Economics 219D
Experimental economics
(Spring 2014)

Quantal Response Equilibrium (QRE)
Lecture V

Quantal Response Equilibrium (QRE)

- Players do not choose best response with probability one (as in Nash equilibrium).

- Players choose responses with higher expected payoffs with higher probability – better response instead of best responses.

- Players have rational expectations and use the true mean error rate when interpreting others' actions.

- Modify Nash equilibrium to incorporate realistic limitations to rational choice modeling of games.

- Provide a statistical framework (structural econometric approach) to analyze game theoretic data (field and laboratory).

- If Nash had been a statistician, he might have discovered QRE rather then Nash equilibrium – Colin Camerer –

In practice, QRE often uses a logit payoff response function:

$$\mathsf{Pr}(a_i) = \frac{\mathsf{exp}[\lambda \sum_{a_{-i} \in A_{-i}} \mathsf{Pr}(a_{-i}) u_i(a_i, a_{-i})]}{\sum_{a_i' \in A_i} \mathsf{exp}[\lambda \sum_{a_{-i} \in A_{-i}} \mathsf{Pr}(a_{-i}) u_i(a_i', a_{-i})]}.$$

The choice of action becomes purely random as $\lambda \longrightarrow 0$, whereas the action with the higher expected payoff is chosen for sure as $\lambda \longrightarrow \infty$.

- QRE does not abandon the notion of equilibrium, but instead replaces perfectly with imperfectly, or noisy, rational expectations.

- Players estimate expected payoffs in an unbiased way (expectations are correct, on average).

- As such, QRE provides a convenient statistical structure for estimation using either field or experimental data.

Normal-form games

Consider a finite n-player game in normal form:

- a set $N = \{1, ..., n\}$ of players,
- a strategy set $A_i = \{a_{i1}, ..., a_{iJ_i}\}$ consisting of J_i pure strategies for each player $i \in N$,
- a utility function $u_i : A \to \mathbb{R}$, where $A = \prod_{i \in N} A_i$ for every player $i \in N$.

Let Δ_i be the set of probability measures on A_i:

$$\Delta_i = \{(p_{i1}..., p_{iJ_i}) : \sum_{ij} p_{ij} = 1, p_{ij} \geq 0\}$$

where $p_{ij} = p_i(a_{ij})$.

The notation (a_{ij}, p_{-i}) represents the strategy profile where i adopts a_{ij} and all other players adopt their components of $p = (p_i, p_{-i})$.

A profile $p = (p_1, ..., p_n)$ is a Nash equilibrium if for all $i \in N$ and all $p_i' \in \Delta_i$

$$u_i(p) \geq u_i(p_i', p_{-i}).$$

Let $X_i = \mathbb{R}^{j_i}$ represent the space of possible payoffs for strategies that i can adopt and let $X = \prod_{i \in N} X_i$.

Then, define the function $\bar{u} : \mathbf{\Delta} \to X$ by

$$\bar{u}(p) = (\bar{u}_i(p), ..., \bar{u}_n(p)),$$

where

$$\bar{u}_{ij}(p) = u_i(a_{ij}, p_{-i}).$$

A quantal response equilibrium

A version of Nash equilibrium where each player's payoff for each action is subject to random error. Specifically:

[1] For each player i and each action $j \in \{1, ..., J_i\}$, and for any $p \in \mathbf{\Delta}$, let

$$\hat{u}_{ij}(p) = \bar{u}_{ij}(p) + \epsilon_{ij}$$

where player i error vector $\epsilon_i = (\epsilon_{i1}, ..., \epsilon_{iJ_i})$ is distributed according to a joint PDF $f_i(\epsilon_i)$.

$f = (f_1, ..., f_n)$ is <u>admissible</u> if, for each i, the marginal distribution of f_i exists for each ϵ_{ij} and $\mathbb{E}(\epsilon_i) = 0$.

[2] For any $\bar{u} = (\bar{u}_1, ..., \bar{u}_n)$ with $\bar{u}_i \in \mathbb{R}^{j_i}$ for each i, define the ij-response set $\mathbf{R}_{ij} \subseteq \mathbb{R}^{j_i}$ by

$$\mathbf{R}_{ij}(\bar{u}_i) = \{\epsilon_i \in \mathbb{R}^{j_i} : \bar{u}_{ij}(p) + \epsilon_{ij} \geq \bar{u}_{ik}(p) + \epsilon_{ik} \forall k = 1, .., J_i\},$$

that is, given p, $\mathbf{R}_{ij}(\bar{u}_i(p))$ specifies the region of errors that will lead i to choose action j.

