

Economics 209A
Theory and Application of Non-Cooperative Games
(Fall 2007)
Strategic Games III

Topics: Bayesian equilibrium (OR 2.6), correlated equilibrium (OR 3.3), evolutionary equilibrium (OR 3.4), equivalence and symmetry, rationalizability (OR 4.1), dominance (OR 4.2, 4.3), trembling hand perfection (OR 12.5.1).

Bayesian equilibrium (OR 2.6)

A Bayesian game consists of a finite set N of players, a finite set Ω of decision-relevant states (characteristics of players), and for each player $i \in N$

- a set A_i of actions
- a finite set T_i of types and a signal function $\tau_i : \Omega \rightarrow T_i$
- a probability measure p_i on Ω (prior belief) for which $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$.
- a preference relation \succsim_i on the set of probability measure over $A \times \Omega$.

$a^* \in \times_{(i,t_i)} A_i$ is a Bayes-Nash equilibrium of a Bayesian game

$$\langle N, \Omega, (A_i), (T_i), (\tau_i), (p_i), (\succsim_i) \rangle$$

if it is a *NE* in which the set of players is the set of all pairs (i, t_i) for all $i \in N$ and $t_i \in T_i$, and for each player (i, t_i)

$$a^* \succsim_{(i,t_i)} b^* \Leftrightarrow L_i(a^*, t_i) \succsim_i L_i(b^*, t_i)$$

where $L_i(a^*, t_i)$ is a *lottery* over $A \times \Omega$ that assigns a probability $\frac{p_i(\omega)}{p_i(\tau_i^{-1}(t_i))}$ to

$$(a^*(j, \tau_j(\omega)))_{j \in N, \omega} \text{ if } \omega \in p_i(\tau_i^{-1}(t_i))$$

and zero otherwise.

Example: *BoS* with one-side imperfect information

		$\omega = y$	
		B	S
B	2, 1	0, 0	
S	0, 0	1, 2	

		$\omega = n$	
		B	S
B	2, 0	0, 2	
S	0, 1	1, 0	

Then, the expected payoffs of player 1 are given by

	(B, B)	(B, S)	(S, B)	(S, S)
B	2	$2p$	$2(1-p)$	0
S	0	p	$1-p$	1

For any belief $p \in (0, 1)$, $(B, (B, S))$ is an equilibrium (B is optimal for player 1 given the actions of the two types of player 2 and his beliefs).

Harsanyi (1973)

Consider a game $G = \langle N, (A_i), (u_i) \rangle$ and let $(\epsilon_i(a))_{i \in N, a \in A}$ be a collection of random variables with support $[-1, 1]$ where

- $\epsilon_i = (\epsilon_i(a))_{a \in A}$ is private information and has well-behaved distribution function, and $\epsilon = (\epsilon_i)_{i \in N}$ are independent.
- The payoff of each player i at the outcome a and state ϵ is $u_i(a) + \epsilon_i(a)$. This defines a Bayesian game $G(\epsilon)$.

For almost any game G and any collection ϵ^* , almost any $\alpha \in NE(G)$ is *approachable* – associated with the limit as $\gamma \rightarrow 0$ of a sequence of pure strategy equilibria of the Bayesian game $G(\gamma\epsilon^*)$ (and visa versa).

Correlated equilibrium (OR 3.3)

A correlated equilibrium of a strategic game $G = \langle N, (A_i), (u_i) \rangle$ consists of a finite probability space (Ω, π) and for each player i ,

- an information partition \mathcal{P}_i of Ω , and
- a function $\sigma_i : \Omega \rightarrow A_i$ s.t. $\sigma_i(\omega) = \sigma_i(\omega')$ for any $\omega, \omega' \in P_i \in \mathcal{P}_i$

such that

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_i(\omega), \sigma_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma'_i(\omega), \sigma_{-i}(\omega))$$

for all $\sigma'_i : \Omega \rightarrow A_i$ s.t. $\sigma'_i(\omega) = \sigma'_i(\omega')$ for any $\omega, \omega' \in P_i \in \mathcal{P}_i$.

Example (BoS)

$\Omega = \{x, y, z\}$ and $\pi = (1 - \eta - \varsigma, \eta, \varsigma)$, $\mathcal{P}_1 = \{\{x, y\}, \{z\}\}$ and $\mathcal{P}_2 = \{\{x\}, \{y, z\}\}$, $\sigma_1(x) = \sigma_1(y) = B$ and $\sigma_1(z) = S$.

