Economics 209A
Theory and Application of Non-Cooperative Games
(Fall 2008)
Strategic Games I

Topics: terminology and notations (OR 1.7), games and solutions (OR 1.1-1.3), rationality and bounded rationality (OR 1.4, 1.6), formalities (OR 2.1), best-response (OR 2.2), Nash equilibrium (OR 2.2), examples (OR 2.3), existence of Nash equilibrium (OR 2.4), mixed strategy Nash equilibrium (OR 3.1, 3.2), Strictly competitive games (OR 2.5).

Terminology and notations (OR 1.7)

Sets For $x, y \in \mathbb{R}^n$,

$$x \geq y \Longleftrightarrow x_i \geq y_i$$

for all i.

$$x > y \Longleftrightarrow x_i \geq y_i \text{ and } x_j > y_j$$

for all i and some j.

$$x >> y \Longleftrightarrow x_i > y_i$$

for all i.

Preferences $\succsim$ is a binary relation on some set of alternatives $A \subseteq \mathbb{R}^n$. From $\succsim$ we derive two other relations on A:

- strict performance relation

$$a \succ b \iff a \succsim b \text{ and not } b \succsim a$$

- indifference relation

$$a \sim b \iff a \succsim b \text{ and } b \succsim a$$

Utility representation $\succsim$ is said to be

- complete if $\forall a, b \in A$, $a \succsim b$ or $b \succsim a$.
- transitive if $\forall a, b, c \in A$, $a \succsim b$ and $b \succsim c$ then $a \succsim c$.

$\succsim$ can be presented by a utility function only if it is complete and transitive (rational).

A function $u : A \to \mathbb{R}$ is a utility function representing $\succsim$ if $\forall a, b \in A$

$$a \succsim b \iff u(a) \geq u(b).$$

$\succsim$ is said to be

- continuous (preferences cannot jump...) if for any sequence of pairs $\{(a^k, b^k)\}_{k=1}^{\infty}$ with $a^k \succsim b^k$, and $a^k \longrightarrow a$ and $b^k \longrightarrow b$, $a \succsim b$.

- (strictly) quasi-concave if for any $b \in A$ the upper counter set $\{a \in A : a \succsim b\}$ is (strictly) convex.

These guarantee the existence of continuous well-behaved utility function representation.

Profiles Let N be a the set of players.

- $(x_i)_{i \in N}$ or simply (x_i) is a profile - a collection of values of some variable, one for each player.

- $(x_j)_{j \in N/\{i\}}$ or simply x_{-i} is the list of elements of the profile $x = (x_j)_{j \in N}$ for all players except i.

- (x_{-i}, x_i) is a list x_{-i} and an element x_i, which is the profile $(x_i)_{i \in N}$.

Games and solutions (OR 1.1-1.3)

A game - a model of interactive (multi-person) decision-making. We distinguish between:

- Noncooperative and cooperative games - the units of analysis are individuals or (sub) groups.
- Strategic (normal) form games and extensive form games - players move simultaneously or precede one another.
- Gams with perfect and imperfect information - players are perfectly or imperfectly informed about characteristics, events and actions.

A solution - a systematic description of outcomes in a family of games.

- Nash equilibrium.

- Subgame perfect equilibrium - extensive games with perfect information.

- Perfect Bayesian equilibrium - games with observable actions.

- Sequential equilibrium (and refinements) - extensive games with imperfect information.

Classic references: Von Neumann and Morgenstern (1944), Luca and Raiffa (1957) and Schelling (1960).

Rational behavior and bounded rationality (OR 1.4, 1.6)

A rational agent

- a set of actions A,
- a set of consequences C,
- a consequence function $g : A \to C$, and
- a preference relation $\succsim$ on the set C.

Given any set $B \subseteq A$ of actions, a rational agent chooses an action $a^* \in B$ such that $g(a^*) \succsim g(a)$ for all $a \in B$. When $\succsim$ are specified by a utility function $U : C \to \mathbb{R}$ $a^* \in \arg\max_{a \in B} U(g(a))$.

With uncertainty (environment, events in the game, actions of other players and their reasoning), a rational agent is assumed to have in mind

- a state space Ω,
- a (subjective) probability measure over Ω, and
- a consequence function $g : A \times \Omega \to C$,

A rational agent is an (vNM) expected utility $u(g(a, \omega))$ maximizer.

Formalities (OR 2.1)

A strategic game A *finite* set N of players, and for each player $i \in N$

- a non-empty set A_i of actions
- a preference relation $\succsim_i$ on the set $A = \times_{j \in N} A_j$ of possible outcomes.

We will denote a strategic game by

$$\langle N, (A_i), (\succsim_i) \rangle$$

or by

$$\langle N, (A_i), (u_i) \rangle$$

when $\succsim_i$ can be represented by a utility function $u_i : A \longrightarrow \mathbb{R}$.