[3] Let the probability that player i will choose action j given $\bar{u}$ be equal

$$\sigma_{ij}(\bar{u}_i) = \int_{\mathbf{R}_{ij}(\bar{u}_i)} f(\epsilon) d\epsilon.$$

The function $\sigma_i : \mathbb{R}^{j_i} \longrightarrow \Delta^{J_i}$ is called the quantal response function (or statistical reaction function) of player i.

Let $G = \langle N, A, u \rangle$ be a normal form game, and let f be admissible. A QRE of G is any $\pi \in \Delta$ such that

$$\pi_{ij} = \sigma_{ij}(\bar{u}_i(\pi))$$

for all $i \in N$ and $1 \leq j \leq J_i$.

The quantal response functions

Properties of quantal response functions σ_{ij}:

[1] $\sigma \in \mathbf{\Delta}$ is non empty.

[2] σ_i is continuous in $\mathbb{R}^{j_i}$.

[1] and [2] imply that for any game G and for any admissible f, there exists a QRE.

[3] σ_{ij} is monotonically increasing in $\bar{u}_{ij}$.

[4] If, for each player i and every pair of actions $j, k = 1, .., J_i$, ϵ_{ij} and ϵ_{ik} are i.i.d., then

$$\bar{u}_{ij} \geq \bar{u}_{ik} \Longrightarrow \sigma_{ij}(\bar{u}) \geq \sigma_{ik}(\bar{u})$$

for all i and all $j, k = 1, .., J_i$.

[4] states that σ_i orders the probability of different actions by their expected payoffs.

A logit equilibrium

For any given $\lambda \geq 0$, the logistic quantal response function is defined, for $x_i \in \mathbb{R}^{j_i}$, by

$$\sigma_{ij}(x_i) = \frac{\exp(\lambda x_{ij})}{\sum_{k=1}^{J_i} \exp(\lambda x_{ik})},$$

and the QRE or logit equilibrium requires

$$\pi_{ij}(x_i) = \frac{\exp(\lambda \bar{u}_{ij}(\pi))}{\sum_{k=1}^{J_i} \exp(\lambda \bar{u}_{ik}(\pi))}$$

for each i and j.

Result I: Let σ be the logistic quantal response function; $\{\lambda_1, \lambda_2, ...\}$ be a sequence such that $\lim_{t\to\infty} \lambda_t = \infty$; $\{p_1, p_2, ...\}$ be a corresponding sequence with $p_t \in \pi^*(\lambda_t)$ for all t where

$$\pi^*(\lambda) = \left\{ \pi \in \Delta : \pi_{ij} = \frac{\exp(\lambda \bar{u}_{ij}(\pi))}{\sum_{k=1}^{J_i} \exp(\lambda \bar{u}_{ik}(\pi))} \forall i, j \right\}$$

is the logit correspondence.

Then, $p^* = \lim_{t\to\infty} p_t$ is a Nash equilibrium.

Proof: Assume p^* is not a Nash equilibrium. Then, there is some player $i \in N$ and some pair a_{ij} and a_{ik} with $p^*(a_{ik}) > 0$ and

$$u_i(a_{ij}, p^*_{-i}) > u_i(a_{ik}, p^*_{-i}) \text{ or } \bar{u}_{ij}(p^*) > \bar{u}_{ik}(p^*).$$

Since $\bar{u}$ is a continuous function, there exists some small ϵ and T, such that for all $t \geq T$,

$$\bar{u}_{ij}(p^t) > \bar{u}_{ik}(p^t) + \epsilon.$$

But as $t \to \infty$, $\sigma_k(\bar{u}_i(p^t))/\sigma_j(\bar{u}_i(p^t)) \to 0$ and thus $p^t(a_{ik}) \to 0$, which contradicts $p^*(a_{ik}) > 0$.

Result II: For almost any game G:

[1] $\pi^*(\lambda)$ is odd for almost all π.

[2] π^* is UHC.

[3] The graph of π^* contains a unique branch which starts at the centroid, for $\lambda = 0$, and converges the a unique NE, as $\lambda \to \infty$.

[3] implies that QRE defines a unique selection from the set of Nash equilibrium (the "tracing procedure" of Harsanyi and Selten, 1988).

