeq_i L_i(b^*, t_i)$$

where  $L_i(a^*, t_i)$  is a *lottery* over  $A \times \Omega$  that assigns a probability  $\frac{p_i(\omega)}{p_i(\tau_i^{-1}(t_i))}$  to

$$(a^*(j, \tau_j(\omega)))_{j \in N, \omega} \text{ if } \omega \in p_i(\tau_i^{-1}(t_i))$$

and zero otherwise.

Example: *BoS* with one-side imperfect information

	$\omega = y$		$\omega = n$	
	$B$	$S$	$B$	$S$
$B$	2, 1	0, 0	2, 0	0, 2
$S$	0, 0	1, 2	0, 1	1, 0

Then, the expected payoffs of player 1 are given by

	$(B, B)$	$(B, S)$	$(S, B)$	$(S, S)$
$B$	2	$2p$	$2(1 - p)$	0
$S$	0	$p$	$1 - p$	1

For any belief  $p \in (0, 1)$ ,  $(B, (B, S))$  is an equilibrium ( $B$  is optimal for player 1 given the actions of the two types of player 2 and his beliefs).

## Harsanyi (1973)

Consider a game  $G = \langle N, (A_i), (u_i) \rangle$  and let  $(\epsilon_i(a))_{i \in N, a \in A}$  be a collection of random variables with support  $[-1, 1]$  where

- $\epsilon_i = (\epsilon_i(a))_{a \in A}$  is private information and has well-behaved distribution function, and  $\epsilon = (\epsilon_i)_{i \in N}$  are independent.
- The payoff of each player  $i$  at the outcome  $a$  and state  $\epsilon$  is  $u_i(a) + \epsilon_i(a)$ . This defines a Bayesian game  $G(\epsilon)$ .

For almost any game  $G$  and any collection  $\epsilon^*$ , almost any  $\alpha \in NE(G)$  is *approachable* – associated with the limit as  $\gamma \rightarrow 0$  of a sequence of pure strategy equilibria of the Bayesian game  $G(\gamma\epsilon^*)$  (and visa versa).

### Correlated equilibrium (OR 3.3)

A correlated equilibrium of a strategic game  $G = \langle N, (A_i), (u_i) \rangle$  consists of a finite probability space  $(\Omega, \pi)$  and for each player  $i$ ,

- an information partition  $\mathcal{P}_i$  of  $\Omega$ , and
- a function  $\sigma_i : \Omega \rightarrow A_i$  s.t.  $\sigma_i(\omega) = \sigma_i(\omega')$  for any  $\omega, \omega' \in P_i \in \mathcal{P}_i$

such that

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_i(\omega), \sigma_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma'_i(\omega), \sigma_{-i}(\omega))$$

for all  $\sigma'_i : \Omega \rightarrow A_i$  s.t.  $\sigma'_i(\omega) = \sigma'_i(\omega')$  for any  $\omega, \omega' \in P_i \in \mathcal{P}_i$ .

### Example (*BoS*)

$\Omega = \{x, y, z\}$  and  $\pi = (1 - \gamma - \delta, \gamma, \delta)$ ,  $\mathcal{P}_1 = \{\{x, y\}, \{z\}\}$  and  $\mathcal{P}_2 = \{\{x\}, \{y, z\}\}$ ,  $\sigma_1(x) = \sigma_1(y) = B$  and  $\sigma_1(z) = S$ .

If  $\omega = x$  then player 2 will play  $B = BR_2(B)$  and if  $\omega = \{y, z\}$  then she will play the mixed strategy

$$\left(\frac{\gamma}{\gamma + \delta}, \frac{\delta}{\gamma + \delta}\right).$$

## Correlated equilibrium and Nash equilibrium

For every mixed strategy Nash equilibrium  $\alpha$  of a finite strategic game  $G = \langle N, (A_i), (u_i) \rangle$  there is a correlated equilibrium

$$\langle (\Omega, \pi), (\mathcal{P}_i), (\sigma_i) \rangle$$

in which for each player  $i \in N$  the distribution on  $A_i$  induced by  $\sigma_i$  is  $\alpha_i$ .

Thus, the set of mixed strategy Nash equilibria is a subset of the set correlated equilibria.

## Evolutionary stability (OR 3.4)

A single population of players. Players interact with each other pair-wise and randomly matched.

Players are assigned modes of behavior (mutation). Utility measures each player's ability to survive.

$\varepsilon$  of players consists of mutants taking action  $a$  while others take action  $a^*$ .

## Evolutionary stable strategy (*ESS*)

Consider a payoff symmetric game  $G = \langle \{1, 2\}, (A, A), (u_i) \rangle$  where  $u_1(a) = u_2(a')$  when  $a'$  is obtained from  $a$  by exchanging  $a_1$  and  $a_2$ .