A two-player finite strategic game can be described conveniently in a bi-matrix.

For example, a 2×2 game

	L	R
T	A_1, A_2	B_1, B_2
B	C_1, C_2	D_1, D_2

Best response (OR 2.2)

For any list of strategies $a_{-i} \in A_{-i}$

$$B_i(a_{-i}) = \{a_i \in A_i : (a_{-i}, a_i) \succsim_i (a_{-i}, a'_i) \ \forall a'_i \in A_i\}$$

is the set of players i's best actions given a_{-i}.

Strategy a_i is i's best response to a_{-i} if it is the optimal choice when i conjectures that others will play a_{-i}.

Nash equilibrium (OR 2.2)

Nash equilibrium (NE) is a steady state of the play of a strategic game.

A NE of a strategic game $\langle N, (A_i), (\succsim_i)\rangle$ is a profile $a^* \in A$ of actions s.t.

$$(a^*_{-i}, a^*_i) \succsim_i (a^*_{-i}, a_i)$$

$\forall i \in N$ and $\forall a_i \in A_i$, or equivalently,

$$a^*_i \in B_i(a^*_{-i})$$

$\forall i \in N$.

In words, no player has a profitable deviation given the actions of the other players.

Examples (OR 2.3)

	L	R
T	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

	L	R
T	$2, 2$	$0, 0$
B	$0, 0$	$1, 1$

	L	R
T	$3, 3$	$0, 4$
B	$4, 0$	$1, 1$

	L	R
T	$3, 3$	$1, 4$
B	$4, 1$	$0, 0$

Existence of Nash equilibrium (OR 2.4)

Let the set-valued function $B : A \longrightarrow A$ defined by

$$B(a) = \times_{i \in N} B_i(a_{-i})$$

and rewrite the equilibrium condition

$$a_i^* \in B_i(a_{-i}^*) \ \forall i \in N$$

in vector form as follows

$$a^* \in B_i(a^*)$$

Kakutani's fixed point theorem gives conditions on B under which $\exists a^*$ such that $a^* \in B(a^*)$.

Kakutani's fixed point theorem

Let $X \subseteq \mathbb{R}^n$ be non-empty compact (closed and bounded) and convex set and $f : X \rightarrow X$ be a set-valued function for which

- the set $f(x)$ is non-empty and convex $\forall x \in X$.

- the graph of f is closed

$$\begin{aligned} y &\in f(x) \text{ for any } \{x_n\} \text{ and } \{y_n\} \text{ s.t.} \\ y_n &\in f(x_n) \forall n \text{ and } x_n \longrightarrow x \text{ and } y_n \longrightarrow y. \end{aligned}$$

Than, $\exists x^* \in X$ s.t.$x^* \in f(x^*)$.

Necessity of conditions in Kakutani's theorem

- X is compact

  $X = \mathbb{R}^1$ and $f(x) = x + 1$

- X is convex

  $X = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ and f is 90° clock-wise rotation.

– $f(x)$ is convex for any $x \in X$

$X = [0, 1]$ and

$$f(x) = \begin{cases} \{1\} & \text{if} \quad x < \frac{1}{2}, \\ \{0, 1\} & \text{if} \quad x = \frac{1}{2}, \\ \{0\} & \text{if} \quad x > \frac{1}{2}. \end{cases}$$

– f has a closed graph

$X = [0, 1]$ and

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 1, \\ 0 & \text{if} \quad x = 1. \end{cases}$$

A strategic game $\langle N, (A_i), (\succsim_i) \rangle$ has a NE if for all $i \in N$

- A_i is non-empty, compact and convex.
- $\succsim_i$ is continuous and quasi-concave on A_i.

B has a fixed point by Kakutani:

- A_i is compact and $\succsim_i$ is continuous $\implies B_i(a_{-i}) \neq \emptyset$.
- $\succsim_i$is quasi-concave on $A_i \implies B_i(a_{-i})$ is convex.
- $\succsim_i$is continuous $\implies B$ has a closed graph.

Randomization (OR 3.1)

Recall that a strategic game is a triple $\langle N, (A_i), (\succsim_i) \rangle$ where

- N is a finite set of players, and for each player $i \in N$
- a non-empty set A_i of actions
- a preference relation $\succsim_i$on the set $A = \times_{j \in N} A_j$ of possible outcomes.

or a triple $\langle N, (A_i), (u_i) \rangle$ when $\succsim_i$ can be represented by a utility function $u_i : A \to \mathbb{R}$.

Suppose that,

- each player i can randomize among all her strategies so choices are not deterministic, and

- player i's preferences over lotteries on A can be represented by vNM expected utility function.