Example I

Consider the game

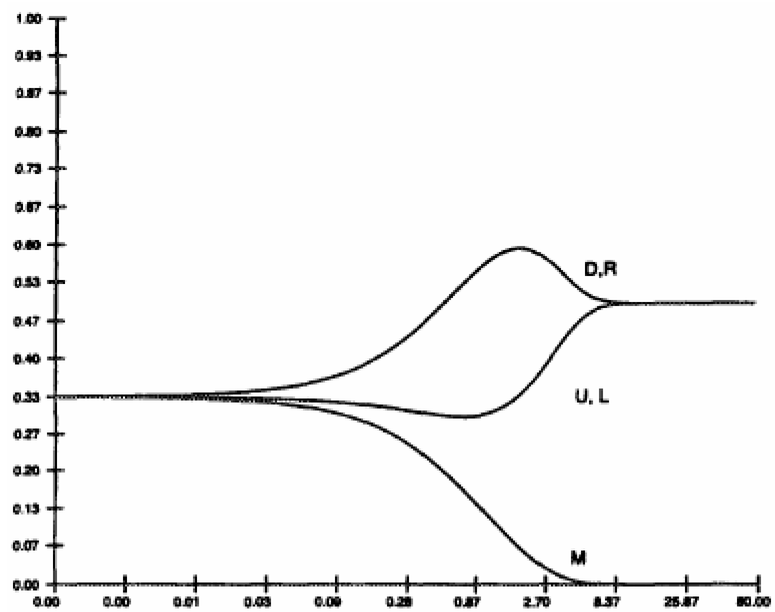
	L	M	R
U	$1, 1$	$0, 0$	$1, 1$
M	$0, 0$	$0, 0$	$0, B$
D	$1, 1$	$A, 0$	$1, 1$

where $A > 0$ and $B > 0$.

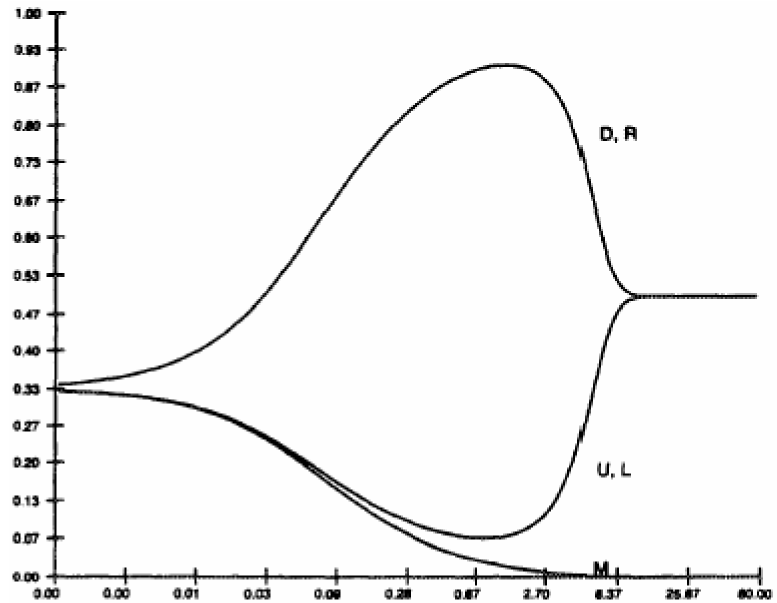
The game has a unique THP (D, R), and the NE consists of all mixtures between U and D (resp. L and R) for player 1 (resp. 2).

The limit logit equilibrium selects $p = (\frac{1}{2}, 0, \frac{1}{2})$ and $q = (\frac{1}{2}, 0, \frac{1}{2})$ as the limit point.