If $\omega = x$ then player 2 will play $B = BR_2(B)$ and if $\omega = \{y, z\}$ then she will play

$$\left(\frac{\eta}{\eta + \varsigma}, \frac{\varsigma}{\eta + \varsigma} \right).$$

Correlated equilibrium and Nash equilibrium

For every mixed strategy Nash equilibrium α of a finite strategic game $G = \langle N, (A_i), (u_i) \rangle$ there is a correlated equilibrium

$$\langle (\Omega, \pi), (\mathcal{P}_i), (\sigma_i) \rangle$$

in which for each player $i \in N$ the distribution on A_i induced by σ_i is α_i .

Thus, the set of mixed strategy Nash equilibria is a subset of the set correlated equilibria.

Evolutionary stability (OR 3.4)

A single population of players. Players interact with each other pair-wise and randomly matched.

Players are assigned modes of behavior (mutation). u measures each player's ability to survive.

ε of players consists of mutants taking action a while others take action a^* .

Evolutionary stable strategy (*ESS*)

Consider a payoff symmetric game $G = \langle \{1, 2\}, (A, A), (u_i) \rangle$ where $u_1(a) = u_2(a')$ when a' is obtained from a by exchanging a_1 and a_2 .

$a^* \in A$ is *ESS* iff for any $a \in A$, $a \neq a^*$ and $\varepsilon > 0$ sufficiently small

$$(1 - \varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) > (1 - \varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

which is satisfied iff for any $a \neq a^*$ either

$$u(a^*, a^*) > u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) > u(a, a)$$

Three results on *ESS*

[1] If a^* is an *ESS* then (a^*, a^*) is a *NE*.

Suppose not. Then, there exists a strategy $a \in A$ such that

$$u(a, a^*) > u(a^*, a^*).$$

But, for ε small enough

$$(1 - \varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) < (1 - \varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

and thus a^* is not an *ESS*.

[2] If (a^*, a^*) is a strict NE ($u(a^*, a^*) > u(a, a^*)$ for all $a \in A$) then a^* is an *ESS*.

Suppose a^* is not an *ESS*. Then either

$$u(a^*, a^*) \leq u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) \leq u(a, a).$$

so (a^*, a^*) can be a *NE* but not a strict *NE*.

[3] A 2×2 game $G = \langle \{1, 2\}, (A, A), (u_i) \rangle$ where $u_i(a) \neq u_i(a')$ for any a, a' has a mixed strategy which is *ESS* (OR 51.1)

	a	a'
a	w, w	x, y
a'	y, x	z, z

If $w > y$ or $z > x$ then (a, a) or (a', a') are strict *NE*, and thus a or a' are *ESS*.

If $w < y$ and $z < x$ then there is a unique symmetric mixed strategy *NE* (α^*, α^*) where

$$\alpha^*(a) = (z - x) / (w - y + z - x)$$

and $u(\alpha^*, \alpha) > u(\alpha, \alpha)$ for any $\alpha \neq \alpha^*$.

Payoff equivalence

Two games

$$G = \langle N, (A_i), (u_i) \rangle \text{ and } G' = \langle N, (A'_i), (u'_i) \rangle$$

are payoff equivalent if for each player i ,

- A'_i may be relabelled such that $A'_i = A_i$, and
- when $A'_i = A_i \exists (x_i, y_i)$ with $y_i > 0$ such that

$$u'_i(a') = x_i + y_i u_i(a).$$

Two results on payoff equivalence

[1] G and G' have the same mixed strategy NE if they are payoff equivalent.

- Affine transformations on u_i : the preference ordering of each i over ΔA is the same as over $\Delta A'$.

[2] G and G' have the same pure strategy NE under any strictly increasing monotonic transformations on u_i .

- Monotonic transformations on u_i : the preference ordering of each i over A is the same as over A' .

A game

$$G = \langle N, (A_i), (u_i) \rangle$$

is payoff symmetric if it is payoff equivalent to a game

$$G' = \langle N, (A'_i), (u'_i) \rangle$$

in which for any two players, i and j ,

- $A'_i = A'_j$, and
- if players i and j exchange strategies, they also exchange payoffs

$$u'_i(a_1, \dots, a_i, \dots, a_j, \dots, a_N) = u'_j(a_1, \dots, a_j, \dots, a_i, \dots, a_N).$$

Strategic equivalence

Two games

$$G = \langle N, (A_i), (u_i) \rangle \text{ and } G' = \langle N, (A'_i), (u'_i) \rangle$$

are strategically equivalent if for each player i ,

- A'_i may be relabelled such that $A'_i = A_i$, and
- when $A'_i = A_i$

$$B'_i(\alpha'_{-i}) = B_i(\alpha_{-i}).$$

Alternatively, the two games

$$G = \langle N, (A_i), (u_i) \rangle \text{ and } G' = \langle N, (A'_i), (u'_i) \rangle$$

are strategically equivalent if for each player i , $\exists y_i > 0$ and a function $f_i : A_{-i} \rightarrow \mathbb{R}$ such that

$$u'_i(a) = f_i(a_{-i}) + y_i u_i(a)$$

for all $a \in A$.