$a^* \in A$  is *ESS* iff for any  $a \in A$ ,  $a \neq a^*$  and  $\varepsilon > 0$  sufficiently small

$$(1 - \varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) > (1 - \varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

which is satisfied iff for any  $a \neq a^*$  either

$$u(a^*, a^*) > u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) > u(a, a)$$

## Three results on *ESS*

[1] If  $a^*$  is an *ESS* then  $(a^*, a^*)$  is a *NE*.

Suppose not. Then, there exists a strategy  $a \in A$  such that

$$u(a, a^*) > u(a^*, a^*).$$

But, for  $\varepsilon$  small enough

$$(1 - \varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) < (1 - \varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

and thus  $a^*$  is not an *ESS*.

[2] If  $(a^*, a^*)$  is a strict  $NE$  ( $u(a^*, a^*) > u(a, a^*)$  for all  $a \in A$ ) then  $a^*$  is an  $ESS$ .

Suppose  $a^*$  is not an  $ESS$ . Then either

$$u(a^*, a^*) \leq u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) \leq u(a, a).$$

so  $(a^*, a^*)$  can be a  $NE$  but not a strict  $NE$ .

[3] A  $2 \times 2$  game  $G = \langle \{1, 2\}, (A, A), (u_i) \rangle$  where  $u_i(a) \neq u_i(a')$  for any  $a, a'$  has a mixed strategy which is *ESS* (OR 51.1)

	$a$	$a'$
$a$	$w, w$	$x, y$
$a'$	$y, x$	$z, z$

If  $w > y$  or  $z > x$  then  $(a, a)$  or  $(a', a')$  are strict *NE*, and thus  $a$  or  $a'$  are *ESS*.

If  $w < y$  and  $z < x$  then there is a unique symmetric mixed strategy *NE*  $(\alpha^*, \alpha^*)$  where

$$\alpha^*(a) = (z - x) / (w - y + z - x)$$

and  $u(\alpha^*, \alpha) > u(\alpha, \alpha)$  for any  $\alpha \neq \alpha^*$ .

## Payoff equivalence

Two games

$$G = \langle N, (A_i), (u_i) \rangle \text{ and } G' = \langle N, (A'_i), (u'_i) \rangle$$

are payoff equivalent if for each player  $i$ ,

- $A'_i$  may be relabelled such that  $A'_i = A_i$ , and
- when  $A'_i = A_i \exists (x_i, y_i)$  with  $y_i > 0$  such that

$$u'_i(a') = x_i + y_i u_i(a).$$

## Two results on payoff equivalence

- [1]  $G$  and  $G'$  have the same mixed strategy  $NE$  if they are payoff equivalent.
  - Affine transformations on  $u_i$ : the preference ordering of each  $i$  over  $\Delta A$  is the same as over  $\Delta A'$ .
  
- [2]  $G$  and  $G'$  have the same pure strategy  $NE$  under any strictly increasing monotonic transformations on  $u_i$ .
  - Monotonic transformations on  $u_i$ : the preference ordering of each  $i$  over  $A$  is the same as over  $A'$ .

A game

$$G = \langle N, (A_i), (u_i) \rangle$$

is payoff symmetric if it is payoff equivalent to a game

$$G' = \langle N, (A'_i), (u'_i) \rangle$$

in which for any two players,  $i$  and  $j$ ,

- $A'_i = A'_j$ , and
- if players  $i$  and  $j$  exchange strategies, they also exchange payoffs

$$u'_i(a_1, \dots, a_i, \dots, a_j, \dots, a_N) = u'_j(a_1, \dots, a_j, \dots, a_i, \dots, a_N).$$

## Strategic equivalence

Two games

$$G = \langle N, (A_i), (u_i) \rangle \text{ and } G' = \langle N, (A'_i), (u'_i) \rangle$$

are strategically equivalent if for each player  $i$ ,

- $A'_i$  may be relabelled such that  $A'_i = A_i$ , and
- when  $A'_i = A_i$

$$B'_i(\alpha'_{-i}) = B_i(\alpha_{-i}).$$

Alternatively, the two games

$$G = \langle N, (A_i), (u_i) \rangle \text{ and } G' = \langle N, (A'_i), (u'_i) \rangle$$

are strategically equivalent if for each player  $i$ ,  $\exists y_i > 0$  and a function  $f_i : A_{-i} \rightarrow \mathbb{R}$  such that

$$u'_i(a) = f_i(a_{-i}) + y_i u_i(a)$$

for all  $a \in A$ .