QRE for example I with *A*=*B*=5



QRE for example I with *A*=*B*=100



Example II

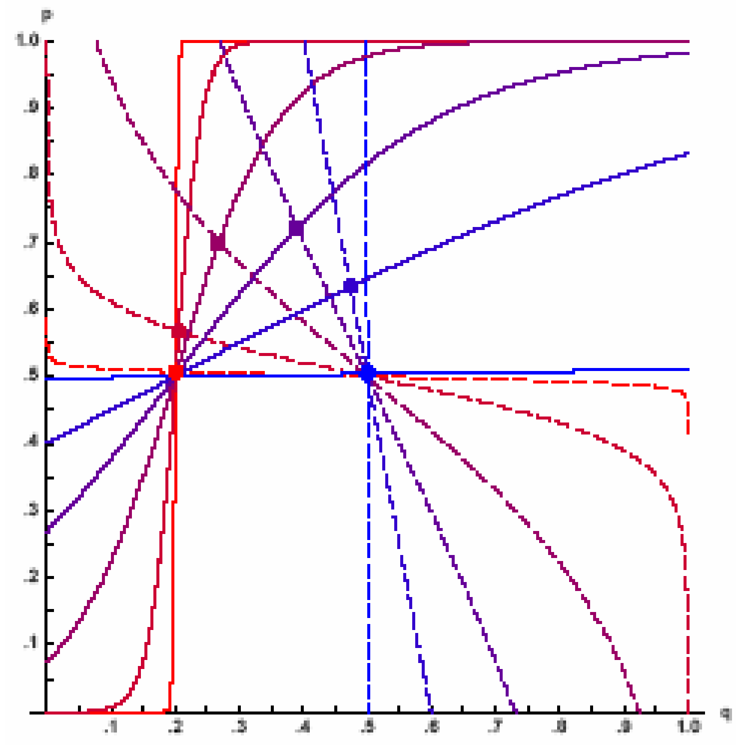
Consider the game

	R	L
T	$x, 1$	$1, 2$
B	$1, 2$	$2, 1$

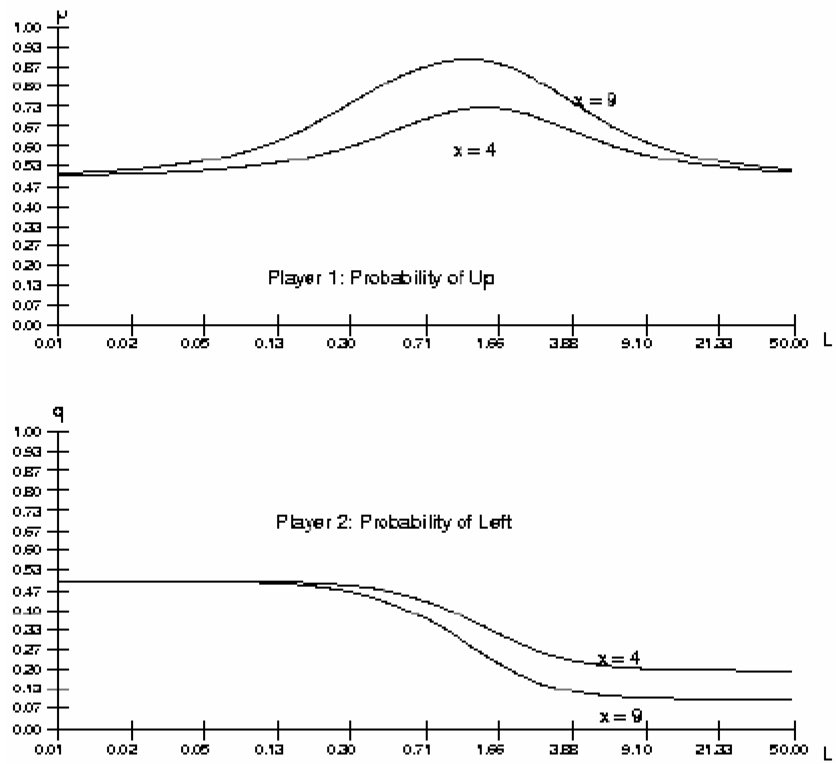
All limit points are Nash equilibria but not all Nash equilibria are limit points (refinement). Computable in small finite games (Gambit).

QRE for example II

Properties of the QRE correspondence



QRE for example II
Own-payoff Effects



Relation to Bayesian equilibrium

In a Bayesian game (Harsanyi 1973), ϵ_i is viewed as a random disturbance to player i's payoff vector.

Suppose that for each $a \in A$, player i has a disturbance ϵ_{ij} added to $u_i(a_{ij}, a_{-i})$ and that each ϵ_{ij} is i.i.d. according to f.

Harsanyi (1973) assumes a separate disturbance $\epsilon_i(a)$ for i's payoff to each strategy profile $a \in A$, whereas here

$$\epsilon_{ij}(a_{ij}, a_{-i}) = \epsilon_{ij}(a_{ij}, a'_{-i})$$

for all ij and all $a_{-i}, a'_{-i} \in A_{-i}$.

QRE inherits the properties of Bayesian equilibrium:

[1] An equilibrium exists.

[2] Best responses are "essentially unique" pure strategies.

[3] Every equilibrium is "essentially strong" and is essentially in pure strategies.

Example - monotone games

- A monotone game is an extensive-form game with simultaneous moves and an irreversibility structure on strategies.

- It captures a variety of situations in which players make partial commitments.

- We characterize conditions under which equilibria result in efficient outcomes.

- The game has many equilibrium outcomes so the theory lacks predictive power.

The game

- An indivisible public project with cost K and N players, each of whom has an endowment of E tokens.