Then, we need to add theses specifications to the primitives of the model of strategic game $\langle N, (A_i), (\succsim_i) \rangle$.

A mixed strategy of player i is $\alpha_i \in \Delta(A_i)$ where $\Delta(A_i)$ is the set of all probability distributions over A_i.

– A profile $(\alpha_i)_{i \in N}$ of mixed strategies induces a probability distribution over the set A.

– Assuming independence, the probability of an action profile (outcome) a is then

$$\prod_{i \in N} \alpha_i(a_i).$$

A vNM utility function

$$U_i : \times_{j \in N} \Delta(A_j) \to \mathbb{R}$$

represents player i's preferences over the set of lotteries over A.

For any mixed strategy profile $\alpha = (\alpha_j)_{j \in N} \in \times_{j \in N} \Delta(A_j)$

$$U_i(\alpha) = \sum_{a \in A} \Big(\prod_{j \in N} \alpha_j(a_j) \Big) u_i(a)$$

which is linear in α.

The mixed extension of a the strategic game $\langle N, (A_i), (u_i) \rangle$ is the strategic game $\langle N, (\Delta(A_i)), (U_i) \rangle$.

Existence of mixed strategy Nash equilibrium

Every finite (action sets) strategic game has a mixed strategy NE.

- The set of player i's mixed strategies $\Delta(A_i)$

  $$\{(p_k)_{k=1}^{m_i} : \sum_{k=1}^{m_i} p_k = 1 \text{ and } p_k \geq 0 \ \forall k\}$$

  where m_i is the number of $a_i \in A_i$ (pure strategies) is non empty, convex and compact.

- vNM expected utility is linear probabilities so U_i is quasi-concave and continuous.

Therefore, the mixed extension has a NE by Kakutani.

Two results on mixed strategy Nash equilibrium

Let $G = \langle N, (A_i), (u_i) \rangle$ be a strategic game and $G' = \langle N, (\Delta(A_i)), (U_i) \rangle$ be its mixed extension.

[1] If $a \in NE(G)$ then $a \in NE(G')$.

[2] $\alpha \in NE(G')$ if and only if

$$U_i(\alpha_{-i}, a_i) \geq U_i(\alpha_{-i}, a_i')$$

for all a_i' and all $\alpha_i(a_i) > 0$.

[1] If $a \in NE(G)$ then

$$u_i(a_{-i}, a_i) \geq u_i(a_{-i}, a_i') \ \forall i \in N \text{ and } \forall a_i' \in A_i.$$

Then, by the linearity of U_i in α_i

$$U_i(a_{-i}, a_i) \geq U_i(a_{-i}, \alpha_i) \ \forall i \in N \text{ and } \forall \alpha_i \in \Delta(A_i)$$

and thus $a \in NE(G')$.

[2] Let $\alpha \in NE(G')$

Suppose that $\exists a_i \in A_i$ such that $\alpha_i(a_i) > 0$ and

$$U_i(a_{-i}, a_i') \geq U_i(a_{-i}, a_i) \text{ for some } a_i' \neq a_i.$$

Then, player i can increase her payoff by transferring probability from a_i to a_i' so α is not a NE.

This implies that $U_i(a_{-i}, a_i) = U_i(a_{-i}, a_i')$ for all a_i, a_i' in the support of α.

Interpretation of mixed strategy Nash equilibrium (OR 3.2)

Since she is indifferent among all strategies in the support, why should a player choose her NE mixed strategy?

[1] Mixed strategies as objects of choice

[2] Mixed strategy NE as a steady state

[3] Mixed strategies as pure strategies in an <u>extended</u> game

[4] Mixed strategies as pure strategies in a <u>perturbed</u> game (Harsanyi 1973).

Strictly competitive game (OR 2.5)

A strategic game $\langle \{1,2\}, (A_i), (\succsim_i) \rangle$ is strictly competitive if for any $a \in A$ and $b \in A$ we have $a \succsim_1 b$ if and only if $b \succsim_2 a$.

	L	R
T	$A, -A$	$B, -B$
B	$C, -C$	$D, -D$

If (x^*, y^*) is a NE of a strictly competitive game then

$$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y).$$

Maxminimization

A two-player strategic game $\langle\{1,2\},(A_i),(\succsim_i)\rangle$ is *strictly competitive* if for any $a, a' \in A$,

$$a \succsim_1 a' \text{ if and only if } a' \succsim_2 a.$$

When $\succsim_i$ is represented by a utility function u_i then for any $a \in A$ we have

$$u_1(a) = -u_2(a).$$

Thus, a strictly competitive game is sometimes called *zero-sum*.