Fix α_{-i} and let

$$f_i(\alpha_{-i}) \equiv \sum_{a_{-i} \in A_{-i}} \alpha_{-i}(a_{-i}) f_i(a_{-i})$$

Then,

$$\begin{aligned} u'_i(a_i, \alpha_{-i}) &= \sum_{a_{-i} \in A_{-i}} \alpha_{-i}(a_{-i}) u'_i(a_i, a_{-i}) \\ &= \sum_{a_{-i} \in A_{-i}} \alpha_{-i}(a_{-i}) [f_i(a_{-i}) + y_i u_i(a_i, a_{-i})] \\ &= f_i(\alpha_{-i}) + y_i \sum_{a_{-i} \in A_{-i}} \alpha_{-i}(a_{-i}) u_i(a_i, a_{-i}) \\ &= f_i(\alpha_{-i}) + y_i u_i(a_i, \alpha_{-i}) \end{aligned}$$

Thus, $a_i \in B_i(\alpha_{-i})$ iff $a_i \in B'_i(\alpha_{-i})$ and thus G and G' have the same set of NE (the converse is also true if no player has any dominated strategies).

A game

$$G = \langle N, (A_i), (u_i) \rangle$$

is strategically symmetric if it is strategically equivalent to a game

$$G' = \langle N, (A'_i), (u'_i) \rangle$$

in which for any two players, i and j ,

- $A'_i = A'_j$, and
- $B'_i(\alpha_{-i}) = B'_j(\alpha_{-j})$ for all $\alpha \in \Delta A$ with $\alpha_i = \alpha_j$.

A mixed strategy NE of a strategically symmetric game in which each player plays the same strategy is called a symmetric NE .

Rationalizability (OR 4.1)

In equilibrium, each player knows the other players' equilibrium strategies.

Rationalizability and dominance solvability are solution concepts that do not entail this assumption.

Players' beliefs about each other's action are not assumed to be correct, but are constrained by (some) considerations of rationality.

An action a_i is rationalizable for player i in $G = \langle N, (A_i), (U_i) \rangle$ if for each $j \in N$ there exists a set of actions $Z_j \subseteq A_j$ such that

- $a_i \in Z_i$ and
- every action $a_j \in Z_j$ is a best response to a belief of player j that assigns positive probability only to $a_{-j} \in Z_{-j}$.

If $(Z_j)_{j \in N}$ and $(Z'_j)_{j \in N}$ satisfy the definition then so does $(Z_j \cup Z'_j)_{j \in N}$ so the set of profiles of rationalizable actions is $\times_{j \in N} Z_j$.

Every action a_i such that $\alpha_i^*(a_i) > 0$ and $\alpha^* \in NE(G)$ is rationalizable.

On the other hand, a strictly dominated action is never rationalizable (OR Lemma 60.1).

An example (OR 57.1)

	b_1	b_2	b_3	b_4
a_1	0, 7	2, 5	7, 0	0, 1
a_2	5, 2	3, 3	5, 2	0, 1
a_3	7, 0	2, 5	0, 7	0, 1
a_4	0, 0	0, -2	0, 0	10, -1

The rationalizable actions are a_1, a_2, a_3 for player 1 and b_1, b_2, b_3 for player 2.

Dominance (OR 4.2)

An action $a_i \in A_i$ of player i is strictly dominated if there exists a mixed strategy α_i such that

$$U_i(a_i, a_{-i}) < U_i(\alpha_i, a_{-i})$$

for all $a_{-i} \in A_{-i}$.

An action $a_i \in A_i$ of player i is weakly dominated if there exists a mixed strategy α_i such that

$$U_i(a_i, a_{-i}) \leq U_i(\alpha_i, a_{-i})$$

for all $a_{-i} \in A_{-i}$ and the inequality is strict for some $a_{-i} \in A_{-i}$.

Two results on dominated strategies

- [1] An action of a player in a finite strategic game G is never a best response *iff* it is strictly dominated.
- [2] Consider G' obtained by iterated removal of all (weakly and strictly) dominated strategies from G then
- if $a \in NE(G')$ then $a \in NE(G)$, and
 - the converse holds for the iterated removal of only strictly dominated strategies.

Trembling hand perfection (THP) (OR 12.5.1)

A THP equilibrium of a finite strategic game is a mixed strategy profile α such that there exists $(\alpha^k)_{k=1}^{\infty}$ of completely mixed strategy profiles such that

- $(\alpha^k)_{k=1}^{\infty}$ converges to α , and
- $\alpha_i \in BR_i(\alpha_{-i}^k)$ for each player i and all k .

A strategy profile α^* in a two-player game is a THP equilibrium *iff* it is a mixed strategy NE and the strategy of neither player is weakly dominated.