- The players simultaneously make *irreversible* contributions to the project at a sequence of dates $t = 1, ..., T$.

- The project is carried out if and only if the sum of the contributions is large enough to meet its cost.

- If the project is completed, each player receives A tokens *plus* to the number of tokens retained from his endowment.

[1] The aggregate endowment is greater than the cost of the project (completion is feasible)

$$NE > K.$$

[2] The aggregate value of the project is greater than the cost (completion is efficient)

$$NA > K.$$

[3] The project is not completed by a single player (either it is not feasible or it is not rational)

$$\min\{A, E\} < K.$$

The static game

The game is essentially the same as the static game in which all players make simultaneous binding decisions.

Proposition (one-shot) *(i) There exists a pure-strategy Nash equilibrium with no completion. Conversely, there exists at least one pure-strategy equilibrium in which the project is completed with probability one. (ii) The game also possesses mixed-strategy equilibria in which the project is completed with positive probability.*

The indivisibility of the public project makes each contributing player "pivotal" (Bagnoli and Lipman (1992)).

The extensive-form game

The sharpest result is obtained for the case of pure-strategy sequential equilibria.

Proposition (pure strategy) *Suppose that $A > E$ and $T \geq K$. Then, under the maintained assumptions, in any pure strategy sequential equilibrium of the game, the public project is completed with probability one.*

In any pure strategy equilibrium, the probability of completion is either zero or one, so it is enough to show that the no-completion equilibrium is not sequential.

Mixed strategies expand the set of parameters for which there exists a no-completion equilibrium.

Proposition (mixed strategy) *Suppose that $A > E$ and $T \geq K$. Then there exists a number $A^*(E, K, N, T)$ such that, for any $E < A < A^*$ there exists a mixed strategy equilibrium in which the project is completed with probability zero.*

The use of mixed strategies in the continuation game can discourage an initial contribution and support an equilibrium with no completion.

The games in which $K = NE$ provide a useful benchmark (no possibility of taking a free ride on the contributions of other players).

Proposition (no-free-riding) *Suppose that $K = NE$, $A > E$ and $T \geq K$. Then the project is completed with probability one in any sequential equilibrium of the game.*

The result does not rule out the use of mixed strategies, even along the equilibrium path.

Taking $K = NE$ as a benchmark for the absence of free riding, the free-rider problem must be worse when the total endowment exceeds this level.

Proposition (free-riding) *Suppose that $E > A$ and $T \geq K$. Then under the maintained assumptions, there exists a pure strategy sequential equilibrium of the game in which the public project is completed with probability zero.*

The essential ingredient in the construction of this equilibrium is the self-punishing strategy employed by Gale (2001).

Symmetric Markov perfect equilibrium (SMPE)

The class of SMPE takes a relatively simple form. The main predictions from SMPE can be summarized by four facts:

– There are no pure strategy SMPE, although mixed strategies may only be used off the equilibrium path.

– There is no completion of the public project in early periods when A "high" and no completion at all when A "low."

– The contribution probability at each state when A is "high" is at least as high as when A is "low."

– A game with horizon $T < T'$ is isomorphic to a continuation game starting in period $T' - T$ of the game with horizon T'.

Example 1

$$A = 3, E = 1, K = 2, N = 3, T = 5$$

τ/n	0			1
4	0.00			$--$
3	0.00			0.00
2	0.00			0.00
1	0.56	0.55	0.00	0.00
0	0.00	0.21	0.79	0.67

where n is the total number of contributions and τ is the number of periods remaining after the current period.

Example 2

$$A = 1.5, E = 1, K = 2, N = 3, T = 5$$

τ / n	0	1
4	0.00	$--$
3	0.00	0.00
2	0.00	0.00
1	0.00	0.00
0	0.00	0.33

Example 3

$$A = 3, E = 2, K = 2, N = 3, T = 5$$

$\tau/(n, n_i)$	0	$(1,0)$	$(1,1)$
4	0.00	$--$	$--$
3	0.00	0.00	0.00
2	0.00	0.00	0.00
1	0.50| 0.48| 0.00	0.00	0.00
0	0.00| 0.21| 0.79	0.42	0.42

where n_i is the total number of contributions to date by player i.

Example 4

$$A = 1.5, E = 2, K = 2, N = 3, T = 5$$

$\tau/(n, n_i)$	0	$(1,0)$	$(1,1)$
4	0.00	$--$	$--$
3	0.00	0.00	0.00
2	0.00	0.00	0.00
1	0.00	0.00	0.00
0	0.00	0.21	0.21

The Markov property reduces the set of sequential equilibria, sometimes substantially.