Fix  $\alpha_{-i}$  and let

$$f_i(\alpha_{-i}) \equiv \sum_{a_{-i} \in A_{-i}} \alpha_{-i}(a_{-i}) f_i(a_{-i})$$

Then,

$$\begin{aligned} u'_i(a_i, \alpha_{-i}) &= \sum_{a_{-i} \in A_{-i}} \alpha_{-i}(a_{-i}) u'_i(a_i, a_{-i}) \\ &= \sum_{a_{-i} \in A_{-i}} \alpha_{-i}(a_{-i}) [f_i(a_{-i}) + y_i u_i(a_i, a_{-i})] \\ &= f_i(\alpha_{-i}) + y_i \sum_{a_{-i} \in A_{-i}} \alpha_{-i}(a_{-i}) u_i(a_i, a_{-i}) \\ &= f_i(\alpha_{-i}) + y_i u_i(a_i, \alpha_{-i}) \end{aligned}$$

Thus,  $a_i \in B_i(\alpha_{-i})$  iff  $a_i \in B'_i(\alpha_{-i})$  and thus  $G$  and  $G'$  have the same set of  $NE$  (the converse is also true if no player has any dominated strategies).

A game

$$G = \langle N, (A_i), (u_i) \rangle$$

is strategically symmetric if it is strategically equivalent to a game

$$G' = \langle N, (A'_i), (u'_i) \rangle$$

in which for any two players,  $i$  and  $j$ ,

- $A'_i = A'_j$ , and
- $B'_i(\alpha_{-i}) = B'_j(\alpha_{-j})$  for all  $\alpha \in \Delta A$  with  $\alpha_i = \alpha_j$ .

A mixed strategy  $NE$  of a strategically symmetric game in which each player plays the same strategy is called a symmetric  $NE$ .

## **Rationalizability (OR 4.1)**

In equilibrium, each player knows the other players' equilibrium strategies.

Rationalizability and dominance solvability are solution concepts that do not entail this assumption.

Players' beliefs about each other's action are not assumed to be correct, but are constrained by (some) considerations of rationality.

An action  $a_i$  is rationalizable for player  $i$  in  $G = \langle N, (A_i), (U_i) \rangle$  if for each  $j \in N$  there exists a set of actions  $Z_j \subseteq A_j$  such that

- $a_i \in Z_i$  and
- every action  $a_j \in Z_j$  is a best response to a belief of player  $j$  that assigns positive probability only to  $a_{-j} \in Z_{-j}$ .

If  $(Z_j)_{j \in N}$  and  $(Z'_j)_{j \in N}$  satisfy the definition then so does  $(Z_j \cup Z'_j)_{j \in N}$  so the set of profiles of rationalizable actions is  $\times_{j \in N} Z_j$ .

Every action  $a_i$  such that  $\alpha_i^*(a_i) > 0$  and  $\alpha^* \in NE(G)$  is rationalizable.

On the other hand, a strictly dominated action is never rationalizable (OR Lemma 60.1).

An example (OR 57.1)

	$b_1$	$b_2$	$b_3$	$b_4$
$a_1$	0, 7	2, 5	7, 0	0, 1
$a_2$	5, 2	3, 3	5, 2	0, 1
$a_3$	7, 0	2, 5	0, 7	0, 1
$a_4$	0, 0	0, -2	0, 0	10, -1

The rationalizable actions are  $a_1, a_2, a_3$  for player 1 and  $b_1, b_2, b_3$  for player 2.

## Dominance (OR 4.2)

An action  $a_i \in A_i$  of player  $i$  is strictly dominated if there exists a mixed strategy  $\alpha_i$  such that

$$U_i(a_i, a_{-i}) < U_i(\alpha_i, a_{-i})$$

for all  $a_{-i} \in A_{-i}$ .

An action  $a_i \in A_i$  of player  $i$  is weakly dominated if there exists a mixed strategy  $\alpha_i$  such that

$$U_i(a_i, a_{-i}) \leq U_i(\alpha_i, a_{-i})$$

for all  $a_{-i} \in A_{-i}$  and the inequality is strict for some  $a_{-i} \in A_{-i}$ .

## Two results on dominated strategies

- [1] An action of a player in a finite strategic game  $G$  is never a best response *iff* it is strictly dominated.
  
- [2] Consider  $G'$  obtained by iterated removal of all (weakly and strictly) dominated strategies from  $G$  then
  - if  $a \in NE(G')$  then  $a \in NE(G)$ , and
  - the converse holds for the iterated removal of only strictly dominated strategies.

## Trembling hand perfection (OR 12.5.1)

A trembling hand perfect equilibrium (*THP*) of a finite strategic game is a mixed strategy profile  $\alpha$  such that there exists  $(\alpha^k)_{k=1}^{\infty}$  of completely mixed strategy profiles such that

- $(\alpha^k)_{k=1}^{\infty}$  converges to  $\alpha$ , and
- $\alpha_i \in BR_i(\alpha_{-i}^k)$  for each player  $i$  and all  $k$ .

A strategy profile  $\alpha^*$  in a two-player game is a *THP* equilibrium *iff* it is a mixed strategy *NE* and the strategy of neither player is weakly dominated.

Example (OR 248.1)

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0, 0	0, 0	0, 0
<i>B</i>	0, 0	1, 1	2, 0
<i>C</i>	0, 0	0, 2	2, 2

The Nash equilibria  $(A, A)$  and  $(C, C)$  are not trembling hand perfect equilibria.