A max min mixed strategy of player i is a mixed strategy that solves the problem

$$\max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})$$

A player's payoff in $\alpha^* \in NE(G)$ is at least her max min payoff:

$$\begin{aligned} U_i(\alpha^*) &\geq U_i(\alpha_i, \alpha^*_{-i}) \\ &\geq \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \\ &\geq \max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \end{aligned}$$

and the last step follows since the above holds for all $\alpha_i \in \triangle(A_i)$.

Two min-max results

[1] $$\max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i} \in \Delta A_{-i}} \max_{\alpha_i \in \Delta A_i} U_i(\alpha_i, \alpha_{-i})$$

For every α'

$$\min_{\alpha_{-i}} U_i(\alpha'_i, \alpha_{-i}) \leq U_i(\alpha'_i, \alpha'_{-i})$$

and thus

$$\min_{\alpha_{-i}} U_i(\alpha'_i, \alpha_{-i}) \leq \max_{\alpha_i} U_i(\alpha_i, \alpha'_{-i})$$

However, since the above holds for every α'_i and α'_{-i} it must hold for the "best" and "worst" such choices

$$\max_{\alpha_i} \min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i}} \max_{\alpha_i} U_i(\alpha_i, \alpha_{-i}).$$

[2] $\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha^*)$

$\Leftarrow$ Since $\alpha^* \in NE(G)$

$$U_1(\alpha^*) = \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2^*) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)$$

and since $U_1 = -U_2$ at the same time

$$U_1(\alpha^*) = \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1^*, \alpha_2) \leq \max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2)$$

Hence,

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)$$

which together with [1] gives the desired conclusion.

$\Rightarrow$ Let $\alpha_1^{\max}$ be player 1's $\max\min$ strategy and $\alpha_2^{\min}$ be player 2's $\min\max$ strategy. Then,

$$\begin{aligned}\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) &= \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1^{\max}, \alpha_2) \\ &\leq U_1(\alpha_1^{\max}, \alpha_2) \; \forall \alpha_2 \in \Delta A_2\end{aligned}$$

and

$$\begin{aligned}\min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) &= \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2^{\min}) \\ &\geq U_1(\alpha_1, \alpha_2^{\min}) \; \forall \alpha_1 \in \Delta A_1\end{aligned}$$

But

$$\begin{aligned}\max_{\alpha_1 \in \mathbf{\Delta} A_1} \min_{\alpha_2 \in \mathbf{\Delta} A_2} U_1(\alpha_1, \alpha_2) &= \min_{\alpha_2 \in \mathbf{\Delta} A_2} \max_{\alpha_1 \in \mathbf{\Delta} A_1} U_1(\alpha_1, \alpha_2) \\ &= U_1(\alpha_1^{\max}, \alpha_2^{\min})\end{aligned}$$

implies that

$$U_1(\alpha_1, \alpha_2^{\min}) \leq U_1(\alpha_1^{\max}, \alpha_2^{\min}) \leq U_1(\alpha_1^{\max}, \alpha_2)$$

$\forall \alpha_2 \in \mathbf{\Delta} A_2$ and $\forall \alpha_1 \in \mathbf{\Delta} A_1$.

Hence, $(\alpha_1^{\max}, \alpha_2^{\min})$ is an equilibrium.

Interchangeability

If α and α' are NE in a zero-sum game, then so are (α_1, α_2') and (α_1', α_2).

– Since α and α' are equilibria

$$U_1(\alpha_1, \alpha_2) \geq U_1(\alpha_1', \alpha_2) \text{ and } U_2(\alpha_1', \alpha_2') \geq U_2(\alpha_1', \alpha_2),$$

and because $U_1 = -U_2$

$$U_1(\alpha_1', \alpha_2') \leq U_1(\alpha_1', \alpha_2).$$

Therefore,

$$U_1(\alpha_1, \alpha_2) \geq U_1(\alpha_1', \alpha_2) \geq U_1(\alpha_1', \alpha_2'). \tag{1}$$

and similar analysis gives that

$$U_1(\alpha_1, \alpha_2) \leq U_1(\alpha_1, \alpha_2') \leq U_1(\alpha_1', \alpha_2'). \tag{2}$$

– (1) and (2) yield

$$U_1(\alpha_1, \alpha_2) = U_1(\alpha_1', \alpha_2) = U_1(\alpha_1, \alpha_2') = U_1(\alpha_1', \alpha_2')$$

– Since α is an equilibrium

$$U_2(\alpha_1, \alpha_2'') \leq U_2(\alpha_1, \alpha_2) = U_2(\alpha_1, \alpha_2')$$

for any $\alpha_2'' \in \Delta A_2$, and since α' is an equilibrium

$$U_1(\alpha_1'', \alpha_2') \leq U_1(\alpha_1', \alpha_2') = U_1(\alpha_1, \alpha_2')$$

for any $\alpha_1'' \in \Delta A_1$. Therefore, (α_1, α_2') is an equilibrium and similarly also (α_1, α_2').