Equilibrium outcomes

- The SMPE explains the qualitative patterns of contributions in the game.

- The other results on provision rates are all consistent with the qualitative predictions of the SMPE.

- The deviations from the SMPE contribution probabilities at earlier and later periods go in opposite directions.

- QRE replicates the tendency of early contributions in games, which could not be captured by the SMPE.

Frequencies of contribution

$A=3, E=1, K=2, N=3$

τ / n	0	1	2
4	0.09 (270)		
3	0.08 (207)	0.11 (38)	0 (2)
2	0.11 (165)	0.07 (54)	0.25 (8)
1	0.37 (117)	0.07 (76)	0.10 (10)
0	0.36 (36)	0.60 (94)	0.08 (24)

τ / n	0	1	2
1	0.18 (270)		
0	0.62 (159)	0.54 (54)	0 (9)

( ) - # of obs.

A =1.5, E =1, K =2, N =3

τ / n	0	1	2
4	0.09 (270)		
3	0.05 (207)	0.03 (36)	0 (3)
2	0.06 (177)	0.06 (54)	0.25 (4)
1	0.26 (144)	0.19 (70)	0.17 (6)
0	0.20 (57)	0.48 (88)	0.09 (23)

τ / n	0	1	2
1	0.18 (270)		
0	0.35 (150)	0.33 (64)	0 (7)

( ) - # of obs.

A =3, E =2, K =2, N =3

τ / n	(0,0)	(1,0)	(1,1)
4	0.14 (270)		
3	0.03 (165)	0.02 (52)	0.12 (26)
2	0.07 (153)	0.04 (50)	0.08 (25)
1	0.3 (126)	0.08 (60)	0 (30)
0	0.53 (45)	0.46 (84)	0.26 (42)

τ / n	0	1	2
1	0.34 (270)		
0	0.44 (75)	0.34 (70)	0.11 (35)

( ) - # of obs.

$A=1.5$, $E=2$, $K=2$, $N=3$

τ / n	(0,0)	(1,0)	(1,1)
4	0.06 (270)		
3	0.05 (228)	0.09 (22)	0.00 (11)
2	0.13 (195)	0.05 (40)	0.15 (20)
1	0.21 (126)	0.07 (70)	0.00 (35)
0	0.04 (63)	0.39 (92)	0.07 (46)

τ / n	(0,0)	(1,0)	(1,1)
1	0.26 (270)		
0	0.13 (111)	0.38 (70)	0.00 (35)

( ) - # of obs.

The relative frequencies of contributions from the different histories

$E=1$, $K=2$, $N=3$, $T=5$

A	(n,τ)	$h(1)$	$h(2)$	$h(3)$	$h(4)$	p-value
1.5	(1,2)	0.03 (34)	0.10 (20)	–	–	0.63
	(1,1)	0.06 (32)	0.25 (16)	0.32 (22)	–	0.05
	(1,0)	0.54 (28)	0.25 (8)	0.30 (10)	0.52 (42)	0.30
3	(1,2)	0.00 (30)	0.17 (24)	–	–	0.07
	(1,1)	0.00 (30)	0.06 (18)	0.14 (28)	–	0.21
	(1,0)	0.47 (30)	0.75 (18)	0.60 (20)	0.64 (28)	0.27

$E=2$, $K=2$, $N=3$, $T=5$

A	(n,τ)	$h(1)$	$h(2)$	$h(3)$	$h(4)$	p-value
1.5	(1,2)	0.56 (18)	0.45 (22)	–	–	0.25
	(1,1)	0.00 (10)	0.05 (20)	0.10 (40)	–	0.50
	(1,0)	0.50 (10)	0.33 (18)	0.47 (32)	0.31 (32)	0.12
3	(1,2)	0.05 (44)	0.00 (6)	–	–	0.10
	(1,1)	0.11 (38)	0.00 (6)	0.06 (16)	–	0.60
	(1,0)	0.43 (30)	0.67 (6)	0.57 (14)	0.41 (34)	0.53

Quantal Response Equilibrium (QRE)

For simplicity, suppose that each player has an endowment of one token ($E = 1$).

The contribution behavior of each uncommitted player at state (n, τ) follows a binomial logit distribution:

$$\lambda_{(n,\tau)} = \frac{1}{1 + \mathsf{exp}(-\beta_{(n,\tau)}\Delta_{(n,\tau)})},$$

where $\Delta_{(n,\tau)}$ is the difference between the expected payoffs from contributing and not contributing, and $\beta_{(n,\tau)}$ is a coefficient.

The calculation of QRE proceeds by backward induction, beginning with the final period.

QRE estimation results and the probability of contribution

$A=3$, $E=1$, $K=2$, $N=3$

β=10.05 (0.78), Log_lik =-472.52

τ / n	0	1	2
4	0.11		
3	0.14	0.07	0.00
2	0.18	0.10	0.00
1	0.20	0.17	0.00
0	0.75	0.65	0.00

β = 10.51 (1.27), Log_lik = -278.55

τ / n	0	1	2
1	0.19		
0	0.76	0.65	0

A =1.5, E =1, K =2, N =3

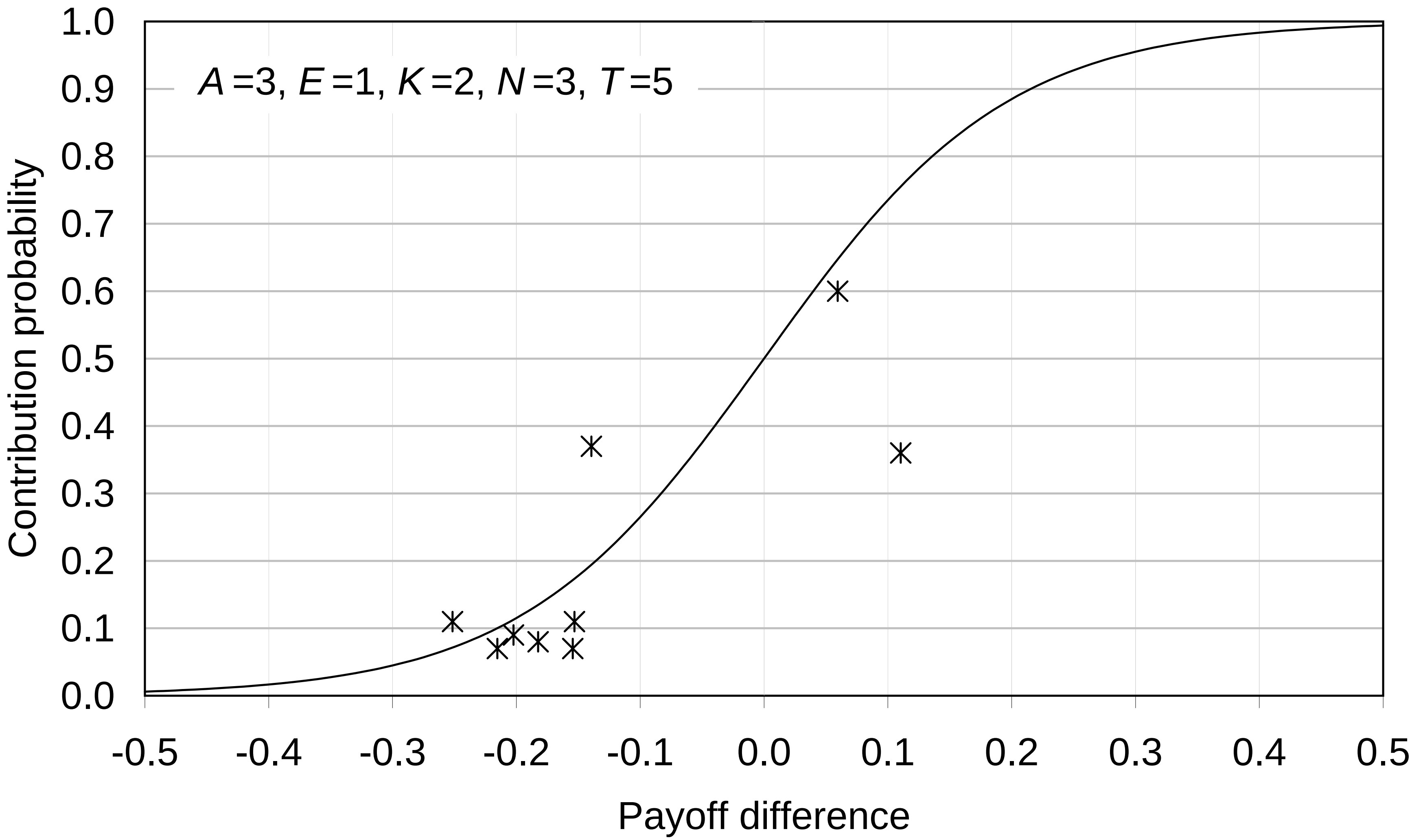
β=12.34 (0.83), Log_lik =-475.01

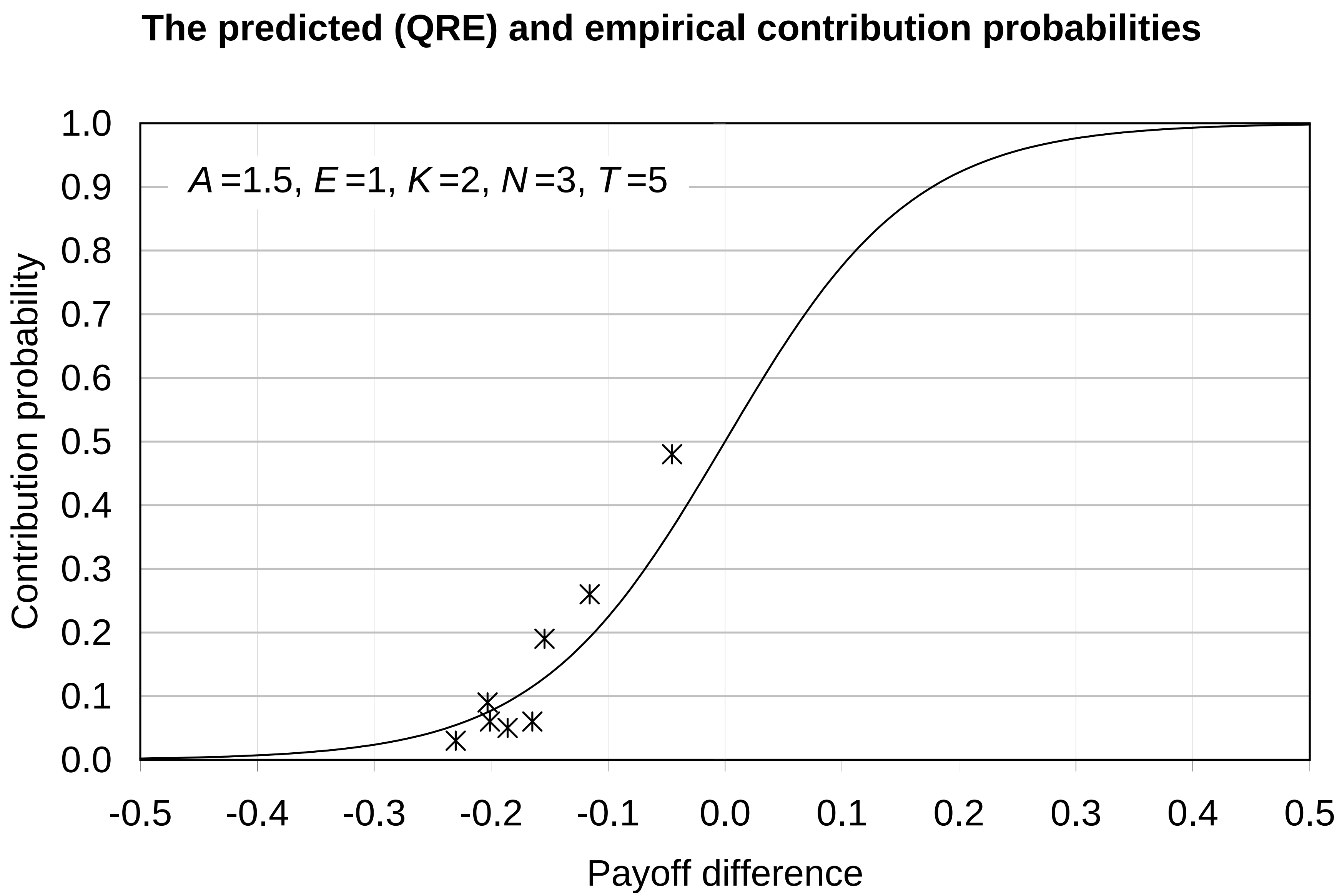
τ / n	0	1	2
4	0.08		
3	0.09	0.06	0.00
2	0.12	0.08	0.00
1	0.19	0.13	0.00
0	0.00	0.36	0.00

β = 2.26 (0.20), Log_lik = -296.41

τ / n	0	1	2
1	0.4		
0	0.3	0.42	0.09

The predicted (QRE) and empirical contribution probabilities




The predicted (QRE) and empirical contribution probabilities
A =1.5, E =1, K =2, N =3, T =5
Contribution probability
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.5
-0.4
-0.3
-0.2
-0.1
0.0
0.1
0.2
0.3
0.4
0.5
Payoff difference